Guarantee and embedded options

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Abstract

Recent economic events, the changing attitude of regulators, and a possible global change in reporting to shareholders have put the options and guarantees included in today's insurance products at the top of the list of management's concerns. Indeed, many senior insurance executives around the world have indicated that they regard the identification, pricing, management, and valuation of guarantees and options embedded in insurance contracts as the most important and difficult financial challenge they face.

On too many occasions over the years, insurance company executives have been surprised by the financial costs of embedded guarantees and options that were not properly understood. An understanding of options and guarantees has become essential to the sound financial management of insurance companies. This paper provides an introduction to the valuation of guarantees and options embedded in life insurance products. It explains the link between investment guarantees and embedded options, and consequently their measurement on a market-value basis. It also explores the implications of guarantees and options for asset and liability valuation, product pricing, and managing balance-sheet volatility.

To show practical application of the concepts introduced we also show a case study through the investigation of a class of investment guarantees that prevail in continental Europe.

Keywords

Guarantees and embedded option, risk management, option pricing, minimum interest rate guarantee, Monte Carlo simulation, Lattice, Replication

1. Introduction

1.1 Guarantees Matter for Insurers

Options and guarantees are common in insurance contracts everywhere in the world. They have been offered for a number of reasons, including:

- Enhanced competitiveness, by providing the policyholder with additional certainty
- Regulatory requirements (such as guaranteed minimum cash values)
- Requirements to receive favorable tax treatment (either for the issuer or for the policyholder)

The additional certainty provided to the policyholder comes at a cost that may or may not be passed on to the policyholder when pricing the contract.

Accounting standards increasingly require fair valuation. Under Canadian GAAP, for example, the cost of the option must be reflected in the valuation of the contract. In other parts of the world, similar requirements are being considered. Examples of guarantees that have received increased attention recently include guaranteed minimum death benefits (GMDBs) in the US, and guaranteed annuity options (GAOs) in the UK.

The Draft Standard of Principles (DSOP), which is the working document for valuing insurance contracts under the new International Accounting Standards (IAS), states in principle 5.6 that such guarantees should be valued in a manner consistent with option-pricing techniques. In addition, IAS 39: Financial Instruments: Recognition and Measurement requires embedded derivatives to be valued and disclosed at fair value.

Good business management requires an understanding of their cost or associated risk

The proposed valuation approach based on an option-pricing model is likely to have a significant impact on the valuation of the liabilities, with consequent balance-sheet and income statement volatility. It follows that companies will reconsider their investment, pricing and discretionary policy benefits strategies as they develop a more thorough understanding of the costs and dynamics of options.

This paper provides an introduction to the valuation of guarantees and options embedded in life insurance products. It explains the link between investment guarantees and embedded options, and consequently their measurement on a market-value basis. It also explores the implications of guarantees and options for asset and liability valuation, product pricing, and managing balance-sheet volatility.

2. Understanding the Costs of Guarantees

2.1 COMPONENTS OF THE GUARANTEE COST

Where an insurer promises to pay a certain quantity, the valuation of policy benefits is usually straightforward. It becomes more complicated when the promise is to pay the greater of two quantities, say, A and B, where it is not certain which amount will be greater. For example, when the guarantee is defined as the greater of the accumulation of premiums invested in an equity index-tracking fund, or a refund of the premiums paid, then it is uncertain which will be the greater amount and therefore the final benefit. In this case, the valuation should reflect the uncertainty and potential for

either amount to be paid, which will have an additional cost over considering just one component.
The principles of IAS recognize the need to reflect the associated additional cost since the valuation must be at fair value and consistent with option-pricing techniques. In the above example, the additional cost is related to the uncertainty around the performance of the equity index-tracking fund.

Where there is uncertainty, the cost of the guarantee can be calculated as the cost of the normal benefit under the contract (say, quantity A) plus the residual cost of the second benefit, i.e., the guaranteed cost (excess of B over A). This is described in simple Example 1.
Expressing the contract as a host contract plus embedded option leads one to apply market valuation techniques to value the guarantee as an embedded option, with simpler techniques applied to the host contract. Some examples of the quantities being compared in a typical guarantee and option are summarized in Table 1.
However, options and guarantees do not always need to be split out to value the associated liability. For example, consider a policy that pays a fixed cash sum at a particular policy anniversary. If there is an asset that is guaranteed to pay the same fixed sum, then the policy may be valued by discounting the guaranteed payment at the appropriate market rate. The value of the liability would be equal to the market value of the asset. Equivalently, the policy may also be valued by splitting out the liability into the accumulation of premiums plus the excess of the fixed cash sum over the accumulated premiums. The latter approach may encourage the company to invest part of its premiums in an asset or derivative to match the excess benefit.

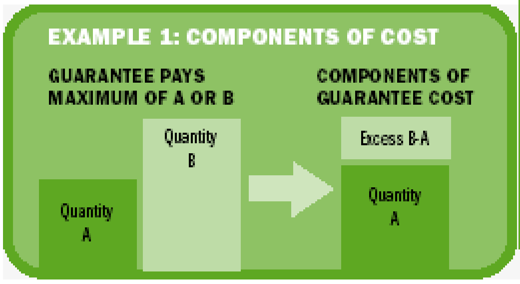


TABLE 1: SAMPLE COMPONENTS

SAMPLE CONTRACT		QUANTITY A	QUANTITY B
UK	Traditional participating insurance with guaranteed sums assured and reversionary bonuses (paid-up additions to sums assured in US parlance)	Asset Share	Sum assured plus vested bonuses
Canada	Segregated fund (i.e., unit linked) with a minimum maturity benefit of return of premium accumulated at a guaranteed minimum crediting rate	Account Balance	Premium accumulated at the guaranteed rate over the term of the contract
US	Guaranteed Minimum Income Benefit (similar to Guaranteed Annuity Option in the UK)	Account Balance	Funds required to purchase the guaranteed annuity, using current terms.

2.2 The Link Between Insurance Guarantees and Traded Options

Options are derivative instruments where one party has the right, but not the obligation, to buy or sell a specified underlying financial instrument at a future date or dates on predefined terms (e.g., at a fixed strike price). The option to buy is known as a call option, and the option to sell is known as a put option. These options can be shown to be very similar to guarantees implicit in many insurance contracts.
Identifying the embedded option, if any, in a more general situation involves bifurcating the benefits into ‘normal benefits’ and the ‘extra guarantee benefit’.
When the guarantee is expressed as the greater of A and B, this is equivalent to expressing the normal benefit as A, plus the positive excess, if any, of B minus A. The

extra guarantee benefit is also equivalent to the payoff of an option. For example, when the normal benefits are A, with a guaranteed minimum of B, this is equivalent to a put option on A, with a strike price of B.
This bifurcation of benefits into normal and excess, and the link between excess and option payoffs, are illustrated in Example 2.

EXAMPLE 2: LINKING GUARANTEE COSTS AND OPTION-PRICING

On the day the benefits become payable:

Value of Total Benefits = Normal Benefits plus Guarantee Cost
= Quantity A
plus (Excess, if any, of Quantity B over Quantity A)
= A + Greater of (B-A) and zero

The Link

Option Payoff = Greater of (B-A) and zero
= Guarantee Cost

EXAMPLE 3: PUT OPTION

Consider a put option to sell stock in a company at \$6.00 in 12 months. In most cases the options are settled for the net gain (option payoff) at maturity rather than requiring a physical trade. In our put option example, the payoff or net gain would be the excess of \$6.00 over the company's share price, if any, at the expiration of the option.

This is very similar to an insurance guarantee that has promised to invest the policyholder's premium in the relevant stock and has guaranteed to provide a benefit to the policyholder in 12 months' time, based on a minimum stock price of \$6.00. If, 12 months later, the stock price dropped below \$6.00, (say, to \$5.50) then the guarantee would have value and the insurance company would have to provide benefits to the policyholder based on the stock price of \$6.00, even though the price actually was at \$5.50 at the payment date. The insurance company would be able to apply the final value of the stock, \$5.50, toward paying the policyholder's benefit, but would have to pay the balance from its own resources.

If, however, the insurance company had recognition of the value of the guarantee at policy issuance, included this value in the policyholder's premium, and invested that value in the put option, then it would have exactly the right amount of funds to pay the policyholder's benefits, regardless of the final stock price. If the stock price finished below \$6.00, the put option would pay off exactly the right amount of funds to enable the insurance company to meet its shortfall when the guarantee bit. If the stock price culminated at above \$6.00, then the put option would not pay anything to the insurance company. The guarantee would have expired without value and there would be no shortfall to make up. In addition, since the product's price would have included the cost of the put option, there would be no ultimate net cost to the company.

