Pricing and Hedging Guaranteed Annuity Options via Static Option Replication[1]

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Abstract

In this paper we derive a market value for with-profits Guaranteed Annuity Options using martingale modelling techniques. Furthermore, we show how to construct a static replicating portfolio of vanilla interest rate swaptions that replicates the with-profits Guaranteed Annuity Option. Finally, we illustrate with historical UK interest rate data from the period 1980 until 2000 that the static replicating portfolio would have been extremely effective as a hedge against the interest rate risk involved in the GAO, that the static replicating portfolio would have been considerably cheaper than up-front reserving and also that the replicating portfolio would have provided a much better level of protection than an up-front reserve.

JEL Codes: G13, G22

1. Introduction

Recently, considerable publicity is drawn to with-profits life-insurance policies with Guaranteed Annuity Options (GAO's). Equitable, a large British insurance office, had to close for new business as a portfolio of old insurance policies with GAO's became an uncontrollable liability. In this paper we want to propose a hedging methodology that can help insurance companies to avoid such problems in the future.

During the last few years, many authors have applied no-arbitrage pricing theory from financial economics to calculate the value of embedded options in (life-)insurance contracts. Initially, the work was focussed on valuing return guarantees embedded in equity-linked insurance policies, see for example Brennan and Schwartz (1976), Boyle and Schwartz (1977), Aase and Persson (1994), Boyle and Hardy (1997) and Bacinello and Persson (2002). In equity-linked contracts, the minimum return guarantee can be identified as an equity put option, and hence the "classical" Black-Scholes (1973) option pricing formula can be used to determine the value of the guarantee.

Many life-insurance policies are not explicitly linked to the value of a reference equity fund. Traditionally, life-insurance policies promise to pay a nominal amount of money to the policyholder at expiration of the contract. In order to compensate the policyholder for the relatively low base-rates which are used for premium calculation, various profit-sharing schemes have been employed by insurance companies. Through a profit-sharing scheme, part of the excess return (i.e. return on investments above the base rate) that the insurance company makes is being returned to the policyholders. However, since only the excess return is being shared with the policyholders and not the shortfall, having a profit-sharing scheme in place is equivalent to giving a minimum return guarantee (at the level of the base rate) to the policyholders. This type of embedded return guarantees has only recently been analysed in the literature, see for example Aase and Persson (1997), Grosen and Jørgensen (1997), (2000a) and (2002), Miltersen and Persson (1999) and (2000) and Bouwknegt and Pelsser (2002).

Guaranteed Annuity Options are another example of minimum return guarantees, but in the case of GAO's the guarantee takes the form of the right to convert an assured sum into a life annuity at the better of the market rate prevailing at the time of conversion and a guaranteed rate. Many life-insurance companies in the UK issued pension-type policies with GAO's in the 1970's and 1980's. During this time UK interest rates were very high, above 10% between 1975 and 1985.

Hence, adding GAO's with implicit guaranteed rates around 8% was considered harmless at that time due to the fact that these option were so far "out-of-the-money". Due to the fall of UK interest rates far below 8% (currently UK interest rates are at a level of 5%), the GAO's have become an uncontrollable liability which caused the downfall of Equitable in 2000. The issue of determining the value of GAO's has been addressed in recent years by Bolton *et al.* (1997), Lee (2001), Cairns (2002), Ballotta and Haberman (2002), Wilkie, Waters and Yang (2003) and Boyle and Hardy (2003).

As is evident from the literature overview provided here, the main focus has been given to determining the *value* of embedded options. With the downfall of Equitable it has, in our view, become apparent that not only the valuation should be addressed, but also the *hedging* of embedded options. Although the hedging issue seems trivial at first sight: any derivative can be replicated by executing a delta-hedging strategy. However, the options written by insurance companies have such long maturities and the insured amounts are so high that executing a delta-hedging strategy can have disastrous consequences.

Typically, an insurance company has sold put options to its policy holders. To create a delta-neutral position the insurance company has to sell the underlying asset of the put option. If markets fall, the insurance company has to sell off more of its asset position to remain delta-neutral. This will create more downside pressure on the asset prices, especially if the insurance company is trying to rebalance a large position. Hence, executing a delta-hedging strategy for a short put position can create dangerous "feedback loops" in financial markets which can have disastrous consequences. Similar feedback loops were present in Portfolio Insurance strategies which used delta-hedging to create synthetic put options and were very popular during the 1980's. Automated selling orders generated by computers trying to follow blindly the delta-hedging strategy have been blamed for triggering the October 1987 crash. After the 1987 crash, Portfolio Insurance strategies very quickly lost their appeal. A second complication with executing a delta-hedging strategy is that delta hedging required frequent rebalancing of the hedging assets in order to remain delta-neutral. Especially for long maturity options, this can be quite expensive because of the transactions costs involved.

We want to propose the use of *static option replication* as a viable alternative for insurance companies to hedge their embedded options. A static option replication can be set up if a portfolio of actively traded options can be found that (approximately) replicates the payoff of the derivative under consideration. Once the payoff of the derivative has been replicated, the no-arbitrage condition implies that also for all prior times the value of the derivative is replicated by the static portfolio. Static replication hedging techniques for exotic equity options have been introduced by Bowie and Carr (1994), Derman, Ergener and Kani (1995) and Carr, Ellis and Gupta (1998). The advantages of static replication are obvious: once the initial static hedge has been set up, no rebalancing is needed in order to keep the derivative hedged. In practice, it is not always possible to find a set of actively traded options that perfectly replicates the payoff of a given derivative. However, if the approximation is close enough the static replication portfolio will track the value of the derivative under a wide range of market conditions.

In this paper we want to show how Guaranteed Annuity Options can be statically replicated using a portfolio of vanilla interest rate swaptions. Interest rate swaptions are actively traded for a wide variety of maturities and single trades can be executed for large notional amounts. Using the history of UK interest rates, we demonstrate that a judiciously chosen static portfolio of swaptions can hedge GAO's over a long time horizon and under a wide range of market conditions. Hence, we illustrate that static replication offers a realistic possibility for insurance companies to hedge their exposure to embedded options in their portfolios.

The remainder of this paper is organised as follows. In Section 2 we describe the payoff of Guaranteed Annuity Options and we derive a pricing formula using martingale modelling. In Section 3 we construct the static replication portfolio consisting of vanilla swaptions. In Section 4 we illustrate the effectiveness of the static portfolio with a hypothetical back test using UK interest rate data from 1980 until 2000. Finally, we conclude in Section 5.

