OPERATIONALITY OF A MODEL FOR THE ASSET LIABILITY MANAGEMENT

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Summary

This paper is related to the possible applications of a stochastic model of ALM (J.JANSSEN(1992)) to real life situations.

To begin with, we recall the model and we give supplementary theoretical results. Then, we give the statistical estimators of the five basic parameters of the model and we treat a numerical example with data coming from the balance sheet of a big belgian insurance company. We also propose a test of the validity of the model.

Finally, we show how to really use the model for the asset liability management of an insurance company in relation with the MARKOWITZ portfolio theory and with a new concept of duration.

Opérationnalité d'un modèle pour la gestion actif-passif

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Résumé

Le présent article envisage les applications possibles d'un modèle stochastique d'ALM (J.JANSSEN (1992) à des situations réelles.

Nous commençons par rappeler le modèle et par donner des résultats théoriques supplémentaires. Puis, nous donnons les estimateurs statistiques des cinq paramètres fondamentaux du modèle et nous traitons un exemple numérique avec des données provenant du bilan d'une grande compagnie d'assurances belge. Nous proposons également un test de la validité du modèle.

Enfin, nous montrons comment utiliser le modèle pour la gestion actif-passif d'une compagnie d'assurances en relation avec la théorie de portefeuille de Markowitz et avec un nouveau concept de la duration.

1. Introduction

This paper is related to the possible applications of a stochastic model of ALM (J.JANSSEN(1992)) to real life situations.

To begin with, we recall the model and we give supplementary theoretical results. Then, we give the statistical estimators of the five basic parameters of the model and we treat a numerical example with data coming from the balance sheet of a big belgian insurance company. We also propose a test of the validity of the model.

Finally, we show how to really use the model for the asset liability management of an insurance company in relation with the MARKOWITZ portfolio theory and with a new concept of duration.

2. JANSSEN's Model

2.1. Presentation of the model

In this section, we will present the model of JANSSEN (1992). This model supposes that the assets A(t) and the liabilities B(t) are governed by the following stochastic differential equations :

$$\begin{aligned} dA(t) &= \mu_A\, A(t)\, dt + \sigma_A\, A(t)\, dZ_A(t) + \beta_A\, A(t)\, dW(t) \\ dB(t) &= \mu_B\, B(t)\, dt + \sigma_B\, B(t)\, dZ_B(t) + \beta_B\, B(t)\, dW(t) \end{aligned} \qquad (1)$$

where

(i) $W = (W(t), t \geq 0)$ is a standard Brownian motion process (or Wiener process),

(ii) $Z = (Z(t), t \geq 0)$,where $Z(t) = (Z_A(t), Z_B(t))$, is a bidimensional Brownian motion process independent of W with :

$$E\{dZ(t)\, dZ(t)^T\} = Q \;\; dt$$

$$Q = \begin{pmatrix} 1 & \varphi \\ \varphi & 1 \end{pmatrix} \quad (|\varphi| \leq 1)$$

(iii) $A(0) = A_o$ and $B(0) = B_o$, $A_o \geq B_o$,

(iv) $\mu_A, \mu_B, \sigma_A, \sigma_B, \beta_A, \beta_B$ are positive constants.

These processes are as usual, defined on a complete probability space (Ω, F, IP) with as filtration (F_t), the one defined as

$$F_t = \sigma(Z_A(s), Z_B(s), W(s), s \leq t).$$

Let us remind that a continuous process $(Z(t), t \geq 0)$ is an unidimensional Wiener process if it satisfies :

(i) $Z(0) = 0$ with probability one,
(ii) $(Z(t), t \geq 0)$ has stationnary independent increments,
(iii) $Z(t) - Z(s)$ is normally distributed for every $t \geq s$ (with mean zero and variance $(t-s)$).

This process is mean-square continuous on $(0,\infty)$ but nowhere mean-square differentiable (see JAZWINSKI (1970 p. 73)).

We will need the well-known Itô's lemma (see JAZWINSKI (1970, p. 112)) :

Lemma (ITO) : Let $X(t)$ be the solution of the following stochastic vector differential equation :

$$dX = f(X,t)\,dt + G(X,t)\;\;dZ$$

where X and f are n-vectors with regular assumptions on f and G to assure the existence of solution,

G is an n x m matrix,

$(Z(t), t \geq 0)$ is an m-vector Brownian motion process with

$$E\{dZ(t)\,dZ(t)^T\} = Q(t)\;\;dt$$

where $Q(t)$ is for every t an m x m matrix.

Let h(X(t), t) be a real fonction continuously differentiable in t with a continuous second mixed partial derivatives with respect to the elements of X. Then h satisfies the following differential equation:

$$dh = h_t \, dt + h_X^T \, dX + \frac{1}{2} \; tr\left[GQG^T h_{XX}\right] \; dt \tag{2}$$

where h_t is the partial derivative of h with respect to t,

h_X is the vector of the partial derivatives of h with respect to the elements of X,

h_{XX} is the hessian matrix,

G^T is the transposed matrix of G and "tr" denotes the trace.

So, by the use of Itô's lemma, we can determine the stochastic evolution of a sufficiently regular fonction of the assets A and of the liabilities B.

Let us consider the process a = (a(t), t ≥ 0) defined as follows :

$$a(t) = \ln(A(t) / B(t)) \tag{3}$$

and let $a_0 = \ln(A_0 / B_0)$.

The process a has the same interpretation as the classical *surplus* in ruin theory (see GERBER (1979)).