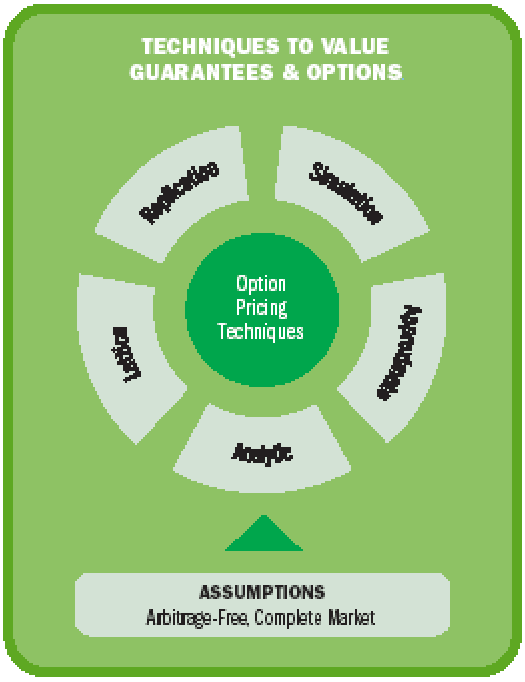
3. Approaches to Valuing Guarantees

There are several approaches and methods to valuing guarantees and options in a manner consistent with the financial markets. These include:

- Replicating portfolio techniques
- Analytic (closed-form) solutions
- Simulation methods, including use of risk-neutral models and deflators
- Lattice methods
- Approximate methods

All of these rely on the principle that the market is arbitrage-free, which means that benefits cannot be guaranteed above a risk-free return without there being an additional cost to someone. In practice, any price variations between different assets that have the same cash flows are likely to be unsustainable and short-lived. The cheaper asset would be bought, and the more expensive asset sold until the arbitrage opportunity was eliminated.

Note that all of the valuation methods should produce the same market value result, assuming the market is arbitrage-free and sufficiently complete to identify traded assets that will replicate the cash flows of the guarantee or option. However, it may be that the market is not sufficiently complete with respect to insurance risks, and there is no uniquely defined market-consistent value, although there may be upper and lower bounds on the possible values. In this case assumptions would have to be made, and the range of possible values would have to be considered carefully.



3.1 Replicating Portfolio Techniques

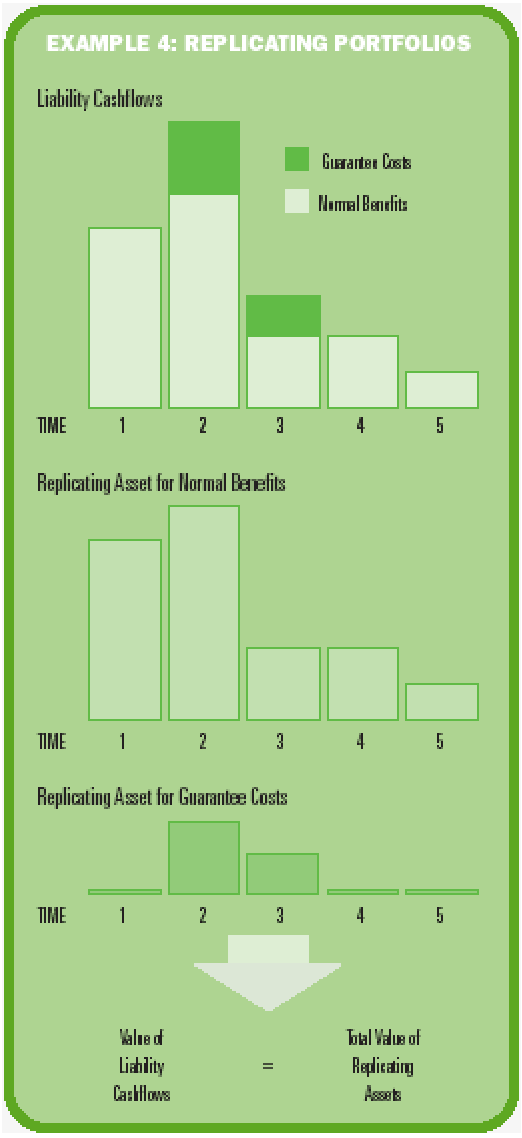
In certain situations, it is possible to find assets that will have cash flows that exactly match those of a liability under each and every possible future scenario. These assets are then described as a replicating portfolio for that liability (see Example 4). An example might be a put option to match a minimum maturity guarantee liability.

The market value of these assets must also be the market-consistent or fair value of that liability, since otherwise arbitrage between the two values for the same cash flows would be possible. Unfortunately, finding a matching portfolio is not always easy. This is especially the case when demographic risk factors and dynamic policyholder behaviors are taken into account.

3.1.1 Static v.s. Dynamic

It is important to distinguish between a static replicating portfolio (a portfolio that once bought will match all future cash flows) and a dynamic hedge portfolio, which requires continual rebalancing to match exposure. For instance, put options could replicate a minimum maturity guarantee, while a dynamic hedge portfolio might consist of futures contracts. Both replication and dynamic hedging are, at least in theory, valid investment strategies to offset the risks associated with a liability.

In practice, due to frictional costs and the inability to rebalance continuously, dynamic hedge portfolios are less likely to exactly replicate the guarantee, although the value of the dynamic hedge portfolio may still provide a good estimate of the value of the guarantee.



3.1.2 Using options to replicate

As explained earlier when considering the link between guarantees and options, the cash flow for the investment guarantee shares the same basic formula as the cash flow for an option (i.e., the greater of B-A and zero). Because the option and guarantee share the same basic formula, the question then arises as to whether it is possible to find an option whose cash flow is identically equal to the guarantee cash flow under each and every scenario. In certain situations, the answer is yes.

The ability to find an option (or other derivative) whose cash flows match those of the guarantee (under each and every possible scenario) can be extremely helpful to provide information on how to invest in order to match the liability and reduce balance-sheet volatility. If two items, (the financial derivative and the investment guarantee under the insurance guarantee) can be shown to have exactly the same cash flows under all possible scenarios, and if one of the items (the financial derivative) has a known market value, then, to preserve the arbitrage-free concept, the other item (the investment guarantee under the insurance guarantee) must have the same market value. In other words, where asset combinations in the market can replicate guarantees, their prices converge. As mentioned previously, replicating strategies may be static (buy and hold) or dynamic (which involves actively rebalancing the portfolio as financial conditions change).

Finding the matching derivative may be complex. However, just because a matching derivative cannot be easily found does not necessarily mean that it does not exist or could not be constructed. Some of the complex areas for consideration when applying replicating techniques are shown in Table 2.

TABLE 2: REPLICATING TECHNIQUES COMPLEX AREAS

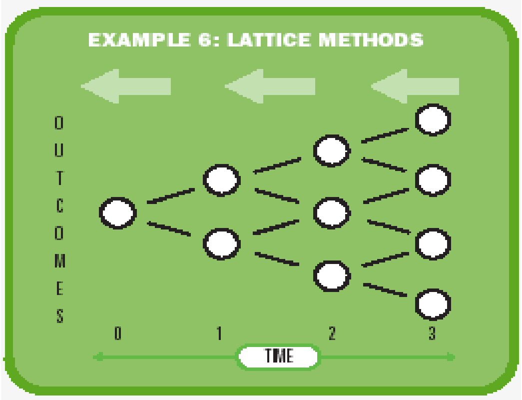
- Liability Cash Flow Profiles
- Embedded Options
- Interest Sensitive Products
- Dynamic Policyholder Behavior
- Availability of Matching Derivatives
- Availability of Long Duration Assets
- Correlation Factors
- Static vs. Dynamic Approach
- Transaction and Rebalancing Costs
- Systems Availability

4. Analytic (Closed-Form) Solutions

If we are able to find a replicating strategy then this often helps us find an analytic solution. For example, if a mix of put options are found to replicate a guarantee then the Black Scholes formula could be used to value the replicating assets and the guarantees they replicate. Unfortunately, embedded options in insurance products are often too complex for analytic solutions to be found. However, as noted later in the section Practical Issues and Shortcuts, simulations or approximations to closed-form solutions can be very useful even when there is no exact solution.

5. Lattice Methods

Under this approach, the possible paths that the assets can take are structured to form a lattice, and any one scenario is equivalent to the assets' following a specific path within the lattice. Values are calculated along all the paths within the lattice, working backward from the end period to the present (see example 6).



It follows that the estimated values of embedded options are also available at all future periods. Lattice methods are therefore especially useful for assessing embedded options where future exercise will depend on future option values. American options fall into this category.

Lattice methods are an intuitive way of covering a large number of asset price scenarios, and can be computationally efficient because a limited number of nodes can represent a large number of price paths. Unfortunately this advantage quickly breaks down if multiple asset classes need to be modeled (interest rates, equity returns, etc.). The lattice concept can extend to multiple dimensions but the number of paths required to obtain accurate results significantly increases. In addition, where payouts depend on prices over a window of time it is no longer valid to condense in one node information that must be used for multiple price paths passing through that node.

6. Simulation Methods

In the situations where replicating portfolios or closed-form solutions cannot be found, the valuation of the guarantee requires one to revert to the principles that underlie option-pricing methods. In this case, the idea is to simulate the cash flows that a matching derivative would have produced, had there been such an asset. This could involve projecting the guaranteed liability cash flows under several thousand scenarios, and computing the discounted present value under each and every scenario. The mean (or average) of the numerous scenarios provides an estimate of the fair value of the embedded option.