2. Guaranteed Annuity Options

Let us consider the market value of annuities at the moment when they are bought. An annuity is financed by a single premium, in our case this single premium equals the lump sum payment of the capital policy. Suppose the annuity is bought at time T by a person of age x. Conditional on the survival probabilities ${}_np_x$ from the mortality table we can write the market value of the annuity $\ddot{a}_x(T)$ with an annual payment of 1 as

$$\ddot{a}_x(T) = \sum_{n=0}^{\omega-x} {}_n p_x D_{T+n}(T), \qquad (2.1)$$

where ${}_np_x$ denotes the probability that an x year old person survives n years and $D_{T+n}(T)$ denotes the market value at time T of a discount factor with maturity $T+n$. Also note that, the sum is truncated at age ω, the maximum age in the mortality table.

In this paper we will make the assumption that the survival probabilities ${}_np_x$ evolve deterministically over time. This allows for trends in the survival probabilities, which are important to take into consideration given the long time horizons for this type of product. Although in practice we know that the survival probabilities are stochastic, the "volatility" of the survival probability process is much smaller than volatility of the discount bond processes. Hence, the main risk factor driving the uncertainty in the value of annuities is the market risk, which we analyse in this paper.

Given the market value $\ddot{a}_x(T)$, the market annuity payout rate $r_x(T)$ over an initial single premium of 1 is given by

$$r_x(T) = 1/\ddot{a}_x(T). \tag{2.2}$$

Note, that we assume that the lump sum payment L at time T is a deterministic quantity. This may seem inconsistent with the fact that GAO's have been issued on unit-linked and with-profits contracts, because in these types of contracts the value of the capital policy at time T is unknown. The papers by Ballotta and Haberman (2002), Wilkie, Waters and Yang (2003) and Boyle and Hardy (2003) explicitly model the uncertainty of the capital policy at time T by treating the policies as unit-linked contracts. In this paper we take a different approach. Our approach exploits the fact that most of the policies offered, especially the policies of Equitable, are with-profits policies. Bolton *et al.* (1997, Appendix 2) report that with-profits policies account for 80% of the total liabilities for contracts which include GAO's.

In the case of with-profits policies, the capital payment L to be paid out at time T depends on the bonuses declared. Under a traditional UK with-profits contract profits are assigned using reversionary and terminal bonuses. Reversionary bonuses are assigned on a regular basis as guaranteed additions to the basic maturity value L and are not distributed until the maturity date T. The terminal bonuses are not guaranteed. Via the profit-sharing mechanism, the amount L can therefore only increase and never decrease. In each year t the reversionary bonus will add an additional "layer" L_t to the contract with an additional GAO. For the remainder of the contract this layer L_t is fixed. Hence, the analysis we offer in this paper is valid for with-profits policies, since each layer L_t of profit-sharing can be valued and hedged at time t when the reversionary bonus is declared.

Suppose that an x year old policyholder has an amount of money L at his disposal at time T which is the payout of his capital policy. The GAO option gives the policyholder the right to choose either an annual payment of $Lr_x(T)$ based on the current market rates (see formula (2.2)) or an annual payment Lr_x^G using the Guaranteed Annuity r_x^G. A rational policyholder will select the highest annuity payout given the current term structure of interest rates. Therefore, we can rewrite the value of the GAO at the exercise date T as

$$\begin{aligned}
&L \max(r_x^G , r_x(T)) \Sigma\, {}_np_x\, D_{T+n}(T) = \\
&L \,(\, r_x(T)\, \Sigma\, {}_np_x\, D_{T+n}(T)\,) + L \max(r_x^G - r_x(T)\, , 0)\, \Sigma\, {}_np_x\, D_{T+n}(T) = \\
&L + L \max(r_x^G - r_x(T)\, , 0)\, \ddot{a}_x(T) \qquad (2.3)
\end{aligned}$$

Hence, the market value of the GAO policy at the exercise date is equal to the lump sum payment L plus L times the value of the GAO put-option.

In the remainder of this paper we will focus only on the value V^G of the GAO put-option

$$V^G(T) = \max(r_x^G - r_x(T)\, , 0)\, \ddot{a}_x(T) \qquad (2.4)$$

To calculate the market value $V^G(0)$ of the GAO put-option today at time 0, we can proceed along several paths. The uncertainty about the value of the option is due to the fact that the discount factors $D_S(T)$ at time T are unknown quantities at time 0. One possible approach therefore, is to model the complete term-structure of interest rates with a term-structure model, like the Heath-Jarrow-Morton (1992) model (HJM model), to obtain an option value. The disadvantage of such an approach is that the option price cannot be determined analytically. Results have to be obtained through numerical approximations which provide us with relatively little insight in the behaviour of the GAO.

To obtain a better handle on the behaviour of the GAO, we draw an analogy between the GAO and a swaption. A swaption gives the holder of the option the right, but not the obligation, to enter into the underlying swap contract for a given fixed rate. As the value of the swap depends on the term-structure of interest rates, we could use a term-structure model to determine the value of the bond option. In the case of a swap, all uncertainty about the term-structure of interest rates is reflected in a single quantity: the par swap rate. Hence, the value of a swaption can be determined more direct by modelling the bond-price itself as a stochastic process. This is exactly the approach that financial markets adopt to calculate the prices of swaptions with the Black (1976) formula.

In the case of the GAO put-option, all the uncertainty about the term-structure of interest rates is reflected in the market annuity payout rate $r_x(T)$. Hence, if we model the market annuity payout r_x directly as a stochastic process, we have sufficient information to price the GAO option. The approach of using market rates, such as LIBOR rates and swap rates, has been applied in recent years with great success to term-structure models. This type of models, which have become known as *market models*, was introduced independently by Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997) and Jamshidian (1998).

The main mathematical result on which this modelling technique is based is the *martingale pricing theorem* which states that, given a *numeraire* (i.e. a reference asset that is used as a new basis to express all prices in the economy in terms of this asset), an economy is arbitrage-free and complete if and only if there exists a unique equivalent probability measure such that all numeraire rebased price processes are martingales under this measure. For a proof of the martingale pricing theorem we refer to the original paper by Geman *et al.* (1995). For a general introduction into the mathematics involved and the application of martingale methods to financial modelling we refer to Musiela and Rutkowski (1997). The books by Hunt and Kennedy (2000) and Pelsser (2000) focus more explicitly on interest rate derivatives.