Theorem 1 : The stochastic process a is solution of the stochastic differential equation :

$$da = \mu \; dt + \sigma \, d\overline{W} \tag{4}$$

where

$$\mu = \mu_A - \mu_B - \frac{1}{2}\left(\sigma_A^2 - \sigma_B^2 + \beta_A^2 - \beta_B^2\right), \tag{5}$$

$$\sigma^2 = \sigma_A^2 + \sigma_B^2 + \beta_A^2 + \beta_B^2 - 2\left(\varphi \, \sigma_A \, \sigma_B + \beta_A \, \beta_B\right). \tag{5'}$$

$\overline{W} = \left(\overline{W}(t), t \geq 0\right)$ denotes a standard Brownian motion .

Proof: The proof is obviously based on Itô's formula.

We consider the tridimensional Brownian motion process (Z_A, Z_B, W) and we define h(A, B, t) as follows :

$$h(A, B, t) = \ln (A / B)$$

so, we have :

$$h_t = 0,$$

$$h_X^T = \left(\frac{1}{A}, \frac{-1}{B}, 0\right),$$

$$h_{XX} = \begin{pmatrix} \frac{-1}{A^2} & 0 \\ 0 & \frac{1}{B^2} \end{pmatrix},$$

$$G = \begin{pmatrix} \sigma_A A & 0 & \beta_A A \\ 0 & \sigma_B B & \beta_B B \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & \varphi & 0 \\ \varphi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the expression (2) becomes :

$$\begin{aligned} dh = & \left[(\mu_A - \mu_B) - \frac{1}{2}\left(\sigma_A^2 - \sigma_B^2 + \beta_A^2 - \beta_B^2\right) \right] \; dt \\ & + \left(\sigma_A \, dZ_A - \sigma_B \, dZ_B + (\beta_A - \beta_B) \; dW\right). \end{aligned} \tag{6}$$

Since Z_A and Z_B are the two elements of a bidimensional Brownian motion process, we can find U_A independent of Z_B so that we have :

$$\begin{pmatrix} Z_A \\ Z_B \end{pmatrix} = \begin{pmatrix} \sqrt{(1-\varphi^2)} & \varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U_A \\ U_B \end{pmatrix}$$

where $U_B = Z_B$.

Then it follows from (6) that :

$$dh = \left(\sigma_A \, dZ_A - \sigma_B \, dZ_B + (\beta_A - \beta_B) \;\; dW\right),$$
$$= \left(\sigma_A \sqrt{(1-\varphi^2)} \, dU_A + (\sigma_A \varphi - \sigma_B) \, dU_B + (\beta_A - \beta_B) \;\; dW\right).$$

The three processes U_A, U_B and W are independent. Therefore we can write dh as : $\sigma \, d\overline{W}(t)$

where $\sigma^2 = \sigma_A^2 + \sigma_B^2 + \beta_A^2 + \beta_B^2 - 2(\varphi \, \sigma_A \, \sigma_B + \beta_A \, \beta_B)$

$\overline{W} = (\overline{W}(t), t \geq 0)$ represents a standard Brownian motion .

This completes the proof.

From theorem 1, it easily follows that the process a satisfies the following relation :

$$a(t) = a_0 + \mu \, t + \sigma \, \overline{W}(t). \tag{7}$$

Corollary : The assets A(t) and liabilities B(t) follow a lognormal law. Accurately :

$$\text{(i)} \quad A(t) = A_0 \quad e^{\left(\mu_A - \frac{1}{2}\sigma'^2_A\right)t \; + \; \sigma'_A Z'_A(t)}$$
$$\text{(ii)} \quad B(t) = B_0 \quad e^{\left(\mu_B - \frac{1}{2}\sigma'^2_B\right)t \; + \; \sigma'_B Z'_B(t)} \tag{8}$$

where :

$Z'_A = (Z'_A(t), t \geq 0)$ and $Z'_A = (Z'_B(t), t \geq 0)$ are two standard Brownian motions,

$\sigma'^2_A = (\sigma_A^2 + \beta_A^2)$ and $\sigma'^2_B = (\sigma_B^2 + \beta_B^2)$.

Proof : the proof is obvious from theorem 1.

Formulas (7) and (8) allow one to construct confidence intervals for the variables a(t), A(t) and B(t) (see JANSSEN(1992)).

In practice, the lognormality of the asset value A(t) is commonly admitted (see Merton(1990)). For the liability at time t, B(t), we follow CUMMINS (1988 and 1990) who presents the lognormality as rational.

2.2. Probability of ruin

The ruin occurs if for some $t \geq 0$ the asset value A(t) becomes lower than the liability value B(t) or equivalently if a(t) becomes negative (see JANSSEN (1992)). Therefore, we define the time of ruin T as :

$$\begin{aligned} T &= \inf\{t : t \geq 0 \quad \text{et} \quad a(t) \leq 0\}, \\ &= \inf\{t : t \geq 0 \quad \text{et} \quad \mu\, t + \sigma\, \overline{W}(t) \leq -a_0\}. \end{aligned}$$

So T is the hitting time for the process a(t) by the region

$$(-\infty,\ 0].$$

Let $X(t) = \mu\, t + \sigma\, \overline{W}(t)$ $[X(0) = 0]$. Then T is the hitting time for the process X = (X(t), t≥0) by the region $(-\infty,\ -a_0]$. So we suppose that there is an absorbing barrier at -a.