To illustrate, consider a simplified situation where the number of potential scenarios is limited rather than unlimited, as is usually the case. Under each investment scenario, there may be corresponding management and policyholder responses that lead to a different net result of benefits paid. The projected management actions and policyholder behaviors are scenario-related, as shown in Example 5.

The objective is to calculate the market consistent value for the benefits paid, allowing for possible variations across the scenarios. In the absence of known replicating assets, one constructs a model, or economic scenario generator, for projecting non-insurance assets (since the insurance assets do not have readily observable market values). The scenario generator must be calibrated to current market prices. This may be done by using the generator to provide a set of scenarios such that the expected value of the relevant cash flows under a large number of scenarios reproduces the market value of readily available assets. The expected value is a weighted average calculation.

These weights can be seen as the product of a present value discount factor and a probability of the occurrence of an event. If the discount factors are calculated using risk-free discount rates, there is one set of corresponding probabilities. These are the probabilities corresponding to a risk-neutral model (described later).

If, alternatively, the probability factors are fixed at the best-estimate probabilities, there is an implied set of present value discount factors (varying by scenario), different from the risk-free discount factors. These discount factors will be the deflators, which are also discussed in more detail later. Having determined the scenario weights, one can then value any set of potential cash flows using the weighted average across the scenarios. This same approach can be generalized to the situation with an arbitrarily large range of potential outcomes.

EXAMPLE 5: MANAGEMENT & POLICYHOLDER RETURNS UNDER EACH SCENARIO

SCENARIO	MANAGEMENT AND POLICYHOLDER RESPONSES	BENEFITS PAID
1	R_1	B_1
2	R_2	B_2
..	...	...
N	R_n	B_n

6.1 Risk-neutral approach

To use this approach, the economic scenario generator needs to be calibrated to market conditions and needs to be what is known as “risk-neutral”.

Risk-neutral valuations are so-called because they involve calculating the expected value of future cash flows as though in a world where investors are risk-neutral, i.e., they do not require a premium to be induced to take risks, and therefore the expected return on all assets is the risk-free rate. The use of this approach does not necessarily imply a belief that investors are actually risk-neutral. As noted above, the combination of risk-neutral probabilities and discounting at the risk-free rate is just a convenient method for finding the scenario weightings that are consistent with market prices.

In practice, the risk-neutral approach involves calibrating the model so that the benchmarking assets have been assigned volatility parameters consistent with current market pricing (e.g. implied volatility in quoted option prices) but their mean investment returns are all equal to the risk-free rate. The risk-free interest rate is also used to convert future liability cash flows to present values.

A potential disadvantage of the risk-neutral approach to valuing guarantees is that the scenarios are derived from a risk-neutral scenario generator, under which all asset classes have the same expected return, regardless of their volatility. Such a scenario set may not be useful for other purposes, such as risk-reward analyses involving ‘realistic’ probabilities to measure risk and influence the likelihood of reward outcomes.

6.2 Deflators

An alternative approach is a deflator approach. This also involves stochastically generating several thousand economic scenarios, computing the present value of the asset’s (or liability’s) cash flows under each scenario, and taking as the fair value the mean of these several thousand scenarios. However, under this approach the economic scenario generator used to project the cash flows is a ‘realistic’ one and the discount factors used to convert the projected cash flows to present-day money values are not the risk-free discounts. The factors that are used to convert the cashflows to present-day money values are known as deflators and they vary by scenario.

Note, however, that one does not have freedom in the choice of deflators. For instance, one key constraint on any set of deflators is that the mean deflator across all scenarios for any given time period must equal the risk-free discount factor for that time period. In a complete arbitrage-free market, there is a unique set of deflators, that reconciles the present value of the cash flows to market values. If different deflators are used, the present value derived will not be equal to the arbitrage-free value and therefore will not be equal to the fair market value.

Working out what the deflators are for a general realistic scenario generator is not a trivial mathematical task, although deflators have been found for a number of popular models such as lognormal models of stock price variation or asset price variation.
It should also be noted that using realistic probabilities with deflators would produce exactly the same result as using risk-neutral probabilities and risk-free discount rates, under consistent calibration to current market values.

7. Interest rate guarantees on saving account by Monte Carlo simulation

7.1 Purposes and structure

The purposes of this sector are:

1) to show the complexity of the whole issue of guarantees valuation through four simplified minimum interest rate guarantees based on four contract conditions;
2) to show how the intuitive simulation methods are used in pricing complex guarantees; and also discuss the inherent drawbacks of simulation methods;
3) to give visual impression about the guarantee prices from ‘guarantee term structures’, in order to help insurers in product design and setting up reserve and risk management.

The structure of this section:

In Section 7.2, we first categorize and define two main Interest rate guarantees, namely the maturity guarantees and multi-period guarantees. Then we give a brief literature review over the recent treatments of this complex issue. In section 7.3, the transformed Hull&White short rate model, which acts as the economic scenario generator in our Monte Carlo simulation, is described, together with its simulation procedures. In section 7.4, four general pricing formulas for interest rate guarantees based on four contract conditions are given. Section 7.5 presents numerical results of guarantee term structures, followed by an application to a hypothetical policy portfolio. Section 7.6 concludes Section 7 with remarks and proposed future works.

7.2 An overview and literature review on interest rate guarantees

Minimum interest rate guarantees (IRG), which cause structural solvency weakness across the European life industry, are embedded in various life insurance contracts, such as unit-linked products and profit-sharing contracts. Example guaranteed returns on legacy business were 4% in The Netherlands and Germany, 4,75% in Belgium, 5% in Italy. Even on current new business, guarantees can be high relative to risk-free rate, for example 3,25% in Germany, 3% in The Netherlands, and up to 3,75% in Belgium. In addition, participation in upside profits on a formulaic basis, as in Italy and The Netherlands, increases the effective floor and the complexity of the guarantee. Policyholders have the best of all worlds: a valuable floor in difficult circumstances, upside participation if markets perform well, and the ability to cash out on fixed terms if more attractive returns are available elsewhere. Someone must pay for these benefits. In first instances, the guarantees must be prices in market consistent base.

Broadly speaking, interest rate guarantees embedded in Unit-linked products are categorized and modeled as maturity guarantees; whereas the interest rate guarantees embedded in profit-sharing products are categorized and modeled as multi-period guarantees.

To illustrate the difference of the two guarantees, we take a pure endowment contract as example. Now we first fix some notations used in Section 7. Rate of returns are expressed in continuous compounding manner. The contract holding period, *[0,T]*, is divided into *n* sub-periods $0 = t_0 \le t_i < t_n = T$, for ($i = 0,1,...,n-1$).

In a Unit-linked product, a maturity guarantee secures the policyholder a minimum rate of return over the holding period till the termination of the contract (e.g. at the maturity of the policy or upon death or surrender). The benefit payable for the contract with a maturity guarantee over the holding period [0, T] can be expressed, with initial investment of 1, as: $\exp\left(\max\left[\sum_{i=0}^{n-1}\alpha_i, gT\right]\right)$, where α_i denotes investment return during the i -th period [t_{i-1}, t_i), and *g* denotes guaranteed rate of return, say 3%.

In a profit-sharing product, where the policyholder participates in upside profit, the benefit is periodically adjusted according to the performance of the reference fund and a minimum is guaranteed to the policyholder. Thus the multi-period guarantee secures the policyholder a minimum rate of return in *each* period[1]. The benefit payable for the pure endowment contract with a profit-sharing scheme (that is a multi-period guarantee) can be expressed as: $\exp\left(\sum_{i=0}^{n-1}\max[\alpha_i, g(t_{i+1} - t_i)]\right)$.

Literature review:
Many publications discuss Interest Rate (or Rate of Return) Guarantees embedded in Unit-Liked products and Profit-sharing products. Most or all of the publications focus on analytical valuation method and numerical methods (e.g. Monte Carlo simulation, lattice method). When analytical solutions are not immediate attainable, people resort to numerical methods.

An incomplete review on analytical solutions published recently:

Persson and Aase (1997) consider and derive analytical solution for maturity guarantees on saving account in a Vasicek stochastic interest rate environment. Miltersen and Persson (1999) (hereon M&P99) present analytical solutions for maturity guarantees and 2-period guarantees on stock account in a general stochastic interest rate environment. Snorre Lindset extends the work of M&P99 in deriving analytical formula of multi-period guarantees and implementing a numerical example up to 5 periods. The above publications all avoid the problem of considering contracts which pay regular premium instead of single premium. Schrager and Pelsser (2003) consider the rate of return guarantees in regular premium in Unit-Linked products and give analytical solution to the problem.

[1] Readers might not find such contracts in the market. However our prototype formulations can be seen as references for real business.

Pricing interest rate guarantees using numerical methods:

Grosen and Jorgensen (2000) proposed a recursive binomial lattice method in valuing surrender option in profit-sharing contract.

In this paper , we use Monte Carlo simulation to price interest rate guarantees embedded in both unit-linked and profit-sharing products, for both single premium and regular premium payments.