In the economy we are considering, the traded assets are the discount bonds D_S for the different maturities S. Any arbitrage-free interest model can be embedded in the HJM framework. Under the risk-neutral measure $\mathbf{Q}^*$ (which is the probability measure associated with the money-market account as the numeraire) the process for D_S in the HMJ framework is given by

$$dD_S(t) = D_S(t)\big(r(t)dt + b_S(t)dW^*(t)\big), \tag{2.5}$$

where $r(t)$ denotes the spot interest rate, $W^*(t)$ denotes Brownian Motion under the measure $\mathbf{Q}^*$ and $b_S(t)$ denotes the volatility of the discount bond. Note that in the HJM framework $b_S(t)$ is allowed to be stochastic. Different specifications of $b_S(t)$ lead to different interest rate models. For example, the choice $b_S(t) = \sigma/\kappa\left(1-e^{-\kappa(S-t)}\right)$ leads to the well-known Vasicek-Hull-White model that is used in the papers by Ballotta and Haberman (2002), Wilkie, Waters and Yang (2003) and Boyle and Hardy (2003) to determine prices of GAO's.

To illustrate the change of numeraire approach, we will also consider the processes of discount bond process under the T-forward measure $\mathbf{Q}^{\mathbf{T}}$. This is the probability measure associated with the maturity T discount bond D_T as the numeraire, see Geman *et al.* (1995). For a proof of the results we derive below, we refer to Musiela and Rutkowski (1997, Section 13.2.2). The Radon-Nikodym derivative ρ_T for the change of measure is given by the ratio of numeraires

$$\rho_T(t) \overset{def}{=} \frac{d\mathbf{Q}^T}{d\mathbf{Q}^*} = \frac{D_T(t)/D_T(0)}{\exp\left\{\int_0^t r(s)ds\right\}} = \exp\left\{-\tfrac{1}{2}\int_0^t b_T^2(s)ds + \int_0^t b_T(s)dW^*(s)\right\}. \tag{2.6}$$

Hence, the Radon-Nikodym kernel $\kappa_T(t) = b_T(t)$ and we have that under the T-forward measure the process $dW^T(t)=dW^*(t)-b_T(t)dt$ is a standard Brownian Motion. This implies that under the T-forward measure the process for a discount bond D_S with maturity $S>T$ is given by

$$\begin{aligned} dD_S(t) &= D_S(t)\left(r(t)dt + b_S(t)\left(dW^T(t) + b_T(t)dt\right)\right) \\ &= D_S(t)\left(\left(r(t) + b_S(t)b_T(t)\right)dt + b_S(t)dW^T(t)\right). \end{aligned} \tag{2.7}$$

An application of Itô's Lemma confirms that the T-forward discount bond price $D_S(t)/D_T(t)$ is indeed a martingale under the T-forward measure:

$$d\left(\frac{D_S(t)}{D_T(t)}\right) = \left(\frac{D_S(t)}{D_T(t)}\right)\left(b_S(t) - b_T(t)\right)dW^T(t)\,. \tag{2.8}$$

A particular convenient choice of the numeraire for the GAO put-option is the annuity $\ddot{a}_x(t) = \Sigma\ {}_np_x\ D_{T+n}(t)$. Note, that under the assumption that the survival probabilities ${}_np_x$ are deterministic, this is a portfolio of traded assets (the discount bonds) and hence a permissible choice as numeraire.[2]

The annuity payout $r_x(T)$ rate for time T was defined in (2.2). At times t prior to T we can consider the value of the portfolio of discount bonds that replicates the cash flows of an annuity starting from T. A person that will be x years old at time T, has at time t an age of x-$(T$-$t)$. Hence, the market value at time t of a *forward annuity* starting at T is given by

$$\sum_{n=0}^{\omega-x} {}_{n+(T-t)}p_{x-(T-t)} D_{T+n}(t) = {}_{(T-t)}p_{x-(T-t)} \sum_{n=0}^{\omega-x} {}_np_x D_{T+n}(t) = {}_{(T-t)}p_{x-(T-t)} \ddot{a}_x(t)\,, \qquad (2.9)$$

where we have used the actuarial identity ${}_{n+m}p_x = {}_mp_x\ {}_np_{x+m}$ (see, e.g., Bowers *et al.* (1997), Chapter 3).

At time t, an insurance company can finance the forward annuity by borrowing money from time t until time T. Only in the cases the insured survives until time T, will the insurance company have to repay the loan. Hence, the market value at time t of this loan is given by

$${}_{(T-t)}p_{x-(T-t)} D_T(t)\,. \qquad (2.10)$$

Combining equations (2.5) and (2.6), we can define the *forward annuity rate* as

$$r_x(t) = D_T(t)\ /\ \ddot{a}_x(t), \qquad (2.11)$$

where we see that the survival probability factor ${}_{(T-t)}p_{x-(T-t)}$ in the numerator and the denominator has cancelled. Note, that if $t=T$ this definition coincides with (2.2) since $D_T(T)\equiv 1$. Also note that the forward annuity rate $r_x(t)$ is the numeraire rebased price of the discount bond $D_T(t)$ using the numeraire $\ddot{a}_x(t)$.

[2] Although this is a dividend paying numeraire, no dividends are paid before the maturity date T of the GAO, and this is therefore a valid choice of numeraire to analyse the price of the GAO. Note,

The change of numeraire theorem states that under the martingale probability measure $\mathbf{Q}^{\mathbf{A}}$ associated with the numeraire $\ddot{a}_x(t)$, all $\ddot{a}_x$-rebased price processes are martingales. Hence, also the price process for the forward annuity $r_x(t)$ is a martingale under the measure $\mathbf{Q}^{\mathbf{A}}$.