For the process X, we define now the probability transition density $p(x_o, x; t)$ as follows :

$$p(x_o, x; t)\ dx \quad = \quad P[x \leq X(t) \leq x + dx \ \mid\ X(0) = x_o] \tag{9}$$

We know that $p(x_o, x; t)$ satisfies the two following equations : (see COX and MILLER (1965, p 208)) :

$$\frac{1}{2}\sigma^2\ p_{xx} - \mu\, p_x \quad = \quad p_t \qquad [\text{forward equation}], \tag{10a}$$

$$\frac{1}{2}\sigma^2\ p_{x_o x_o} + \mu\, p_{x_o} \quad = \quad p_t \quad [\text{backward equation}]. \tag{10b}$$

Let $g(t \mid x_O, a_0)$ be the "probability density" of T.

Remark : The ruin is not necessarily certain. In this case, $g(t \mid x_o, a_0)$ is not really a density because his integral on $(0, \infty)$ is not equal to 1.

Nevertheless, his interpretation is given by the following relation :

$$g(t \mid x_o, a_0)\, dt = P[\, t \leq T \leq t + dt]. \tag{11}$$

Then it is easy to verify that :

$$g(t|x_o, a_0) = -\frac{\partial}{\partial t}\int_{-a_0}^{\infty} p(x_o, y; t)\; dy. \tag{12}$$

We will need the Laplace Transform $g^*(s \mid x_O, a_0)$ of the fonction $g(t \mid x_o, a_0)$ defined by :

$$g^*(s|x_o, a_0) = \int_0^{\infty} e^{-st}\, g(t|x_o, a_0)\; dt. \tag{13}$$

For purpose of simplification we will adopt the following notation (see COX and MILLER (1965, chapter five)) :

$$\gamma(x_O) = g^*(s \mid x_O, a_0). \tag{14}$$

By performing the Laplace Transform on (10b) we obtain :

$$\frac{1}{2}\,\sigma^2\,\gamma'' + \mu\,\gamma' = s\,\gamma \tag{15}$$

where prime denotes differentiation (with respect to x_O). The equation (15) is a linear differential equation of which solution can be written as :

$$\gamma(x_o) = C\; e^{x_o \theta_1(s)} + D\; e^{x_o \theta_2(s)}$$

$$\text{where} \quad \theta_1(s) = \frac{-\mu - \sqrt{\mu^2 + 2 s \sigma^2}}{\sigma^2}, \tag{16}$$

$$\theta_2(s) = \frac{-\mu + \sqrt{\mu^2 + 2 s \sigma^2}}{\sigma^2},$$

C and D are constants to be determined from :

$$\text{(i)}\quad \gamma(-a_0) = g^*(s|-a_0,\ a_0) = \int_0^\infty e^{-st}\, g(t|-a_0,\ a_0)\, dt = 1, \qquad (17)$$

$$\text{(ii)}\quad \gamma(x_o) \le \int_0^\infty g(t|x_o,\ a_0)\, dt \le 1. \qquad (18)$$

The first relation results from a well-known property on the Wiener process (see KARLIN and TAYLOR (1975, p.348)) and the second is evident.

Finally the solution of (15) is :

$$\gamma(x_O) = g^*(s\,|\,x_O,\ a_0) = e^{(x_o + a_0)\theta_2(s)} \qquad (19)$$

In our concern, the value of the process X(t) in zero is always zero and consequently :

$$\gamma(0) = g^*(s\,|\,0,\ a_0) = e^{a_0\,\theta_2(s)} \qquad (20)$$

If μ is negative, the ruin is certain ($g^*(s\,|\,0,\ a_0) = 1$) and by performing the Inverse Laplace Transform on $g^*(s\,|\,0,\ a_0)$ (using the formulas related in table 1) we obtain $g(t\,|\,0,\ a_0)$:

$$g(t\,|\,0, a_0) = \frac{a_0}{\sigma\sqrt{2\,\pi\, t^3}}\ e^{\left(\frac{-(a_0 + \mu\, t)^2}{2\sigma^2\, t}\right)}. \qquad (21)$$

Table I :

$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\ dt$
$\frac{b}{\left(2\sqrt{\pi\, t^3}\right)}\ \exp\left(\frac{-b^2}{4\,t}\right)$	$\exp\left(\frac{-b}{\sqrt{s}}\right)$
$b\, f(b\, t)$	$F\left(\frac{s}{b}\right)$
$e^{(bt)}\, f(t)$	$F(s-b)$

We see that the random variable T is inverse gaussian. So we can find the mean and the variance of T (see JANSSEN (1992)):

$$E(T) = \frac{a_0}{|\mu|}, \qquad Var(T) = \frac{a_0\,\sigma^2}{|\mu|^3}. \tag{22}$$

On the other hand if μ is positive, the ruin is not certain and we get :

$$g^*(0\,|\,0, a_0) = P[T < \infty] = e^{-\frac{2\mu a_0}{\sigma^2}}. \tag{23}$$

So, the probability of ultimate survival is :

$$1 - e^{-\frac{2\mu a_0}{\sigma^2}}. \tag{24}$$

The expression (23) has a very interessant interpretation. If we consider that the process a(t) gives a relevant mesure of the company surplus, we see that the factor $(2\mu/\sigma^2)$ plays the same role as the adjustment coefficient in classical ruin theory (see DUFRESNE (1989) or GERBER (1979)). Furthermore, this factor satisfies one of the definitions of the adjustment coefficient (see DUFRESNE (1989,p. 140)) : "the adjustment coefficient is the value R so that $\{\exp(-R\, a(t))\ ;\ t \geq 0\}$ is a martingale". This is shown in the following theorem.