7.3 Asset Modeling

7.3.1 Asset assumption – the stochastic saving account

In principle, a guarantee may be connected to any specified rate of return, referred to as the rate of return process or simply the return process. Real-life examples include rates of return on stocks and mutual funds, various indexes, or interest rates.

To highlight the valuation framework, we only work with one underlying asset in Section 7, that is the saving account, for the minimum interest rate guarantees. It is also called money market account. It represents the market value at date *t* of one unit of account where interest is accrued according to the short-term interest rate. Let r_s denote the short-term interest rate at time *s*. Let $\beta(t)$ denote the cumulated return of the short-term interest rate process, i.e. $\beta(t) = \int_0^t r_s ds$. The saving account is defined as $A_\beta(t) = e^{\beta(t)}$. We adopt the notations in M&P99.

7.3.2 Model setup

Forward rate models and spot rate models of the term structure have been intensively analyzed in literatures (see for example Moraleda and Pelsser 2000). We favor the use of the Hull&White (H&W) model, because of its analytical tractability and easy implementation. It gives analytical prices of bond options and hence of caplets/floorlets, so that the model can be calibrated to market observations easily.

The original H&W short rate model:

Hull and White (1990) ("Pricing interest rate derivative securities")assume that the spot interest rate r follows the process

$$dr_t = (\theta(t) - ar)dt + \sigma_r dW_t \,, \qquad (1)$$

under the equivalent martingale measure *Q* (i.e. the risk-neutral measure),
where $\theta(t)$ is an arbitrary function of time, the reversion speed a, the volatility σ_r are treated as constants, and W_t is a Brownian Motion under *Q*.

The Transformed H&W short rate model

Pelsser (1999) proposed an equivalent formulation of the process of *r* as:

$$r_t = \alpha_t + x_t \qquad (2)$$

$$dx_t = -ax_t dt + \sigma_r dW_t \,,\; x_0 = 0 \qquad (3)$$

where $\alpha_t \equiv e^{-at} r_0 + \int_0^t \theta_s e^{-a(t-s)} ds$, a deterministic function of time, is to be determined from the initial term structure. It is shown in Pelsser (1999) that,

$$\alpha_t = -\frac{\partial}{\partial t} \ln D(0,t) + \frac{\sigma_r^2}{2} \frac{(1-e^{-at})^2}{a^2}, \tag{4}$$

where *D(0,t)* is the discount bond price at time zero.

Solutions to the interest rate process under risk neutral measure

The Stochastic Differential Equation (3) has the following solution (after apply Itô's Lemma to $e^{at} x_t$): (See Itô's Lemma in Appendix A)

$$x_t = \sigma_r \int_0^t e^{-a(t-s)} dW_s \tag{5}$$

If conditional on the information at time *t*, (*T* > *t*),

$$x_T = e^{-a(T-t)} x_t + \sigma_r \int_t^T e^{-a(T-s)} dW_s \tag{6}$$

We get for $\int_0^t r_s ds$:

$$\begin{aligned}\int_0^t r_s ds &= \int_0^t \alpha_s ds + \int_0^t x_s ds \\ &= \int_0^t \alpha_s ds + \sigma_r \int_0^t \int_0^s e^{-a(s-u)} dW_u ds \\ &= \int_0^t \alpha_s ds + \sigma_r \int_0^t B(s,t) dW_s \end{aligned} \tag{7}$$

where $B(s,t) = \frac{1-e^{-a(t-s)}}{a}$.

The time *t* value of the saving account (with initial investment equals 1) is then determined by:

$$A_\beta(t) = e^{\beta(t)} = \exp(\int_0^t r_s ds) \tag{8}$$

However, in simulations, we need a discrete time model, instead of the continuous model like (7). In 7.3.3 we explain further about the discrete model.

7.3.3 Monte Carlo simulation from the transformed Hull&White model

Here we briefly review the simulation approach suggested in D. Schrager (2002). When pricing complex derivatives, only in special cases analytical formulas are available. When these formulas are not available, one solution to calculate the risk neutral expectation is to use Monte Carlo simulation. In general when one want to calculate the expectation of a function of a random variable X, $f(X)$, and X has a known distribution (e.g. Normal distribution). One can approximate *E[f(X)]* by $\frac{1}{N}\sum_i f(X_{(i)})$.

Where $X_{(i)}$'s are drawn (simulated) from the known distribution. This result is a consequence from the Law of Large Numbers, as the average converges to the expectation. In this section we will discuss how to simulate the Hull&White model.

We want to simulate interest rate paths. From the solution of (6), we know that, conditional on the value of the short rate at time *t*, x_t, we can write for x_T as

$$x_T = e^{-a(T-t)} x_t + \sigma_r \int_t^T e^{-a(T-s)} dW_s$$

Hence we write x_t as a function of x_{t-h}, where h is the time step

$$x_t = \beta x_{t-h} + \varepsilon_t \tag{9}$$

where $\beta = e^{-ah}$ and $\varepsilon_t \sim i.i.d.N\left(0, \frac{\sigma_r^{\,2}}{2a}(1-e^{-2ah})\right)$[2].

The discrete version of the value process of the saving account $\exp(\int_0^t r_s ds)$ can be approximated by:

$$A_\beta(t) \cong \exp(\sum_{i=1}^{t/h-1} r_i * h) \tag{10}$$

7.4 IRG valuation formulas

7.4.1 mortality assumption and deterministic lapses assumption

We adopt the common practice and assume independence between mortality and financial risk. This enables us to consecutively take expectations with respect to mortality and financial risk. Furthermore, the independence determines that risk-neutral mortality probabilities equal real-world mortality probabilities.

We adopt the view that mortality risk can be diversified by increasing the number of policies. In practice, the real world mortality probabilities are known and are part of the products. Thus we treat the real-world mortality probabilities as known constants and take them outside of the risk-neutral expectations. Examples are the survival probability ${}_{T-t}\,p_{x+t}$ and the death probability ${}_{T-t}\,q_{x+t}$.

Recent attempts have been made to value surrender option as optimal stopping time problem (such as, the early exercise of American Option). There, surrender is purely triggered by investment performance. When the fund value is much higher than the guaranteed value (minimum surrender value), that is, the guarantee is deeply out of money, the policyholder is assumed to sell back the policy (lapse). Therefore the surrender behavior is stochastic and interest-rate-dependent.

In section 7, however, we don't adopt the above-mentioned approach, but restrict ourselves within the deterministic surrender behaviors, which are quantified by best-estimated lapse rates. The reasons why we treat surrender behavior deterministic are mainly:

1) Surrender decisions are actually partially triggered by investment performance. Besides benefiting from upside gains, other reasons for surrender could be that, for example, the policyholder needed cash or couldn't afford the premium.
2) Policies with profit-sharing mechanism attract policyholders to stay, since upside gains are locked and thus less necessary to surrender.
3) Contracted penalties also prevent policyholders from surrendering.
4) Occasionally, regulations and other facts can force policyholders to stop contracts prior to maturity (e.g. a product line might be stopped by regulators). Such non-foreseeable facts cannot be modeled purely by 'best stopping time' problem.

[2] $i.i.d.$ denotes identical and independent distributed

Due the above listed reasons, we work with deterministic surrender probabilities in this section[3]. Thus it can be taken outside of the risk-neutral expectation and treated as known constants (see formulas (11) through (14) below).

Let lapse(i) be the lapse rate of the i-th period $[t_{i-1}, t_i)$. We denote $p_{t_i}^{stay}$ as the probability that the insured stays within the pool till the end of i-th period (t_i), hence $p_{t_i}^{stay} = \prod_{k=1}^{i}(1 - lapse_k)$. The probability that a contract reaches its maturity without surrender, taking both mortality and lapse rate into account, can be expressed as $\prod_{i=1}^{n} p_{x+t_i}(1 - lapse(t_i))$ which equals ${}_T p_x \cdot p_{t_n}^{stay}$, $(0 \le t_i < t_n = T)$.

7.4.2 The general interest rate guarantee pricing formula

Before going into four explicit interest rate guarantee pricing formulas based on four contract conditions, we first give the general pricing formula as the following.

For an endowment contract, the amount payable (the amount assured) is either upon the survival of the insured till the contracted maturity date, or upon death or surrender prior to the maturity date. Here we adopt the same actuarial notations for survival probability ${}_{t_i - t} p_{x+t}$ and death probability ${}_{(t_{i+1}-t_i)} q_{x+t_i}$. Let ${}_{T-t} p_{x+t}$ denote at time t the survival probability till time T of an insured, who is x-year old when the contract starts. At time zero, this simplifies to ${}_T p_x$. The probability that the insured dies during $[t_i, t_{i+1})$ is ${}_{(t_{i+1}-t_i)} q_{x+t_i}$.