The Radon-Nikodym derivative $\rho_A(t)$ for the change of measure to $\mathbf{Q}^{\mathbf{A}}$ is given by the ratio of numeraires:

$$\rho_A(t) \stackrel{def}{=} \frac{d\mathbf{Q}^A}{d\mathbf{Q}^*} = \sum_{n=0}^{\omega-x} \frac{{}_n p_x}{\ddot{a}_x(0)} \frac{D_{T+n}(t)}{\exp\left\{\int_0^t r(s)ds\right\}}. \tag{2.12}$$

By an application of Itô's Lemma we obtain that $\rho_A(t)$ follows the process

$$d\rho_A(t) = \sum_{n=0}^{\omega-x} \frac{{}_n p_x}{\ddot{a}_x(0)} \frac{D_{T+n}(t)}{\exp\left\{\int_0^t r(s)ds\right\}} b_{T+n}(t) dW^*(t). \tag{2.13}$$

The Radon-Nikodym kernel $\kappa_A(t)$ is the volatility of $\rho_A(t)$. Hence, we can identify $\kappa_A(t)$ from (2.13) as

$$\kappa_A(t) = \sum_{n=0}^{\omega-x} w_n b_{T+n}(t) \quad \text{with} \quad w_n(t) = \frac{{}_n p_x D_{T+n}(t)}{\sum_{m=0}^{\omega-x} {}_m p_x D_{T+m}(t)} \tag{2.14}$$

and we have that under $\mathbf{Q}^{\mathbf{A}}$ the process $dW^A(t)=dW^*(t)-\kappa_A(t)dt$ is a standard Brownian Motion. We can now derive that the forward annuity rate is a martingale under the measure $\mathbf{Q}^{\mathbf{A}}$ and follows the process

$$dr_x(t) = -r_x(t) \underbrace{\left(\sum_{n=0}^{\omega-x} w_n(t) \big(b_{T+n}(t) - b_T(t) \big) \right)}_{\sigma_r(t)} dW^A(t). \tag{2.15}$$

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that similar numeraires are used in Swap Market Models to analyse the price of swaptions. See, e.g., Jamshidian (1998).

From this expression we see that the forward annuity rate volatility $\sigma_r(t)$ is a weighted average of the forward discount bond volatilities (2.8).

Furthermore, the numeraire rebased market value $V^G/\ddot{a}_x$ of the GAO put-option is also a martingale process under the probability measure $\mathbf{Q^A}$. Using equation (2.4) which gives the value of the GAO put-option at time T, the value of the GAO option for any time $t \leq T$ can be expressed as

$$\begin{aligned}\frac{V^G(t)}{\ddot{a}_x(t)} &= \mathbf{E}^A\left[\frac{V^G(T)}{\ddot{a}_x(T)}\right] = \mathbf{E}^A\left[\frac{\max(r_x^G - r_x(T),0)\ddot{a}_x(T)}{\ddot{a}_x(T)}\right] \\ &= \mathbf{E}^A\left[\max(r_x^G - r_x(T),0)\right],\end{aligned} \tag{2.16}$$

where $\mathbf{E^A}[]$ denotes an expectation under the probability measure $\mathbf{Q^A}$. Multiplying both sides of equation (2.16) by $\ddot{a}_x(t)$ leads to the following expression for the market price of the GAO:

$$V^G(t) = \ddot{a}_x(t)\mathbf{E}^A\left[\max(r_x^G - r_x(T),0)\right]. \tag{2.17}$$

Given the process (2.15) for $r_x(t)$ under the measure $\mathbf{Q^A}$, we can use expression (2.17) to calculate the value of the GAO option explicitly. However, since the weights $w_n(t)$ are stochastic, it is quite complicated to evaluate (2.17) analytically.

An alternative approach is to approximate the process (2.15) as $dr_x(t) = -r_x(t)\overline{\sigma}_r dW^A(t)$ with deterministic volatility $\overline{\sigma}_r$. This implies that we approximate the probability distribution of $r_x(T)$ by a lognormal distribution. Given such an approximation, we can infer $\overline{\sigma}_r$ from (2.15) by "freezing" the stochastic weights at their current values $w_n(t)$. If the discount bond volatilities $b_S(t)$ are deterministic functions (like in the Vasicek-Hull-White model), we can then approximate $\overline{\sigma}_r^2(T-t)$ by the quadratic variation of $\ln r_x(T)$ as

$$\overline{\sigma}_r^2(T-t) = \int_t^T \left(\sum_{n=0}^{\omega-x} w_n(t)\left(b_{T+n}(s) - b_T(s)\right)\right)^2 ds. \tag{2.18}$$

Instead of presuming a particular functional form for the discount bond volatilities $b_S(t)$, we can also estimate $\bar{\sigma}_r$ directly from historical observations of the forward annuity rate. Given a value for $\bar{\sigma}_r$, we can approximate the price for the GAO put-option via the Black (1976) formula as:

$$
\begin{aligned}
V^G(t) &= \ddot{a}_x(t)\left(r_x^G N(-d_2) - r_x(t)N(-d_1)\right) \\
d_{1,2} &= \frac{\ln\left(\frac{r_x(t)}{r_x^G}\right) \pm \frac{1}{2}\bar{\sigma}_r^2 (T-t)}{\bar{\sigma}_r\sqrt{T-t}}.
\end{aligned}
\tag{2.19}
$$

We have adopted the latter approach in Section 4 of this paper.

3. Static Replicating Portfolio

The GAO put-option we have discussed in the previous section, is not a standard interest rate option. To hedge the risk of such a non-standard option, an insurance company can execute a dynamic replication strategy (delta hedging). This replication strategy requires continuous rebalancing of a portfolio of discount bonds. Discussions on how to set up delta hedging strategies can be found in Boyle and Hardy (2003) and Wilkie, Waters and Yang (2003). Executing such a trading strategy in practice can be costly due to transaction costs or even unsuccessful due to inconsistencies in the model assumptions and the actual behaviour of the market. Especially the long time horizons that are typically involved in life-insurance products make the implementation of a delta hedging strategy a challenging task.

We therefore want to propose a static options replication strategy that can be used to hedge the risk of GAO's. In a static options replication strategy one sets up a portfolio of actively traded options such that the payoff of the GAO at maturity is exactly replicated. Due to the fact that this portfolio matches the payoff of the GAO at maturity, the portfolio will also accurately track at all previous times the value of the GAO. Were this not the case, an arbitrage opportunity would arise. Hence, once the initial portfolio of options is bought, its composition never needs to be adjusted until the time that the GAO expires. Even when the actual behaviour of the market is inconsistent with the model assumptions of the underlying options, this has still no impact on the hedge effectiveness of the static replicating portfolio. In other words, not only the market risk but also the "model risk" is eliminated by a static hedge portfolio.

In the remainder of this section we show how a static replication portfolio of vanilla interest rate swaptions can be set up for with-profits GAO's. In interest rate markets, interest rate swaptions are the most actively traded options contracts and can be traded in large quantities for a wide variety of maturities and exercise prices. The construction we propose for GAO's is inspired by the static replication strategy proposed by Hunt and Kennedy (2000, Ch. 15) for irregular swaptions.