Theorem 2 : if $(P_t, t \geq 0)$ denotes the stochastic process defined by :

$$P_t = \exp(-2\mu/\sigma^2\, a(t)).$$

Then $(P_t, t \geq 0)$ is a martingale.

Proof :

It is sufficient to prove that for every $t \geq s$, we have :

$$E[P_t \,|\, P_s] = P_s. \tag{25}$$

We know that a(t) satisfies the following relation :

$$a(t) = a(s) + \mu\,(t - s) + \sigma\,(\overline{W}(t) - \overline{W}(s)).$$

Conditionally to the knowledge of P_S, a(t) is normally distributed with mean (a(s) + μ (t-s)) and variance (σ^2 (t-s)). His moment generating function, denoted by M(x), is :

$$M(x) = E(\exp (x\, a(t))) = \exp [(a(s) + \mu (t-s))\, x + (1/2) (\sigma^2 (t-s))\, x^2].$$

Setting $x = -(2\mu/\sigma^2)$ gives then (25).

■

2.3. *Parameters estimation*

For the remainder of this paper, we will suppose for simplification that the coefficients β_A and β_B are equal to zero. Then the system (1) becomes :

$$\begin{aligned} dA &= \mu_A\, A\, dt + \sigma_A\, A\, dZ_A, \\ dB &= \mu_B\, B\, dt + \sigma_B\, B\, dZ_B. \end{aligned} \qquad (26)$$

So we have to estimate five parameters : μ_A, μ_B, σ_A, σ_B, φ. Let us recall that we suppose these parameters to be constant. Let us note that ideally they should be considered as functions of t, A(t) and B(t). But this supposes developments that are beyond the scope of this article.

Let us now investigate the problem of their estimation. The equations (26) are similar to those considered by Black-Scholes. This allows us to use the classical estimators in option theory (see ROURE and BUTERY(1989, chapter 4) and ROURE (1992, p. 472)).

It follows from equations (8) that :

$$\text{(i)}\quad d[\ln(A(t))] = \left(\mu_A - \frac{1}{2}\sigma_A^2\right) dt \;+\; \sigma_A\, dZ_A(t),$$

$$\text{(ii)}\quad d[\ln(B(t))] = \left(\mu_B - \frac{1}{2}\sigma_B^2\right) dt \;+\; \sigma_B\, dZ_B(t),$$

where Z_A and Z_B are two standard Brownian motions.

If we dispose of daily estimations of the assets A(t) and liabilities B(t), we can estimate $(\mu_A - (\sigma_A)^2/2)$, σ_A, $(\mu_B - (\sigma_B)^2/2)$ and σ_B by the known estimators (in option theory). So we can consider the last thirty observations (or estimations) of the assets or liabilities (or more exactly of their logarithms) :

$$\ln(A(1)), \ln(A(2)), \ldots\ldots\ldots\ldots, \ln(A(30)),$$

$$\ln(B(1)), \ln(B(2)), \ldots\ldots\ldots\ldots, \ln(B(30)).$$

Then we define X(i) and Y(i), i = 1,...,29 as

$$X(i) = \ln(A(i+1)) - \ln(A(i)),$$

$$Y(i) = \ln(B(i+1)) - \ln(B(i)).$$

Let us remark that { (X(i), Y(i)), i = 1,...,29 } represents a random sample from a bivariate normal population. Therefore we know (see LINDGREN (1976)) that the joint maximum likekihood estimator of the parameters is given by the minimal sufficient statistic :

$$\left(\overline{X}, \overline{Y}, S_X^2, S_Y^2, r\right)$$

Then we can estimate :

$$(\mu_A - (\sigma_A)^2/2) \quad \text{by} \quad \sum_{i=1}^{29} X(i) \Big/ 29 \quad (= \overline{X}),$$

$$(\mu_B - (\sigma_B)^2/2) \quad \text{by} \quad \sum_{i=1}^{29} Y(i) \Big/ 29 \quad (= \overline{Y}),$$

$$(\sigma_A)^2 \quad \text{by} \quad \sum_{i=1}^{29} \left(X(i) - \overline{X}\right)^2 \Big/ 28 \quad (= S_X^2), \qquad (28)$$

$$(\sigma_B)^2 \quad \text{by} \quad \sum_{i=1}^{29} \left(Y(i) - \overline{Y}\right)^2 \Big/ 28 \quad (= S_Y^2).$$

Let us note that :

$$\overline{X} = \ln\left(A(30)/A(1)\right),$$
$$\overline{Y} = \ln\left(B(30)/B(1)\right).$$

The parameter φ will be estimated by the Bravais-Pearson coefficient:

$$\left[\sum_{i=1}^{29}\left(X(i)-\overline{X}\right)\ \left(Y(i)-\overline{Y}\right)\right] \Big/ \sqrt{\left(\sum_{i=1}^{29}\left(X(i)-\overline{X}\right)^2\right)\left(\sum_{i=1}^{29}\left(Y(i)-\overline{Y}\right)^2\right)} \quad (= r).$$

Remark 1 : we estimate the parameters on a daily basis. So, we obtain the yearly variance [respectively the yearly mean] by multiplying the daily variance $(\sigma_A)^2$ [respectivement μ_A] by the number of "significant" days in the year (one can prove that for the assets (see ROURE (1992, p. 472)) it is more relevant to consider a year of 250 days in lieu of 365 (number of working days on the market) ; for the liabilities, this number (N) is also near of 250.