Supposing no lapse happens for all contracts, then the value of the contract at time zero, C_0, equals the survival probability (or death probability) times the expected amount payable at maturity (or at death) discounted back to present time. Hence, we have

$$C_0 = {}_T p_x \cdot E^Q[D_T \cdot C_T] + \sum_{i=1}^{n-1} {}_{t_i} p_x \cdot {}_{(t_{i+1}-t_i)} q_{x+t_i} \cdot E^Q[D_{t_{i+1}} \cdot C_{t_{i+1}}],\ n \ge 2 \qquad (11)$$

where D_T denotes the discount factor and $E^Q[\cdot]$ denotes risk-neutral expectation. The first term can be seen as the actuarial present value of a pure endowment contract and the second term can be interpreted as the actuarial present value of a term insurance contract.

In the same way, the IRG element within an endowment contract is given by:

$$IRG_0 = {}_T p_x \cdot E^Q[D_T \cdot IRG_T] + \sum_{i=1}^{n-1} {}_{t_i} p_x \cdot {}_{(t_{i+1}-t_i)} q_{x+t_i} \cdot E^Q[D_{t_{i+1}} \cdot IRG_{t_{i+1}}], n \ge 2 \qquad (12)$$

If we also take the deterministic lapse rates into account, the general pricing formula becomes:

$$IRG_0 = {}_T p_x \cdot p_T^{stay} \cdot E^Q[D_T \cdot IRG_T] + \sum_{i=1}^{n-1} {}_{t_i} p_x \cdot p_{t_i}^{stay} \cdot ({}_{(t_{i+1}-t_i)} q_{x+t_i} + lapse_{t_{i+1}}) \cdot E^Q[D_{t_{i+1}} \cdot IRG_{t_{i+1}}] \qquad (13)$$

---

[3] For future research, the non-deterministic lapses is an interesting topic, to investigate the inter-relation of the policy discontinuance with other stochastic factors

In the rest of Section 7.4, we focus on the expectation components $E^{Q}[D_t \cdot IRG_t]$ in expressions (12) and (13). These components form the interest rate guarantee "term structure like" curves presented in Section 7.5. The mortality factors and lapse rates can be incorporated with the "term structure" afterwards according to (12) and or (13). In Section 7.6, we apply the guarantee "term structure" to a pure endowment policy portfolio according to the first term in (12). Therefore instead of valuating IRG in an endowment contract, we are going to valuate IRG in a pure endowment contract. However, the extensions to (12) and (13) are straightforward.

7.4.3 Single premium maturity guarantee

The single premium paid by the insured at time zero is denoted as L_0. After some cost deduction, a certain percentage of L_0, which equals π_0, is invested into the saving account. The liability is financed by the value of the asset at expiration date, *T*, of the contract.

The maturity guarantee guarantees an averaged rate of return, denoted as *g*, say 3%, over the holding period [0, *T*]. We denote the total return on the asset over a time interval [0, *t*] as *R(0, t)*. Express this in term of short rate r_s, we have $R(0,t) = \exp(\beta(0,t)) = \exp\left(\int_0^t r(s)ds\right)$. Here we implicitly define a quantity $\beta(0,t) = \int_0^t r(s)ds$. Furthermore, we normalize the initial investment premium to 1, that is, $\pi_0 = 1$.

Without incorporate the survival probability, the present value of the interest rate guarantee, IRG_0, is then given by

$$1 + IRG_0 = E^{Q}\left[\exp\left(-\int_0^T r(s)ds\right) \cdot \exp\left(\max[\int_0^T r(s)ds, gT]\right)\right]$$

Hence,

$$IRG_0 = E^{Q}\left[\exp\left(\max[0, gT - \int_0^T r(s)ds]\right)\right] - 1, \tag{14}$$

Notice that IRG_0 is proportional to π_0.

The IRG_0 can be interpreted as difference between the present value of a contract with interest rate guarantee V_0^{IRG} and the present value of a contract without any guarantee $V_0 = \pi_0 = 1$. Thus, $IRG_0 = V_0^{IRG} - 1$.

7.4.4 regular premium maturity guarantee

Let the start of the contract at time $t_0 = 0$ and let $t_i, i = 0,...,n-1$ be the time points at which a premium P_i is credited to the reserve. With premium we mean *investment* premium. P_i could be path-dependent, due to the cost reduction scheme and the changing asset value over time. For simplicity, we keep P_i constant, which equals a fixed percentage of the gross premium.

The maturity guarantee guarantees an averaged rate of return (known as internal rate of return), denoted as *g*, say 3%, over the holding period [t_i, T] for every paid premium P_i. We denote K_T as guaranteed amount payable at maturity date of a pure endowment contract. Thus we define $K_T = \sum_{i=0}^{n-1} P_i \cdot \exp(g(T - t_i))$, $i = 0,...,n-1$.

Further we denote the total asset value at maturity time as F_T, which evolves according to the total return process. We define the return on the investment over sub-periods of $[t_i, T)$ as $R(t_i, T) = \exp(\beta(t_i, T)) = \exp\left(\int_{t_i}^{T} r(s)ds\right)$. Thus we define $F_T = \sum_{i=0}^{n-1} P_i \cdot \exp(\int_{t_i}^{T} r(s)ds)$.

Without incorporate the survival probability, the present value of the interest rate guarantee, IRG_0, equals to the present value of the expected put option payoff at maturity.

$$IRG_0 = E^Q\left[\exp\left(-\int_0^T r(s)ds\right) \cdot \max[0, K_T - F_T]\right]$$

Plug in the expressions for K_T and F_T, without incorporate the survival probability, the present value of the interest rate guarantee, IRG_0, within a regular premium payment scheme, is then given by

$$IRG_0 = E^Q\left[\exp\left(-\int_0^T r(s)ds\right) \cdot \max\left(0, \sum_{i=0}^{n-1} P_i \cdot \exp(g(T - t_i)) - \sum_{i=0}^{n-1} P_i \cdot \exp(\int_{t_i}^{T} r(s)ds)\right)\right] \quad (15)$$

7.4.5 Single premium multi-period guarantee

The multi-period guarantee, which is embedded in a profit-sharing contract, entitles the holder to a minimum rate of return g_i in sub-periods of $[t_{i-1}, t_i)$.

We define the real return on the investment over sub-periods of $[t_{i-1}, t_i)$ as $R_i = R(t_{i-1}, t_i) = \exp(\beta(t_{i-1}, t_i)) = \exp\left(\int_{t_{i-1}}^{t_i} r(s)ds\right)$. Here we implicitly define a quantity $\beta_i = \beta(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} r(s)ds$. We still normalize the initial single premium to 1, that is, $\pi_0 = 1$.

Without incorporate the survival probability, the present value of the interest rate guarantee, IRG_0, is then given by

$$1 + IRG_0 = E^Q\left[\exp\left(-\int_0^T r(s)ds\right) \cdot \exp\left(\sum_{i=1}^{n} \max[\beta_i, g_i(t_i - t_{i-1})]\right)\right]$$

Since $\int_0^T r(s)ds = \sum_{i=1}^{n} \beta_i$ and $\beta_i = \beta(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} r(s)ds$, hence we combine the two exponential terms and get

$$1 + IRG_0 = E^Q\left[\exp\left(\sum_{i=1}^{n} \max[0, g_i(t_i - t_{i-1}) - \beta_i]\right)\right]$$

Hence

$$IRG_0 = E^Q\left[\exp\left(\sum_{i=1}^{n}\max[0, g_i(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} r(s)ds]\right)\right] - 1 \qquad (16)$$

7.4.6 regular premium multi-period guarantee

The same as in 7.4.4, denote the start of the contract at time $t_0 = 0$ and let $t_i, i = 0,...,n-1$ be the time points at which a premium P_i is credited to the reserve. With premium we mean *investment* premium. P_i could be path-dependent, due to the cost reduction scheme and the changing asset value over time. For simplicity, we keep P_i constant, which equals a fixed percentage of the gross premium.

The multi-period guarantee embedded in this profit-sharing contract entitles the holder to a minimum rate of return g_i in sub-periods of $[t_{i-1}, t_i)$.We define the real return on the investment over sub-periods of $[t_{i-1}, t_i)$ as $R_i = R(t_{i-1}, t_i) = \exp(\beta(t_{i-1}, t_i)) = \exp\left(\int_{t_{i-1}}^{t_i} r(s)ds\right)$.

Here we implicitly define a quantity $\beta_i = \beta(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} r(s)ds$.

If we denote V^{IRG} as the present value of a profit-sharing contract with IRG element, and denote V as the present value of the regular premium investment without the IRG element. Then the value of IRG should be the difference of these two contracts. That is,

$$IRG = V_0^{IRG} - V_0$$

Since

$$V_0^{IRG} = E_0^Q\left[\exp\left(-\int_0^T r(s)ds\right)\cdot\sum_{i=0}^{n-1} P_i \cdot \exp\left\{\sum_{j=i+1}^{n}\max(g_j, \beta_j)\right\}\right]$$

$$V_0 = \sum_{i=0}^{n-1} P_i \cdot D(0, t_i)$$

V_0 is actually the sum of the discounted regular premiums, where discount factors are discount bond prices.