Note that the use of swaptions as a hedging strategy has been proposed previously by Bolton *et al.* (1997), Lee (2001) and Wilkie, Waters and Yang (2003). However, none of the mentioned contributions uses the idea of static hedging. Bolton *at al.* (1997) propose a particular simple approach, where they buy receiver swaptions with a strike equal to the rate of interest underlying the GAO. However since the stream of cash flows associated with an interest rate swap has a radically different structure from the cash flows of an annuity, such a hedging strategy will not be very effective in practice.

At the exercise date T, the GAO put-option gives the holder the right, but not the obligation, to enter into an annuity at the guaranteed rate r^G:

$$V^G(T) = \max\left(r_x^G - r_x(T), 0\right)\ddot{a}_x(T) = \max\left(\sum_{n=0}^{\omega-x} ({}_n p_x r_x^G) D_{T+n}(T) - 1, 0\right), \tag{3.1}$$

where we have substituted the definition $\ddot{a}_x$ given in equation (2.1). Hence, the GAO gives the right to obtain a series of cash payments ${}_np_x\, r_x^G$ at the different dates $T+n$ for the price of 1 at time T. Note that, due to the fact that the annuity payments are made at the beginning of each year, at time T one has to pay 1 but one receives r_x^G immediately so that the net cash flow at time T is equal to 1- r_x^G.

A vanilla interest rate swaption gives the right, but not the obligation, to enter at time T into an interest rate swap in which during N years the floating LIBOR interest rate is exchanged for a fixed interest rate K_N. It is well known that the market value S^N of a *receiver swap* in which the fixed rate is received annually is given by (see, e.g. Hull (2000, Ch. 5))

$$S^N(T) = \left(\sum_{n=1}^{N-1} K_N\, D_{T+n}(T) + (1+K_N) D_{T+N}(T)\right) - 1. \tag{3.2}$$

Hence, the market value V^N of a *receiver swaption* that gives the right to enter into an N-year receiver swap at time T can be expressed as

$$V^N(T) = \max\left(S^N(T), 0\right) = \max\left(\left(\sum_{n=1}^{N-1} K_N\, D_{T+n}(T) + (1+K_N) D_{T+N}(T)\right) - 1, 0\right). \qquad (3.3)$$

From expression (3.3) we see that, similar to the GAO, a swaption also gives the right to obtain a series of cash payments for a price of 1. However, the pattern of the cash payments is very different in the two options. The cash flows ${}_np_x\ r_x^G$ associated with the guaranteed annuity are gradually decreasing over time due to the gradually decreasing survival probabilities ${}_np_x$. The cash flows associated with an N-year swap follow a very different pattern: the first N-1 years one receives an amount of K_N, whereas in the Nth year, a cash amount of $(1+K_N)$ is received.

By combining positions in receiver swap contracts all starting at date T with different maturities N, it is possible to replicate the cash flow pattern ${}_np_x\ r_x^G$ of the guaranteed annuity for all dates $T+n$. To find the right amounts that has to be invested in each swap, we proceed backwards from time $T+(\omega\text{-}x)$ to time $T+1$. To replicate the cash flow ${}_{\omega-x}p_x\ r_x^G$ we have to enter into the $(\omega\text{-}x)$-year receiver swap $S^{\omega-x}$ with fixed rate $K_{\omega-x}$. At time $T+(\omega\text{-}x)$ this swap has a cash flow of $(1+K_{\omega-x})$. Hence, if we invest an amount $L_{\omega-x} = {}_{\omega-x}p_x\ r_x^G / (1+K_{\omega-x})$ in swap $S^{\omega-x}$ we replicate the cash flow of the guaranteed annuity at time $T+(\omega\text{-}x)$.

One year earlier, at time $T+(\omega\text{-}x\text{-}1)$, the guaranteed annuity pays out a cash flow of ${}_{\omega-x-1}p_x\ r_x^G$. From the position $L_{\omega-x}$ in swap $S^{\omega-x}$ we already receive a cash flow of $K_{\omega-x}\, L_{\omega-x} = {}_{\omega-x}p_x\ r_x^G - L_{\omega-x}$. Hence, if we invest an amount $L_{\omega-x-1} = (L_{\omega-x} + r_x^G\ ({}_{\omega-x-1}p_x - {}_{\omega-x}p_x)) / (1+K_{\omega-x-1})$ in swap $S^{\omega-x-1}$ we replicate the cash flow of the guaranteed annuity at time $T+(\omega\text{-}x\text{-}1)$.

Continuing this backward construction, we find that we can replicate the cash flow of the guaranteed annuity at a general date $T+n$ by investing an amount $L_n = (L_{n+1} + r_x^G\ ({}_np_x - {}_{n+1}p_x)) / (1+K_n)$ in swap S^n. Proceeding backwards in this fashion, we continue to match all the cash payments of the guaranteed annuity up until time $T+1$.

However, there is a catch. From equation (3.2) we see that at the start date T of the swap contract we require an initial cash payment of 1. Hence, the total portfolio of receiver swaps constructed above to replicate the cash flows of the guaranteed annuity requires an initial cash payment of $\sum_{n=1}^{\omega-x} L_n$. But in equation (3.1) we derived that the GAO put-option gives the right to enter the guaranteed annuity for an initial net cash payment of $1\text{-}r_x^G$. Fortunately, we can adjust the amounts L_n by considering receiver swaps with different fixed rates K_n. This implies that we have to choose a set of fixed rates K_n* for all the swaps S^n such that the invested amounts L_n* satisfy $\sum_{n=1}^{\omega-x} L_n^* = 1 - r_x^G$.

With the portfolio of swaps $\Sigma\ L_n^*\ S^n$ we have replicated all the cash flows of the guaranteed annuity with rate r_x^G. Hence, the GAO which gives the right, but not the obligation, at time T to enter into the guaranteed annuity is equivalent to the option to enter into the portfolio $\Sigma\ L_n^*\ S^n$. This implies that the value $V^G(T)$ at time T of the GAO can be expressed in terms of swaptions V^n as:

$$V^G(T) = \max\left(\sum_{n=1}^{\omega-x} L_n^*\, S^n(T), 0\right) \leq \sum_{n=1}^{\omega-x} L_n^* \max\left(S^n(T), 0\right) = \sum_{n=1}^{\omega-x} L_n^*\, V^n(T), \tag{3.4}$$

where the inequality stems from the fact that the value an option on a portfolio of swaps is less than or equal to the value of the portfolio of the corresponding swaptions. An intuitive explanation for this fact is that in the option on the portfolio you have only an “all-or-nothing” choice to obtain all underlying swaps at once or none at all, whereas in the portfolio of swaptions you can “cherry pick” the individual swaps that have positive market values at time T.