Remark 2 : considering the rapidity of economical environment evolution, the estimations are valid only for a short delay and have thus to be regularly reestimated. A delay of a week is conceivable in the framework of the dynamical portfolio management.

2.4. Example

We present now an example based on the data of a belgian firm (table II).

Table II : Annual data from 1980 to 1991 (in millions of Belgian Francs).

Year	Assets(A)	Liabilities(B)	ln(A/B)	A-B
1980	24406	22631	0.0755	1775
1981	27805	25714	0.0782	2091
1982	31379	28894	0.0825	2481
1983	36546	33661	0.0822	2885
1984	40162	37083	0.0798	3079
1985	44853	40674	0.0978	4179
1986	49939	44911	0.1061	5028
1987	56753	50283	0.1210	6470
1988	64461	55671	0.1466	8790
1989	73461	59999	0.2024	13462
1990	76683	63137	0.1944	13546
1991	82567	68370	0.1887	14197

These data contain two inadequacies : first we only dispose of the annual data but also these data give us a false face of the reality of the firm. This is due to some techniques of management such as for example *window dressing*.

We obtain an estimation of the parameters μ_A, μ_B, σ_A, σ_B and φ by applications of formulas (28) to the data of table II. We get also an estimation of μ and of σ^2 by adapting these formulas to the process a. Finally, we get :

$$\hat{\mu}_A = 0.1130$$
$$\hat{\mu}_B = 0.1008$$
$$\hat{\sigma}_A = 0.0323$$
$$\hat{\sigma}_B = 0.0261$$
$$\hat{\varphi} = 0.8149$$
$$\hat{\mu} = 0.0103$$
$$\hat{\sigma} = 0.0186$$

We can now estimate the probability of ruin for this company. Since a_0 is equal to 0.1887 (ln(82567/68370)), this probability can be evaluated to :

$$P[T < \infty] = e^{-\frac{2\hat{\mu} a_0}{\hat{\sigma}^2}} = 1.32\,10^{-5}$$

2.5. Validity of the model

Janssen's model used here can have a lot of applications to the asset-liability management. Therefore, before using this model, it is very important to verify the conformity of observed data to the model.

Obviously, we can verify that data satisfy the properties of a Brownian motion (see for example MERTON (1990, pp. 51-78) or GILLET (1991)). So we have to verify that all the processes (a(t), t≥0), (ln(B(t)), t≥0) and (ln(A(t)), t≥0) are solutions of an Itô stochastic differential equation. Nevertheless, knowledge of the values of the processes is needed at every instant t.

Consequently, we propose a method that could allow us to give some *credit* to the model. This method consists on verifying the application of Itô's lemma. Indeed, we have estimators (see formulas 28) for the parameters of the processes (A(t), t≥0) , (B(t), t≥0) and their transformation (a(t), t≥0).

Applying Itô's lemma, the two following relations have to be verified :

$$\hat{\mu} = \hat{\mu}_A - \hat{\mu}_B - \frac{1}{2}\left(\hat{\sigma}_A^2 - \hat{\sigma}_B^2\right),$$
$$\hat{\sigma}^2 = \hat{\sigma}_A^2 + \hat{\sigma}_B^2 - 2\,\hat{\varphi}\,\hat{\sigma}_A\,\hat{\sigma}_B.$$

Our method consists then on considering the values of the right member as correct ones and on carrying out the two following tests of hypothesis:

$$\begin{cases} H_0: & \mu = \hat{\mu}_A - \hat{\mu}_B - \frac{1}{2}\left(\hat{\sigma}_A^2 - \hat{\sigma}_B^2\right), \\ H_1: & \mu \neq \hat{\mu}_A - \hat{\mu}_B - \frac{1}{2}\left(\hat{\sigma}_A^2 - \hat{\sigma}_B^2\right), \end{cases} \tag{29}$$

$$\begin{cases} H_0: & \sigma^2 = \hat{\sigma}_A^2 + \hat{\sigma}_B^2 - 2\,\hat{\varphi}\,\hat{\sigma}_A\,\hat{\sigma}_B, \\ H_1: & \sigma^2 \neq \hat{\sigma}_A^2 + \hat{\sigma}_B^2 - 2\,\hat{\varphi}\,\hat{\sigma}_A\,\hat{\sigma}_B. \end{cases} \tag{30}$$

The test about the mean μ will be performed by the use of the Student's law:

$$t_{n-1} = \frac{\hat{\mu} - \mu}{\hat{\sigma}/\sqrt{n}}, \tag{31}$$

where n denotes the number of observations,
and the test about the variance σ^2 will be performed by the use of the Chi-square law :

$$\chi_{n-1}^2 = \frac{(n-1)\hat{\sigma}^2}{\sigma^2}. \tag{32}$$

2.6. Example (continuation)

The estimations of the different parameters are (n=11) :

$$\hat{\mu}_A = 0.1130, \quad \hat{\sigma}_A = 0.0323, \quad \hat{\sigma} = 0.0186, \quad \hat{\varphi} = 0.8149,$$
$$\hat{\mu}_B = 0.1008, \quad \hat{\sigma}_B = 0.0261, \quad \hat{\mu} = 0.0103.$$

We have to carry out the tests :

$$\begin{cases} H_0: & \mu = 0.0120 \\ H_1: & \mu \neq 0.0120 \end{cases} \quad \text{and} \quad \begin{cases} H_0: & \sigma^2 = 3.5053 \quad 10^{-4} \\ H_1: & \sigma^2 \neq 3.5053 \quad 10^{-4} \end{cases}.$$

The observed value for the Student is :

$$t_{10} = \frac{\hat{\mu} - \mu_0}{\hat{\sigma}/\sqrt{11}} = \frac{0.0103 - 0.0120}{0.0186/3.3166} = -0.3031,$$

and the one for the Chi-square is :

$$\chi^2_{10} = \frac{(10)\hat{\sigma}^2}{\sigma_0^2} = \frac{10 \ 3.4596\, 10^{-4}}{3.5053\, 10^{-4}} = 9.8696.$$

These values lead us to accept H_0 (for significance level $\alpha = 0.05$) for the two tests and therefore not to reject the model.