Therefore,

$$IRG = E_0^Q\left[\exp\left(-\int_0^T r(s)ds\right)\cdot\sum_{i=0}^{n-1} P_i \cdot \exp\left(\sum_{j=i+1}^{n}\max(g_j, \beta_j)\right)\right] - \sum_{i=0}^{n-1} P_i \cdot D(0, t_i) \qquad (17)$$

7.5 Numerical results for Interest rate guarantees

7.5.1 Model parameters and input data

The short rate model should be calibrated using market prices of actively traded interest rate products, like caps, floors or swaps. By using implied volatilities and parameter(s), we obtain market consistent view over the future return processes.

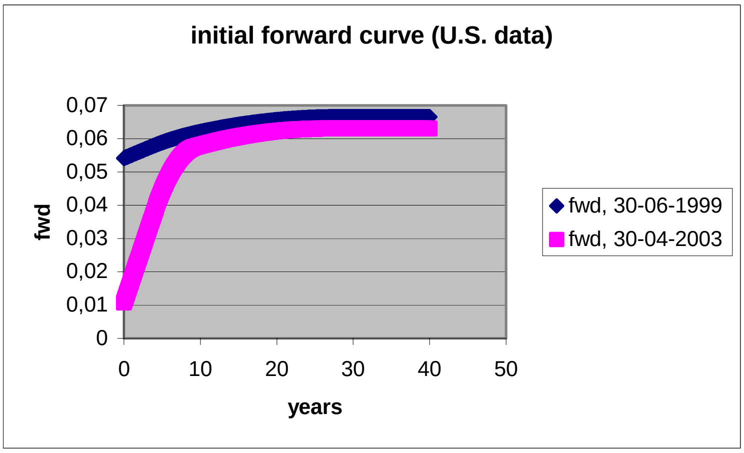
For the single factor H&W model ($r_t = \alpha_t + x_t$; $dx_t = -ax_t dt + \sigma_r dW_t$; $x_0 = 0$), the following parameters are used in the numerical examples shown in section 7.5:

$a = 0.15 \qquad \sigma = 0.015$

The initial zero-coupon bond prices (discount factors) and initial forward rates term structures are U.S. data of April 30, 2003, which can be downloaded from: http://economics.sbs.ohio-state.edu/jhm/ts/ts.html

In section 7.5.2.1 we also use another set of initial forward term structure of U.S. data of June 30, 1999, where higher zero yields were offered at that time and a flatter forward curve took place (see Figure 1). By changing data inputs, we see how the economic environment influences the guarantee values.

Figure 1: Two initial forward curves



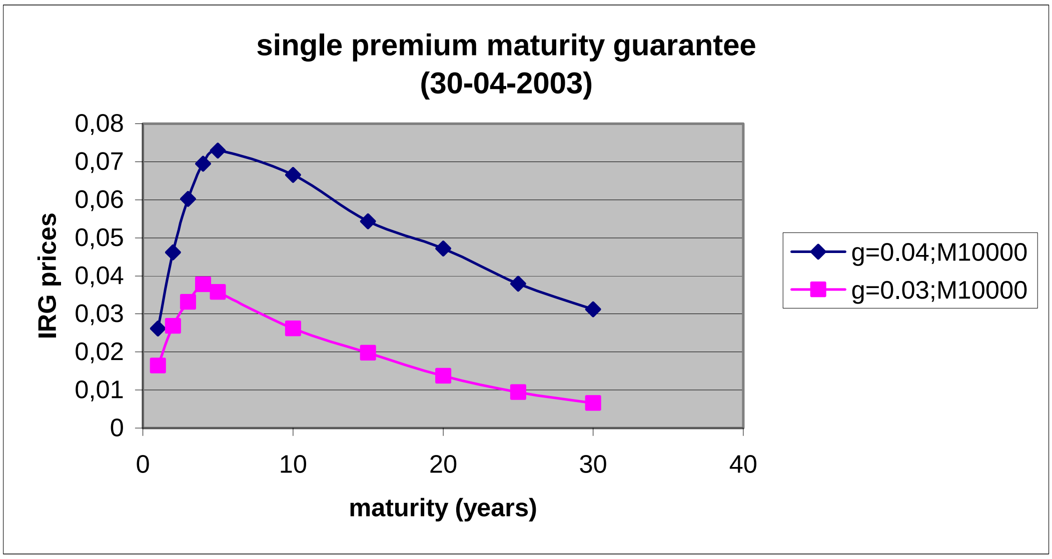
The number of simulation is specified as M in the corresponding figures and tables below. The executing time of the MATLAB program is reported in table 3 and 4.

7.5.2 Term structures of interest rate guarantees

7.5.2.1 Single premium maturity guarantee

A single premium of 1 Euro is paid by the insured at the inception of a life policy and a maturity minimum interest guarantee is proved by the insurer. The guaranteed minimum interest rate 'g' is continuously compounded.

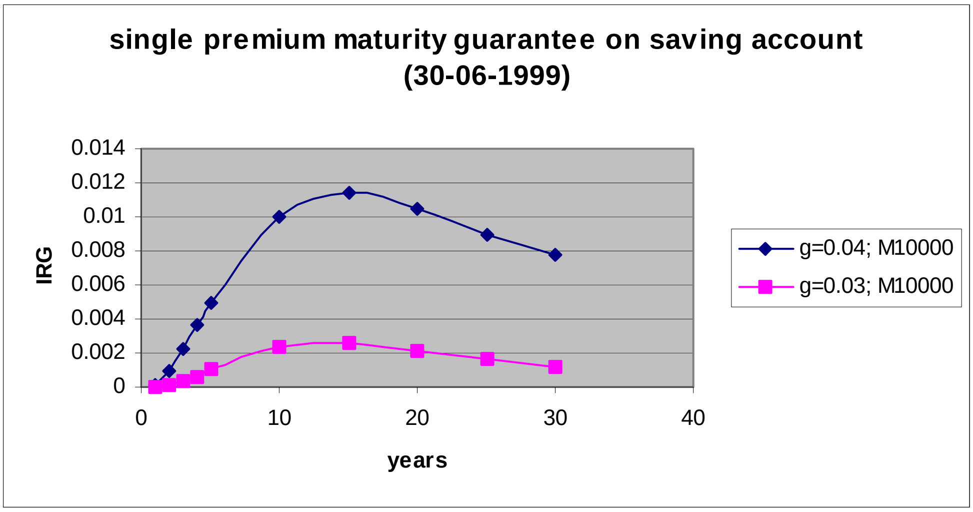
Figure 2: single premium maturity guarantee "term structure" 1



We observe a humped pattern in this guarantee 'term structure'. We think that the sharp increase of the guarantee values within the first 5 years might be caused by the sharp increase of the initial forward rate curve of April 30, 2003, U.S. data.

Hereunder we also present another experiment, where initial term structure are from 1999 June 30, when higher bond yields were prevailing at the time, and flatter forward curve was observed. The resulting guarantee "term structure" shows a humped pattern, however the peak postpones to 15 years away from the valuation time. Another expected observation is that the guarantees were less valuable then current situation, due to a better economic environment in the past.

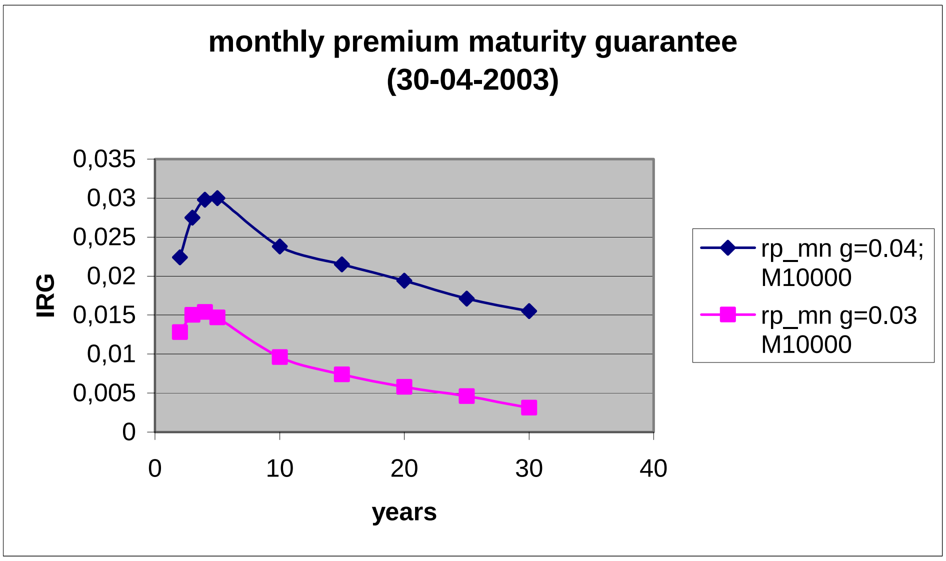
Figure 3: single premium maturity guarantee "term structure" 2



7.5.2.2 regular premium maturity guarantee

A monthly premium of 1 euro is paid by the insured and a maturity guarantee is proved by the insurer at the end of life policy subject to the survival of the insured. The guarantee values shown in this graph can be seen as the relative costs of the guarantee related to the present values of the regular premium.