If all the interest rates in the economy are perfectly correlated, i.e. all interest rates move all the time in perfect lockstep, then there exists only one single set of market swap rates K_n* for which the swaps S^n exactly replicate the cash flow stream of the guaranteed annuity. Due to the perfect correlation of the interest rates, all market swap rates will either be simultaneously above the rates

K_n* or simultaneously below. Hence, in the case of perfectly correlated interest rates, the inequality in equation (3.4) becomes an equality for the set of swaptions with strikes K_n*.[3] But this implies that in the case of perfectly correlated interest rates, we have replicated the payoff of the GAO via a portfolio of vanilla interest rate swaptions and, a fortiori, that we have identified a static options replication for the GAO.

In practice we know that the interest rates in the economy are not perfectly correlated, and therefore that the portfolio of swaptions has a higher price than the GAO due to the inequality in equation (3.4). However, GAO's typically are products with a very long maturity. Therefore, their value depends mainly on the behaviour of interest rates with long maturities and these interest rates are very highly correlated. We therefore conjecture that the price of the static hedge replication will be very close to the true price of the GAO.

4. Historical Test

To test the performance of the static replication strategy we have proposed in Section 3, we have conducted a hypothetical historical test using UK interest rate data. This is only a hypothetical test, because in 1980 the swap market in the UK was not as far developed as it is today. This means that the swaps and swaptions needed to execute the static hedge were not available in 1980. However, since the historical period from 1980 until 2000 does provide a very interesting stress-test for our static hedge approach, we resort to a hypothetical test were we impute swap and swaption prices on the basis of UK Government Bond yield data.

We downloaded from Datastream UK Government Bond yields with maturities 2, 3, 5, 7, 10, 15, 20 and 30 years. We used the data at the last trading day of each year from 1980 until 2000. On the basis of the UK Government Bond yields we constructed hypothetical swap rates by taking the bond yields as proxies for the par swap rates with the same maturities. In each year we used a Nelson-Siegel (1987) parameterisation to obtain a complete term structure of zero-rates. In each year the Nelson-Siegel parameters were obtained by a least squares fit of the swap rates implied by the zero-curve to the observed Government Bond yields. The results of the parameter fits are

[3] This remarkable result was derived for the first time by Jamshidian (1989) where he showed that in a one-factor interest rate model an option on a coupon bearing bond can be expressed as a portfolio of options on zero coupon bonds. Note also, that in the case of perfectly correlated rates the apparent ambiguity in choosing the rates K_n* is resolved.

reported in Table 1. (Table can be found at the end of this paper.) Note that, in order to stress-test our static hedge, we have also allowed the "time-scale" parameter tau to vary over time, to obtain as much as possible variation in the interest rates with long maturities. Practitioners usually keep the value of tau constant to stabilise the long end of the yield curve.

Given the Nelson-Siegel parameterisation, we have zero-rates available for all possible maturities. Using the PMA92 mortality table[4], we determined the forward annuity rates using formula (2.11). In Figure 1 below, we have plotted the forward annuity rates for a male that was 45 years old in 1980 and that would retire at age 65 in 2000. Initially, the forward annuity rate was above the guaranteed level of 11.1%. However, due to the falling interest rates we see that the forward annuity rate dropped below the guaranteed level very quickly after 1980.

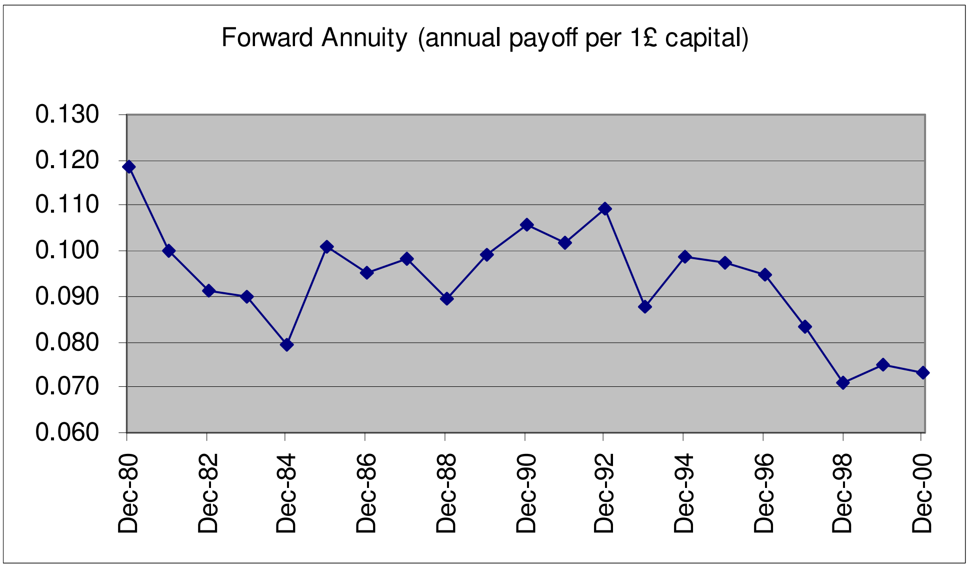

Figure 1: Forward annuity rate for UK data and PMA92 mortality table

From the mortality table, we calculated that the minimum annuity rate $r_{65}*$ is equal to 4.56%. From the time-series of the forward annuity rates, we estimated the volatility of the forward annuity rate process at 11.3%. To account for the fact that implied volatilities are higher than historical volatilities, we multiplied the historical volatility with a factor of 1.25. On the basis of a volatility of 14.2% in formula (2.19), we calculated the market value of the GAO put-option.

[4] The author would like to thank Andrew Cairns for supplying the PMA92 tables.