3. Applications to the asset-liability management.

3.1. Forecasting of the company financial position

Each (strategy) decision at the management level is based on the value of some parameters. Among them is surely of great interest the estimation (at some future instant) of the equity of the company. The level of the equity can be caracterized by the process a. Therefore we have to determine the transition probability density $p(0, x; t)$ of the process $(X(t), t \geq 0)$, denoted from here by $p(x; t)$. This is the solution of the differential equation (see 10.a) :

$$\frac{1}{2}\sigma^2 \, p_{xx} - \mu \, p_x = p_t, \qquad [\text{forward equation}], \qquad (33)$$

that moreover satisfies the two foregoing boundary conditions:

(i) $p(x; 0) = \delta(x)$ where $\delta(x)$ denotes the Dirac function,

(ii) $p(-a_0; t) = 0$.

The first condition is evident as X(0) is always 0. The second condition is justified by a well-known property of the Brownian motion: it vanishes with probability one on every interval (o, t) (see for example KARLIN and TAYLOR (1975, p. 348)). So, if, for t' > 0, X(t') = $-a_0$, we can find with probability one 0 < t" < t' so that X(t") = $-a_0$. Therefore, it follows from the definition of an absorbing barrier that we don't have to consider the process X for t > t" (indeed, the company in state of collapse does not exist any more).

We solve now (33) by applying the method known as "image source " (see COX and MILLER (1965, p 221)).
Let $p_1(x_0, x; t)$ be the function defined as follows :

$$p_1(x_0, x; t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}},$$

where x_0 is a constant.

It is easy to verify that, for every x_0, the function $p_1(x_0, x; t)$ is solution of (33) and moreover satisfies (i). Now we search λ so that the following function p(x,t)

$$p(x,t) = p_1(0, x; t) + \lambda\, p_1(-2\, a_0, x; t)$$

satisfies (ii). For this purpose, we perform the value $p(-a_0,t)$:

$$p(-a_o, t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(a_o+\mu t)^2}{2\sigma^2 t}} + \lambda \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(a_o-\mu t)^2}{2\sigma^2 t}}.$$

We must have $p(-a_0, t) = 0$. Therefore :

$$\lambda = -e^{-\frac{2\mu a_o}{\sigma^2}}.$$

Finally the solution is :

$$p(x, t) = \frac{1}{\sigma\sqrt{2\pi t}} \left[e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} - e^{-\frac{2a_o\mu}{\sigma^2}} e^{-\frac{(x+2a_o-\mu t)^2}{2\sigma^2 t}} \right].$$

3.2. General objectives for the company

We deduce now general implications of the model to the company asset-liability management. Our analysis follows JANSSEN (1992). First we look at financially viable companies ($\mu \geq 0$) and after those for which ruin is certain ($\mu \leq 0$).

a) $\mu \geq 0$

It follows then from (23) that management is as risky as σ is great. So a volatility equal to zero will be the best for this company. This is the case if and only if :

$$\varphi = 1 \quad , \quad \sigma_A = \sigma_B \quad \text{et} \quad \beta_A = \beta_B. \tag{34}$$

That is the goal to be achieved by the company. We give in the following section some tools or techniques to modify parameters.

a) $\mu \leq 0$

Then it follows from relation (21) that the ruin occurs with probability one. This tragic situation impose to the company a global restructuration in order to improve firm rentability (μ). So company interest is in the realisation of the following relation (see thorem 1) :

$$\mu \geq 0 \quad \Leftrightarrow \quad \mu_A - \mu_B \geq \frac{1}{2}\left(\sigma_A^2 + \beta_A^2 - \sigma_B^2 - \beta_B^2\right). \tag{35}$$

This would be the case by (see JANSSEN (1992))

i) increasing ($\mu_A - \mu_B$) or by

ii) reducing asset's volatility $\sigma_A^2 + \beta_A^2$ and by increasing

liability's one ($\sigma_B^2 + \beta_B^2$).

3.3. *Asset-liability management tools*

It follows from the preceding section that the company must reduce volatility (σ^2) or increase rentability (μ). Therefore our goal in this section is to present some methods allowing to achieve these objectives.

a) Volatility reduction

First we carry out a segmentation of the assets A(t) and the liabilities B(t) :

$$\begin{aligned} &A_1(t), A_2(t), \dots\dots\dots, A_n(t) \qquad \sum_{i=1}^{n} A_i(t) = A(t) \\ &B_1(t), B_2(t), \dots\dots\dots\dots\dots\dots, B_m(t) \qquad \sum_{j=1}^{m} B_j(t) = B(t) \end{aligned} \tag{36}$$