Figure 4: monthly premium maturity guarantee "term structure"

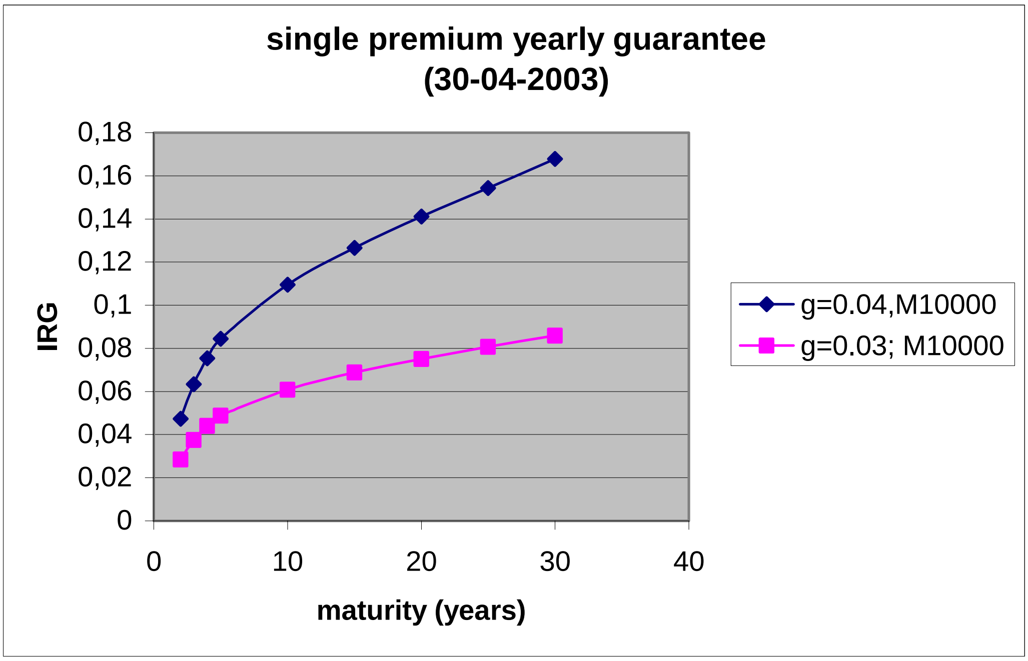


Roughly starting from year 5, the curve of 'guarantee term-structure' decreases. This is mainly because of the declining guarantee cost structure of the single premium maturity guarantee over long period. The regular premium maturity guarantee can be seen as a collection of single premium maturity guarantee. Hence a fraction of the guarantee for the premium paid at an earlier time depreciates as maturity becomes longer.

7.5.2.3 Single premium multi-period guarantee

A single premium of 1 Euro is paid by the insured at the inception of a life policy and a multi-period minimum interest guarantee is proved on yearly basis. The guaranteed minimum interest rate '*g*' is continuously compounded.

Figure 5: single premium yearly guarantee "term structure"



The multi-period guarantee is much more expensive than the maturity guarantee shown in Figure 2. The guarantee becomes more expensive when policy maturity becomes longer. The results presented in Table 3 and Table 4 will be used in a reserve calculation in Section 7.5.3.

Table 3: single premium yearly guarantee "term structure"

		a	sigma	M	g	exe time (sec)			
parameters		0,15	0,015	10000	0,04	89			
maturity	2	3	4	5	10	15	20	25	30
(g=0.04)	0,0473	0,0633	0,0754	0,0844	0,1096	0,1267	0,1412	0,1544	0,1679
(g=0.03)	0,0284	0,0375	0,0441	0,0488	0,0608	0,0688	0,0751	0,0807	0,0859

7.5.2.4 regular premium multi-period guarantee

A yearly premium of 1 euro is paid by the insured and the insurer proves an annual interest rate guarantee. The following graph and results are for yearly premium annual guarantee:

The multi-period guarantee is much more expensive than the maturity guarantee (both happening in single premium and regular premium scheme). The longer the maturity, the higher this guarantee costs.

Figure 6: yearly premium annual guarantee "term structure"

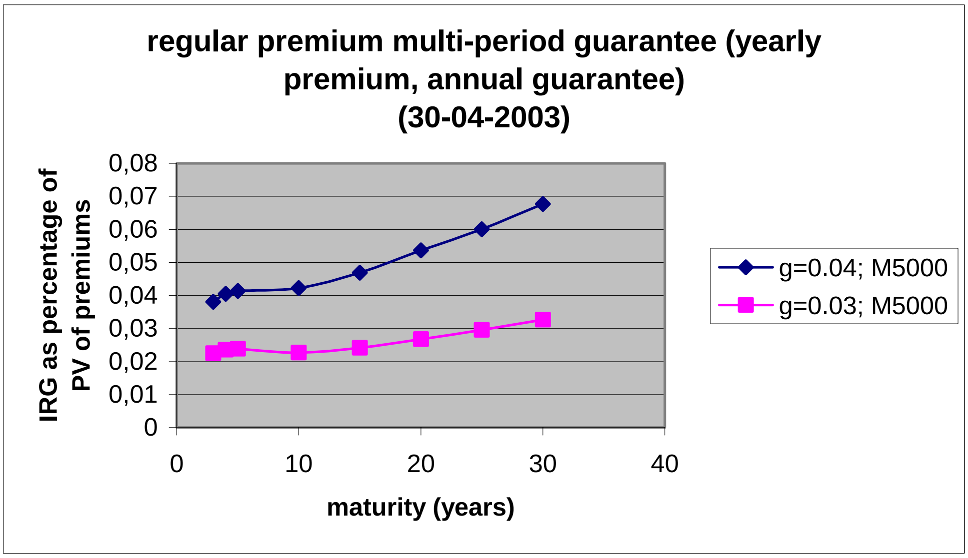


Table 4: Yearly premium annual guarantee "term structure"

	a	sigma	M	g	Exe time (sec)			
parameter	0,15	0,015	5000	0,04	1339			
maturity	3	4	5	10	15	20	25	30
irg_abs	0,1123	0,1573	0,1984	0,3687	0,5488	0,7458	0,9327	1,1305
pv_reg	2,9512	3,8898	4,7964	8,7376	11,7103	13,9112	15,5261	16,7062
IRG (g=0,04)	**0,0380**	**0,0404**	**0,0414**	**0,0422**	**0,0469**	**0,0536**	**0,0601**	**0,0677**
irg_abs	0,0664	0,0916	0,1141	0,1982	0,2823	0,3720	0,4586	0,5458
pv_reg	2,9512	3,8898	4,7964	8,7376	11,7103	13,9112	15,5261	16,7062
IRG (g=0,03)	**0,0225**	**0,0236**	**0,0238**	**0,0227**	**0,0241**	**0,0267**	**0,0295**	**0,0327**

7.5.3 Application to a hypothetical policy portfolio

In this section we construct a hypothetical regular premium profit-sharing policy portfolio. Then we apply the results obtained in Section 7.5.2 to calculate the reserve needed for the interest rate guarantees.

For illustration purpose, the portfolio consists only 20 policies, including 16 existing policies and 4 new policies. The ages of policyholders, (x), at the valuation time, are 20, 30, 40 and 50. The remaining term to policy maturities are denoted by T, ranging from 3 to 20 years. The yearly premium paid by insured varies from 100 euro to 400 euro. By the time of valuation, the credited surrender values of the existing 16 policies are taken from administration system.

Table 5: policy details of the hypothetical policy portfolio

policy data	yearly premium	credited surrender value				
(x)		T=3	T=5	T=10	T=15	T=20
20	100	500	500	250	250	0
30	200	1500	1200	800	500	0
40	300	4000	3000	1500	1000	0
50	400	5000	5000	2000	1000	0

For simplicity, we assume that the 20 policies are pure endowment contracts, where the benefit payable is conditional on the survival of the policyholder, and further assume no lapse happens. The extension to endowment contract is straightforward as given in (13), and the inclusion of deterministic lapses goes in the same straightforward manner as given by (14) (see the guarantee pricing formulas in Section 7.4). The mortality tables and lapse assumptions are company specific. The illustrative survival probabilities shown in Table 6 are based on Dutch GBm9500 mortality base table.

Table 6: survival probabilities ${}_T p_x$ (according to Dutch GBm9500 base table)

	Policy Remaining Term (T)				
(x)	3	5	10	15	20
20	0,99798	0,99659	0,99314	0,98918	0,98370
30	0,99773	0,99601	0,99049	0,98161	0,96720
40	0,99515	0,99103	0,97649	0,95289	0,91467
50	0,98693	0,97583	0,93669	0,87139	0,76847

The reserve of the guarantees are split up into two parts, namely reserve for already credited surrender value (resulted from the paid premiums) and the reserve for the future premiums to be paid till policy maturity. The first part of the reserve can be treated as the reserve for newly issued single premium annual guarantees, where the credited surrender values are seen as the single premiums paid immediately prior to valuation time, as shown in Section 7.5.2.3. The second part of the reserve is treated as the reserve for newly issued regular premium annual guarantees, as shown in Section 7.5.2.4.