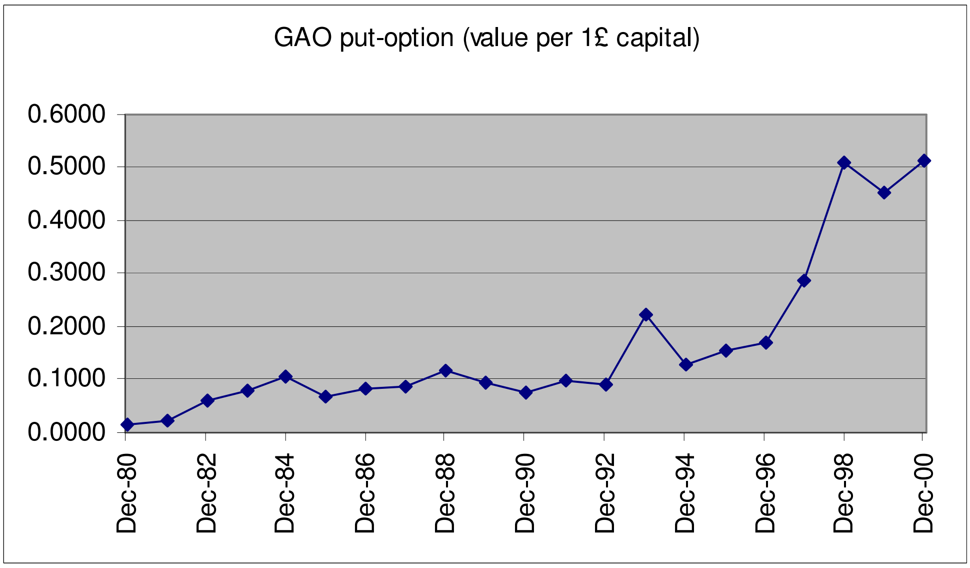

Figure 2: Market value of GAO put-option.

The calculated market values of the GAO put-option have been plotted in Figure 2. Again, we see that the value of the GAO put-option increased dramatically in value with the falling interest rates during the late 1990's. In fact, the value of the GAO increased almost a factor 30: from 1.56% in 1980 to 51.24% in December 2000.

This already indicates what the disadvantages are of "only" reserving for maturity guarantees instead of replication: reserving is very expensive and does not give complete protection. See, for example, the results reported by Wilkie, Waters and Yang (2003, Table 2.5.1). They calculate, on the basis of the 1984 Wilkie model, that the reserve at a 99% level that would have to be set aside in 1980 for the policy with term 20 was equal to 15,36%. As we see here, the actual value of the GAO at the end of the 20-year period (51.24%) was much higher than this 99% reserve. Hence, even reserving at a 99% probability-level would not have provided sufficient protection against the explosive growth in value of the GAO put-option during the 20 year period from 1980 until 2000.

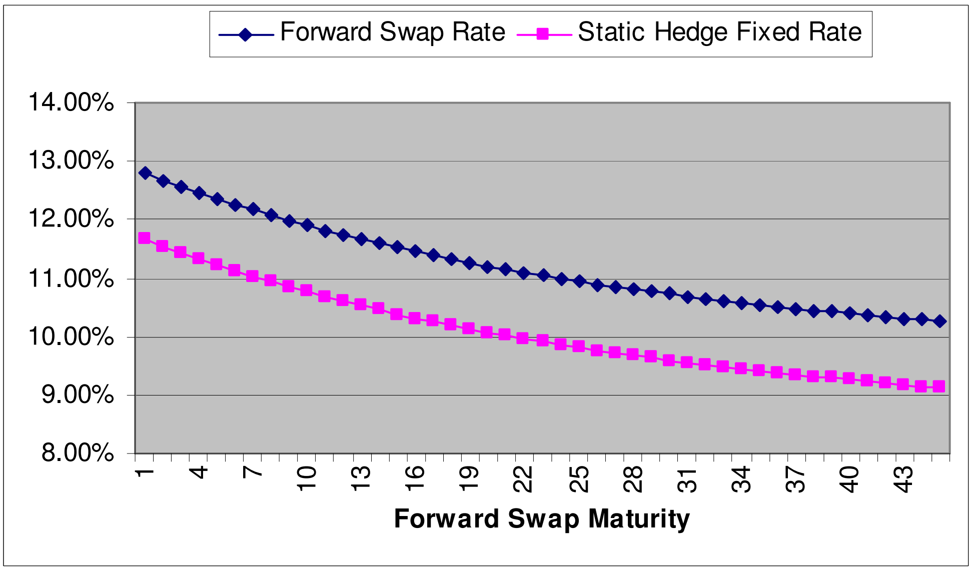

Figure 3a: Forward swap rates and static hedge fixed rates in Dec-1980

Setting up the static replication portfolio of vanilla swaptions would have been considerably cheaper than "only" reserving, and would have provided superior protection. In 1980, the insurance company should have forecasted the annuity payments for a then 45 year old person which would reach the retirement age 65 in the year 2000. In Figure 3a we have plotted the (hypothetical) forward swap rates of Dec-1980. All swap rates are 20 year forward rates, with various swap maturities. We see that the forward swap rates slowly decreased from 12.79% for the 20-year forward 1-year swap rate, until 10.25% for the 20-year forward 45-year swap rate.

As was explained in Section 3, to set up the static replicating portfolio, we have to select a set of fixed rates K_n*. If all the interest rates are correlated perfectly, this will be the swap rates for which the GAO will be exactly "at-the-money". To construct the static hedge portfolio, we have made the assumption that all interest rates are perfectly correlated and also that all interest rates move exactly parallel.[5] Hence, we have shifted all the rates by the same amount until the invested amounts L_n* satisfied $\sum_{n=1}^{\omega-65} L_n^* = 1 - 0.111 = 0.889$. We found that this was achieved for a downward shift of 1.13%-point. The set of fixed rates K_n* obtained by this parallel shift of the swap rates has also been depicted in Figure 3a.