We suppose that the following relations are satisfied :

$$\begin{aligned} dA_i &= \mu_i \; A_i \; dt + \sigma_i \; A_i \; dZ_i \,, 1 \le i \le n, \\ dB_j &= \mu_j' \, B_j \; dt + \sigma_j' \; B_j \; dZ_j',, 1 \le j \le m \end{aligned} \tag{37}$$

where

$Z = \left(Z(t), t \ge 0\right)$ where $Z(t) = \left(Z_1(t), \dots, Z_n(t)\right)$, is a standard n-dimensional Brownian motion process with :

$$E\left\{dZ(t)\, dZ(t)^T\right\} \;=\; \Sigma\, dt$$

where Σ is the variance-covariance (n x n) matrix of Z,

$$\Sigma = \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{1,2} & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \rho_{1,n} & \dots & \dots & 1 \end{bmatrix}.$$

$Z' = \left(Z'(t), t \ge 0\right)$ where $Z'(t) = \left(Z_1'(t), \dots, Z_m'(t)\right)$, is a m-dimensional standard Brownian motion process with :

$$E\left\{dZ'(t)\, dZ'(t)^T\right\} \;=\; \Sigma'\, dt$$

where Σ' is the variance-covariance (m x m) matrix of Z',

$$\Sigma' = \begin{bmatrix} 1 & \rho'_{1,2} & \cdots & \rho'_{1,n} \\ \rho'_{1,2} & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \rho'_{1,n} & \cdots & \cdots & 1 \end{bmatrix}.$$

μ_i , σ'_i, μ'_j and σ'_j are positive constants $\{1 \leq i \leq n \ ; \ 1 \leq j \leq m\}$.

i) <u>Assets volatility reduction</u>

We present two approaches. The first one consists on a generalization of the MARKOWITZ diversification theory (see MÜLLER (1988)) and applies to the *idiosyncratic risk* (see CUMMINS (1990)). The second one applies to riskiest assets and is derived from option theory (see ROURE and BUTERY(1989)).

For a fixed instant t, the infinitesimal assets return $\mu*$ is:

$$\mu^* = \sum_{i=1}^{n} \mu_i \ A_i \tag{38}$$

and the volatility σ^{*2} is :

$$\sigma^{*2} = \sum_{i=1}^{n} \sum_{i'=1}^{n} \left(\rho_{ii'} \ \sigma_i \ \sigma_{i'} \ A_i \ A_{i'}\right) \tag{39}$$

where $\rho_{i,i'}$ denotes the general term of the matrix Σ and satisfies :

$$\rho_{ii'} \ dt = E[dZ_i(t) \, dZ_{i'}(t)].$$

So we have to result the following optimization problem:

$$\min_{A_1,\ldots,A_n} \sigma^{*2} \tag{40}$$

subject to :

i) $A_i \geq 0, 1 \leq i \leq n$,

ii) $\sum_{i=1}^{n} A_i = A(t)$,

iii) $\sum_{i=1}^{n} \mu_i \, A_i \geq \mu^*$.

This produces the asset portfolio (valid on the infinitesimal interval]t, t+dt)) and we insist here on the necessary periodicity of the so defined techniques.

Nevertheless some assets of which detention may be considered as essential for strategic purpose can present a highier risk. In this case hedging can be obtained from option theory. There are different strategies (see ROURE and BUTERY (1989)) and here we devellop only the simplest :

> Let us denote by A' any riskier asset. The strategy to adopt is the following : we buy an option PUT with an exercise price equal or just lower than the actual value of A'. This assures a minimal value of the portfolio composed of the asset A' and the option. But the return is reduced by option price.

ii) Liabilities volatility reduction

We can obviously follow the same procedure. We adopt the following notations for the infinitesimal return and volatility :

$$\mu' = \sum_{j=1}^{m} \mu'_j \, B_j \tag{41}$$

$$\sigma'^2 = \sum_{j=1}^{m} \sum_{j'=1}^{m} \left(\rho'_{jj'} \, \sigma'_j \, \sigma'_{j'} \, B_j \, B_{j'} \right) \tag{42}$$

where $\rho'_{j,j'}$ denotes the general term of the matrix Σ' and satisfies :

$$\rho'_{jj'} \, dt = E[dZ'_j(t) \, dZ'_{j'}(t)].$$

Then we have to solve the following optimization problem :

$$\min_{B_1, \ldots, B_m} \sigma'^2 \tag{43}$$

subject to :

i) $B_j \geq 0,\ 1 \leq j \leq m,$

ii) $\sum_{j=1}^{m} B_j = B(t),$

iii) $\sum_{j=1}^{m} \mu'_j B_j \leq \mu'$.

Remark 1 : let us note it is more difficult to modify liability "portfolio" than the asset one. Moreover this imposes marketing operations of which succes and interest depend on the perfect communication between different sectors of the firm.

Remark 2 : the technique we present here allows us to reduce only the *(liability) idiosyncratic risk* and other tools are necessary to eliminate the systematic risk. Among them there are traditionnal insurance techniques such as Reinsurance or Mutual insurance and also more recent methods derived from *Insurance futures* . This one's consists on the adaptation of strategies based on (Assets) option theory. The definitions, principles and interests of *insurance futures and options on (insurance) futures* are presented in COX and SCHWEBACH (1992), D'ARCY and FRANCE (1992) or NIEHAUS and MANN (1992). These futures are similar to futures written on indices (see ROURE (1992). These new products allow us to benefit from any anticipation on the loss ratio (paid claims divided by earned premiums) evolution.

b) Increasing of rentability

The strategies allowing us to improve rentability (μ) are essentially based on new financial products such as options, swaps,... For example, if we can anticipate the increasing of an asset A we sell a put with the current asset value as exercise price. Actually there is a strategy for any anticipated evolution of any asset (see for example ROURE and BUTERY (1989)).