The calculated reserve is presented in Table 7, for two guaranteed minimum interest rates respectively. The subtotals group the policies with the same maturity time, and give an idea of how many percentage of the reserve will be paid out in certain future. In this hypothetical example, we see that nearly 45% of the reserve is going to be cashed out within 5 years from the valuation date. While the rest will be paid out over 10 to 20 years from the valuation date.

Table 7: technical reserve for the interest rate guarantees

	(x)	3	5	10	15	20
irg_reserve	20	42,79	61,83	63,82	85,61	73,36
g=0,04	30	117,14	140,39	159,86	169,90	144,27
	40	285,49	309,91	268,49	277,57	204,65
	50	356,69	489,24	343,40	301,65	229,25
total reserve	**sub total**	**802,11**	**1001,38**	**835,56**	**834,73**	**651,54**
4125,31		19,44%	24,27%	20,25%	20,23%	15,79%

	(x)	3	5	10	15	20
irg_reserve	20	25,33	35,70	34,78	44,94	36,59
g=0,03	30	69,34	81,08	87,45	89,18	71,96
	40	169,03	179,07	147,14	146,25	102,07
	50	211,17	282,73	188,19	158,34	114,34
total reserve	**sub total**	**474,87**	**578,57**	**457,57**	**438,70**	**324,97**
2274,68		20,88%	25,44%	20,12%	19,29%	14,29%

7.6 Discussions

The simulation method and the guarantee pricing framework presented in section 7 can be easily extended to include an asset mix. We are currently working with the Black&Scholes-Hull&White model to generate interest rate, stock- and bond-price scenarios.

The simulation framework can be connected with Asset Liability Management system naturally. And ALM studies will then take place to better match assets and insurance liabilities and achieve insurers financial strategies.

The pricing formulas adopt product based pricing approach, which avoids calculations done in policy-by-policy basis. The simulation framework can be applied to multiple product lines valuations, as illustrated by four products.

A by-product of the simulation is the shortfall probabilities, which could provide useful information about the solvency status affected by the interest rate guarantees.

Sensitivity analysis can be performed with respect to changes in underlying asset assumptions, economic environments and or contract conditions. The information resulted from these analyses could also be helpful in finding some replication assets and or hedging strategies.

The accuracy and performance of the simulations can be further improved by variance reduction techniques, and or by parallel computing.

8. Practical Issues and Shortcuts

There are a number of practical considerations that must be addressed in calculating values of guarantees and options. All of the methods described in this paper can involve a considerable amount of work, including modeling effort, especially in the first year of valuation of a contract. The determination of a replicating asset may or may not be an easy task. Where the replicating portfolio can be found, the valuation is easy; simply take the value of the replicating assets. Short of asking a potential counter-party to quote, the main challenge involves finding a simple closed-form solution.

8.1 Deepness of guarantee

When the guarantee is very deeply in or out of the money the following could be considered subject to certain constraints:

- *Guarantee is very deeply out of the money*. If the underlying assets are not too volatile, the value of the guarantee is likely to be very small, and not materially different from zero. Only extreme scenarios then need to be considered. However, when the assets exhibit volatile behavior, caution is required as they are likely to have value even though they may be deeply out of the money
- *Guarantee is deeply in the money*. If the underlying assets are not very volatile, the discounted cash flow value of the liability (intrinsic value) will be much larger than the time value of the guarantee. The time value reflects the value relating to the possibility that the guarantee will start to bite, or will bite even more than currently. In this case, ignoring the time value or estimating it crudely as a small percentage of the intrinsic value may not materially misstate the total value of the liability

However, this approach should be exercised with caution and monitored regularly, even for a guarantee deeply out of the money.

LIMITATIONS: OPTION-PRICING TECHNIQUES

Most option-pricing methods are based on a theory of frictionless markets in which there are no transaction, agency, double taxation, or regulatory capital costs. Although there have been efforts to extend the theory to reality, these costs still vary from one counterparty to another and will lead to different fair values for each party.

Tax is a special case of friction within a market. In a gross-of-tax accounting environment this may be overlooked, although as a practical matter, taxes should be considered in the conventional way when making pricing and investment decisions.

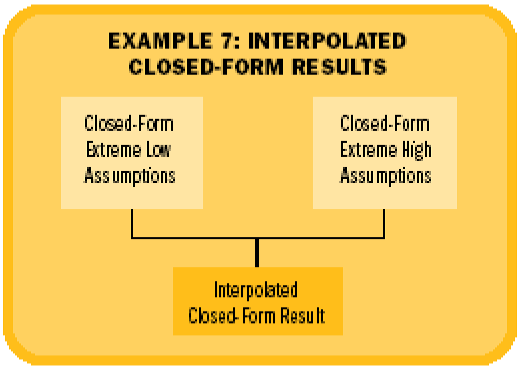
It is interesting that the DSOP uses the words "option-pricing techniques" which is somewhat different, say, to proposing, as the UK regulator has, that the liability should be at least as large the amount a reinsurer or derivative counterparty would charge to assume the risk. This alternative approach allows for such frictional costs, including contingency and profit loadings that the counterparty would charge, which are arguably not included in "option-pricing methods" in their purest form.

8.2 Dealing with demographics

If the source of the complexity for finding a replicating strategy is due to demographic factors (lapses, mortality, etc.), then one approach might involve finding the replicating asset both where the demographic assumptions are much lower and much higher than expected. A closed-form formula such as Black Scholes may then be used to value the option in both cases. The actual value of the option will lie somewhere between these two values. Provided the two values are not too far apart, the actual value of the option may not be materially different from an interpolation between them as shown in Example 7. The interpolation should be based on sensible assumptions for the demographic factors.

8.3 Correlations

If the source of the complexity is the uncertainty around the correlation between multiple asset classes, then the value might also be estimable by interpolating between the upper and lower bounds for the correlation factors.



8.4 Pre-calculated Tables

In some situations, the use of a stochastic model can be avoided by using pre-calculated tables of factors that indicate the relative magnitude of the time value and the intrinsic value appropriate for the relevant contract type.

The tables would be indexed with the relevant financial variable (e.g., current interest rate or share index level) relative to the level at which the guarantee bites, outstanding term of the guarantee, volatility parameter and risk-free rate. The market value of the guarantee can then be estimated by scaling up the intrinsic value by the tabular percentage.

In more complex situations, pre-calculated tables may oversimplify the situation or replicating assets may not be easy to find. Where this is the case, the risk-neutral model will often be the easiest method to apply.

The risk-neutral approach avoids the search for a replicating asset and avoids the additional calculations for the deflators. However it still requires building a stochastic model with a risk-neutral economic scenario generator and choosing risk-neutral parameters.

9. How to Win the Chess Game

An understanding of the different approaches available to calculating a fair value for guarantees and options is only part of the story. In the insurance industry, many companies are now focusing their energies on developing and applying the mechanics of determining the fair value of the liabilities. In addition, companies are gaining an understanding of what needs to be done to prepare for the IAS accounting requirements before the applicable deadlines.

To win the chess game, however, one must think one move ahead of one's competition. So while most others are focusing on how to compute values, the best companies will also be thinking about how they will mitigate the risk of volatility and stabilize earnings results in the new environment. In particular, how should one offset the fluctuations in reported profit caused by varying liability values as market conditions change? A key approach will be to reduce balance-sheet volatility by an appropriate investment strategy, e.g., having asset values move to offset the movement in the value of the liabilities.

Therefore there is a key advantage of the replication approach to valuing the liabilities. Not only does it calculate the fair value of the liability, it also provides information on how to invest in order to match the liability changes and reduce balance-sheet volatility.

When it is not possible to match assets and liabilities, or when a company chooses not to, the question becomes what is the appropriate amount of risk capital required to absorb the unmatched asset/liability fluctuations. To model this, it is necessary to use best-estimate scenario distributions combined with a process for estimating path-wise future fair values, so that one can examine the probabilities of the net asset position falling below threshold levels. In practice, this problem is usually addressed by employing an approximate process for estimating the future reserve values.

Designing new products whose prices properly allow for the value of the guarantees will also be important in ultimately achieving satisfactory levels of profitability.

In summary, and as a general rule, companies should be developing their capabilities (around decision-making processes for bonuses, investments, and product pricing) so as to optimize the trade-off between risk and reward for shareholders in a fair value world.

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Appendix A

Lemma (Itô's) Suppose we have a stochastic process x given by the stochastic differential equation $dy = \mu(t,\omega)dt + \sigma(t,\omega)dW$ and a function $f(t,y)$ of the process *y*, then *f* satisfies

$$df = \left(\frac{\partial f(t,y)}{\partial t} + \mu(t,\omega)\frac{\partial f(t,y)}{\partial y} + \frac{\sigma(t,\omega)^2}{2}\frac{\partial f^2(t,y)}{\partial y^2} \right) dt + \sigma(t,\omega)\frac{\partial f(t,y)}{\partial y} dW$$