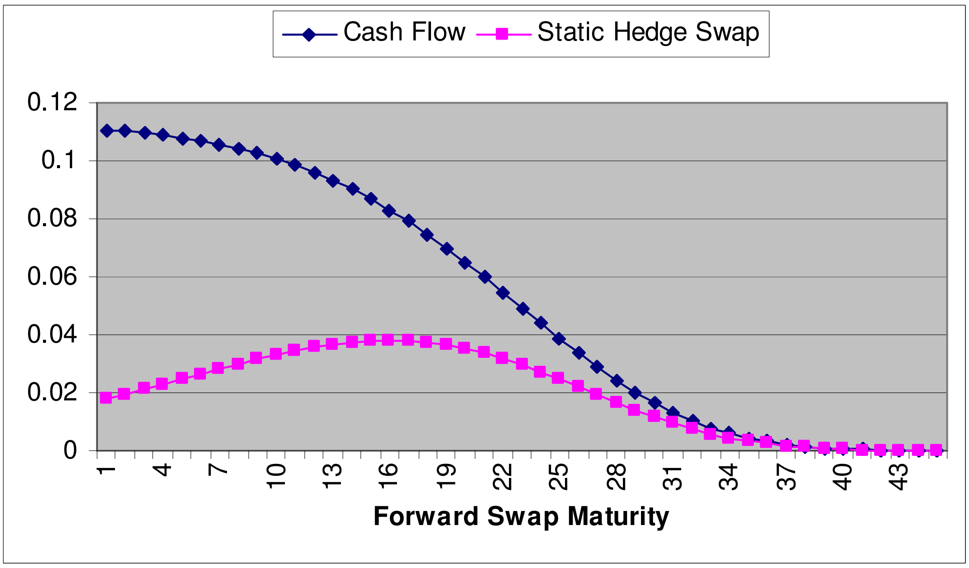

Figure 3b: Static Replication Portfolio of Annuity cash flows

In Figure 3b, we have plotted the projected cash flows for the annuity for the years 2001 until 2045. Also, we have plotted the weights L_n* that would have to be invested in all the swaps with fixed rates K_n* for n=1 to 45. Hence, with the weights L_n* the insurance company could have bought the portfolio of vanilla swaptions Σ_n L_n* V^n. This portfolio of swaptions would have costed[6] 0.0187 per 1£ capital in 1980, which is only 0.0031 per 1£ capital more expensive than the true market value of the GAO put-option. Once this portfolio of swaptions would have been attained, no further buying or selling would have been necessary until December 2000, when the portfolio would have been unwound to cover the cost of the GAO put-option.

[5] A more sophisticated approach would be to select a one-factor interest rate model to model the possible changes in the term structure more accurately. Such an approach would lead to an even lower price for the static hedge. However, for ease of exposition we are using just a parallel shift.
[6] We have calculated the historical volatility of each forward swap-rate. To calculate the price of each swaption we used an implied volatility which was 1.25 times higher than the historical volatility.

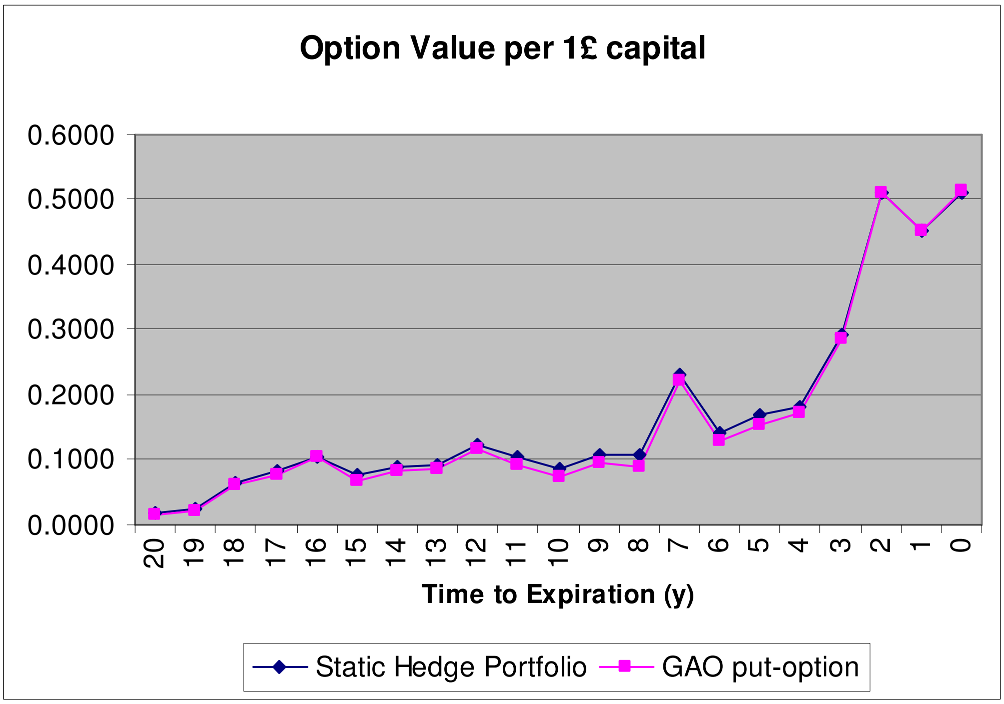

Figure 4: Performance of Static Hedge Portfolio vs. GAO put option

In Figure 4 we have plotted the value of the static replicating portfolio against the market value of the GAO put option for the period Dec-1980 until Dec-2000. The lines with diamonds and squares depict the market value per 1£ capital of the static replicating portfolio and the market value of the GAO put-option respectively. We see that the value of the static replicating portfolio tracks the market value of the GAO extremely closely during the whole period of 20 years.

5. Summary and Conclusion

In this paper we have derived a market value for with-profits Guaranteed Annuity Options using martingale modelling techniques. Furthermore, we have shown how to construct a static replicating portfolio of vanilla swaptions that replicates the with-profits Guaranteed Annuity Option. Finally, we have shown in a hypothetical back test using historical UK interest rate data from 1980 until 2000 that the static replicating portfolio would have been extremely effective as a hedge against the interest rate risk involved in the GAO, and that the static replicating portfolio would have been considerably cheaper than up-front reserving and also that the replicating portfolio would have provided a much better level of protection than a fixed reserve.

Table 1: Nelson-Siegel zero-curves

	beta0	beta1	beta2	Tau
12/31/80	0.0000	0.1255	0.2242	20.2
12/31/81	0.0000	0.1412	0.2675	12.0
12/31/82	0.0374	0.0622	0.1396	10.0
12/30/83	0.0649	0.0269	0.1068	5.0
12/31/84	0.0291	0.0669	0.1696	7.0
12/31/85	0.0873	0.0295	0.0275	3.0
12/31/86	0.0566	0.0524	0.0582	10.0
12/31/87	0.0417	0.0452	0.0993	12.7
12/30/88	0.0531	0.0628	0.0243	10.0
12/29/89	0.1059	0.0252	-0.0852	10.0
12/31/90	0.0845	0.0324	0.0095	10.0
12/31/91	0.0878	0.0100	0.0238	3.0
12/31/92	0.1005	-0.0139	-0.0867	1.6
12/31/93	0.0657	-0.0256	0.0252	4.1
12/30/94	0.0806	-0.0123	0.0430	3.0
12/29/95	0.0644	-0.0087	0.0643	10.0
12/31/96	0.0778	-0.0195	0.0157	3.0
12/31/97	0.0616	0.0106	-0.0064	3.0
12/31/98	0.0440	0.0224	-0.0252	1.5
12/31/99	0.0367	0.0201	0.0552	2.3
12/29/00	0.0241	0.0293	0.0233	10.0

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