For liabilities new instruments such as *insurance futures* give us the possibility to benefit of similar anticipation.

3.4. *A possible new concept of duration*

We try here to generalize the concept of duration. This takes place in the framework of immunization against the interest rate risk (see JANSSEN (1993)).

We consider only the immunization on the horizon [0, H] (H is a known positive value). So let us evaluate the current value $\tilde{A}$ of the asset flow and the current value $\tilde{B}$ of the liability flow. We obtain :

$$\tilde{A} = \int_0^H e^{-\delta t}\, dA_t \qquad \text{and} \qquad \tilde{B} = \int_0^H e^{-\delta t}\, dB_t \tag{44}$$

where δ denotes underlying force of interest.

It is natural to wish that :

$$\tilde{A} \geq \tilde{B} \tag{45}$$

and therefore by taking expectations in (44) that :

$$E\left[\int_0^H e^{-\delta t}\, dA_t\right] \geq E\left[\int_0^H e^{-\delta t}\, dB_t\right]. \tag{46}$$

After permutation of expectation and integration (see JAZWINSKI (1970)) and with the use of corollary of theorem 1 we get :

$$\mu_A\, A_0 \int_0^H e^{(\mu_A - \delta)t}\, dt \geq \mu_B\, B_0 \int_0^H e^{(\mu_B - \delta)t}\, dt \tag{47}$$

or

$$\mu_A\, A_0 \left[\frac{e^{(\mu_A - \delta)H} - 1}{\mu_A - \delta}\right] \geq \mu_B\, B_0 \left[\frac{e^{(\mu_B - \delta)H} - 1}{\mu_B - \delta}\right]. \tag{48}$$

Using Taylor's formula we obtain :

$$\mu_A A_0 \left[\frac{e^{(\mu_A-\delta)H}-1}{\mu_A-\delta}\right] \cong \mu_A A_0 \left[H+(\mu_A-\delta)(H^2/2)\right], \tag{49}$$

$$\mu_B B_0 \left[\frac{e^{(\mu_B-\delta)H}-1}{\mu_B-\delta}\right] \cong \mu_B B_0 \left[H+(\mu_B-\delta)(H^2/2)\right]. \tag{49'}$$

Now we can rewrite (49) and (49') as follows :

$$E\tilde{A} \cong A_0 C_A \qquad \text{and} \qquad E\tilde{B} \cong B_0 C_B \tag{50}$$

where

$$C_A = \mu_A \left[H+(\mu_A-\delta)(H^2/2)\right] \tag{51}$$

and $$C_B = \mu_B \left[H+(\mu_B-\delta)(H^2/2)\right]. \tag{52}$$

We now reconsider the segmentations of A(t) anb B(t) (see section (3.3)). We get then :

$$E\tilde{A} \cong A_0 C'_A \qquad \text{and} \qquad E\tilde{B} \cong B_0 C'_B \tag{53}$$

where

$$C'_A = \sum_{i=1}^{n} (A_{0i}/A_0)\left(\mu_{A_{0i}}\left[H+\left(\mu_{A_{0i}}-\delta\right)(H^2/2)\right]\right) \tag{54}$$

and $$C'_B = \sum_{j=1}^{m} \left(B_{0j}/B_0\right)\left(\mu_{B_{0j}}\left[H+\left(\mu_{B_{0j}}-\delta\right)(H^2/2)\right]\right) \tag{55}$$

We suppose all parameters constants except the force of interest δ. Now the following question arises (see JANSSEN (1993)) :

how to measure the influence of the variations of the force of interest ?

By the use of Taylor expansion we obtain the variation of $E\tilde{A}$ consecutive to an infinitesimal variation $\Delta\delta$ of the force of interest :

$$\Delta\left(E\tilde{A}\right) \cong \sum_{i=1}^{n}\left(\mu_{A_{0_i}}\, A_{0_i}\left(H^2/2\right)\right) \quad \Delta\delta. \tag{56}$$

We can rewrite (56) as follows :

$$\Delta\left(E\tilde{A}\right) \cong A_0\, D_A\, \Delta\delta \tag{57}$$

$$\text{where } D_A = \sum_{i=1}^{n}\left(\mu_{A_i}\left(H^2/2\right)\right)\left(A_{0_i}/A_0\right). \tag{58}$$

We obtain a similar formula for liability :

$$\Delta\left(E\tilde{B}\right) \cong B\, D_B\, \Delta\delta \tag{59}$$

$$\text{where } D_B = \sum_{j=1}^{m}\left(\mu_{B_j}\left(H^2/2\right)\right)\left(B_{0_j}/B_0\right). \tag{60}$$

It follows from (57) and (59) that (mean) immunization can be achieved if :

$$A\, D_A = B\, D_B \tag{61}$$

Thus we obtain a result similar to JANSSEN's one.

4. Conclusion

In conclusion, we can say that the JANSSEN's model presented and developed here is now able to be really used as an **operational tool** for asset liability management for insurance companies or even for banks.

Its use is simple as it only depends on five parameters and it allows with our new concept of duration to improve the immunization of the balance sheet of the company. Its help for the use of off balance sheet financial products will be presented in another paper.

For its optimal use, it is clear that we need at least mensual data for the balance sheet of the company so that we can reestimate the five basic parameters as soon as necessary. It is also possible to use this model as a simulation one as done by the software SIMFIN.

Finally, we may add that supplementary results concerning among others the distribution of dividends will appear in a short future.

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