PRICING GUARANTEED SECURITIES-LINKED LIFE INSURANCE UNDER INTEREST-RATE RISK

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ABSTRACT

This paper analyses the problem of pricing an endowment policy in which the benefits are linked to the realization of a portfolio of securities and a minimum amount guaranteed is provided. Building on the model of Brennan-Schwartz (4), (5) for the case of an all-equity reference fund, we extend it in order to account for interest-rate risk and show how to obtain a close-form solution for the single premium when the value of the reference fund follows a geometric brownian motion and an Ornstein-Uhlenbeck process is employed for the market force of interest. We then consider the case in which the reference fund is composed all by fixed-income securities. Employing both the Vasicek (18) and the Cox-Ingersoll-Ross (8) models for the term structure, we derive close-form solutions for the single premium. We also present some comparative statics results, and hint at other possible further extensions.

1. INTRODUCTION

The life insurance markets have witnessed, in the last two decades, a significant shift from traditional products in which the only relevant uncertainty concerns life contingencies, to contracts in which the financial aspects play a crucial role. In particular, stochasticity enters the benefits not only through the time at which they (may) become due to be paid but also in their amount (see, for instance, Delvaux and Magnée (11) for a description of some products of this type present in the European markets).

A typical example in which these two effects of stochasticity are

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merged is a guaranteed equity-linked policy, where the benefits are expressed in parts of an all-equity reference fund and a minimum-amount-guaranteed provision is present. In their seminal contributions, Brennan and Schwartz (4), (5) have employed the techniques of contingent claims analysis to provide a valuation framework for this type of contracts. Subsequently, Delbaen (10) has presented an alternative approach to the valuation of periodic premium contracts and Bacinello and Ortu (2) have extended these two models in order to account for endogenous minimum guarantees.

The results obtained in the above mentioned papers rest on the fundamental assumption of a nonstochastic (and flat) term structure of interest rates. While this assumption could have been a fairly good first approximation at the times in which Brennan-Schwartz built their model, the volatility of market interest rates in the last decade clearly asks for an extension of their valuation framework in order to account for a stochastic market interest rate.

This paper provides such an analysis, concentrating the attention on guaranteed securities-linked endowment policies for which a single premium is paid. The introduction of a stochastic market interest rate not only extends the Brennan-Schwartz model on equity-linked products but also broadens the scope of our analysis in a very significant way. In the reference fund, in fact, may very well enter fixed-income securities, whose market value before maturity is a stochastic process when interest-rate uncertainty is present. While the most general model for the reference fund would take into account both equities and fixed-income securities, in this paper we choose to concentrate on the two polar cases of an all-equity and an all-fixed-income reference fund respectively.

In the next Section we introduce a one-factor model for the term structure of interest rates in which the single factor is the market force of interest and we briefly recall the way in which pure-discount bonds of every maturity can be priced. We also introduce a fairly general model for pricing single premium guaranteed securities-linked policies. Section 3 is devoted to the case in which the reference fund is composed entirely by equities. Assuming that the reference fund follows a geometric brownian motion and that an Ornstein-Uhlenbeck process is used for the market force of interest, we show how to obtain a close-form solution for the premium. Section 4 tackles the case in which the reference fund is composed only by fixed-income securities. Along with the above mentioned Ornstein-Uhlenbeck process we also consider a mean-reverting process of the more general type analysed by Cox, Ingersoll and Ross.
Pricing guaranteed securities-linked life insurance under interest-rate risk

(8) in their seminal paper on the term structure of interest rates. We show how to obtain close-form solutions for the premium in both cases. Some comparative statics results are presented in Section 5. In this Section we also compare the premium obtained in the Brennan-Schwartz model to the one obtained in our framework. We exhibit, moreover, some measures for the stochastic duration of securities-linked policies with an all-income reference fund, and suggest how some issues related to the riskiness of this line of business can be discussed. In Section 6 we outline some possible extensions of the models presented in Sections 3 and 4, while concluding remarks with an agenda for future research are collected in Section 7.

2. A GENERAL MODEL WITH STOCHASTIC INTEREST-RATE

We consider an economy in which financial and insurance markets are perfectly competitive, frictionless and free of arbitrage opportunities. To model the term structure of interest rates we follow the lines pioneered by Vasicek (18) and Cox-Ingersoll-Ross (8) and assume that all fixed-income securities depend on a single state variable identified in the market force of interest. We assume that this variable follows a stochastic process of the form

$$dr(t) = a[\theta - r(t)]dt + \eta[r(t)]^\beta dZ_1(t),$$

where $dZ_1(t)$ is the infinitesimal increment in a standard brownian motion and $a, \theta, \eta, \beta \in \mathbb{R}^+$. We observe that when $\beta = 0$ we obtain an Ornstein-Uhlenbeck process of the type originally employed by Vasicek while $\beta = 1/2$ yields the mean reverting process introduced by Cox, Ingersoll and Ross. We let $b(r, t, T)$ be the time $t$ price of a pure-discount bond with maturity $T$ when the prevailing (spot) force of interest is $r(t)$. It is well known that, since all fixed-income securities can be expressed as a portfolio of pure-discount bonds, their prices are obtained exploiting the value-additivity principle implied by the arbitrage-free assumption. In what follows the same assumption is also employed in the valuation of some interest-rate-sensitive contingent claims, in particular put and call options on pure-discount bonds.

We assume for simplicity that the parameters in [2.1] are already
risk-neutralized\(^{(1)}\), so that the local expectation hypothesis can be imposed\(^{(2)}\) to obtain the pure-discount bond price as the solution of the following partial differential equation:

\[
22 \quad rb(r, t, T) = \frac{\partial b(r, t, T)}{\partial t} + \frac{\partial b(r, t, T)}{\partial r} a(\theta - r) + \frac{1}{2} \eta^2 r^2 \frac{\partial^2 b(r, t, T)}{\partial r^2},
\]

with boundary conditions

\[
\begin{aligned}
&b(r, t, T) > 0 \quad \forall \ t \leq T \\
&b(r, T, T) = 1.
\end{aligned}
\]

In the following Sections we present the solutions to [2.2] for the cases \( \beta = 0 \) and \( \beta = 1/2 \), and exploit them in our valuation frameworks.

We consider now an endowment policy with entry age \( x \) and term \( N \). The following variables are relevant in our valuation problem:

- \( U \) = single premium
- \( D(t) \) = time \( t \) value of the reference fund, \( t \geq 0 \)
- \( g_n \) = minimum amount guaranteed at time \( n, n = 1, \ldots, N \)
- \( v_n \) = benefit at death or maturity, \( n = 1, \ldots, N \)
- \( \alpha_n \) = probability that \( v_n \) is due to be paid at time \( n, n = 1, \ldots, N \).

In our case, we have

\[
[2.3] \quad \alpha_n = \begin{cases} 
n - 1/1 q_x & n = 1, \ldots, N - 1 \\
N - 1/1 q_x + N p_x = N - 1 p_x & n = N 
\end{cases}
\]

A guaranteed securities-linked endowment policy is characterized by the fact that the benefit \( v_n \) is the value at time \( n \) of the reference fund, insured at level \( g_n \), i.e. \( v_n = \max\{D(n), g_n\}, n = 1, \ldots, N \), so that the following standard decompositions in terms of maturity values of either call or put options on \( D(n) \) applies:

\[
[2.4] \quad v_n = g_n + \max \{D(n) - g_n, 0\} = D(n) + \max \{g_n - D(n), 0\}, \quad n = 1, \ldots, N.
\]

\(^{(1)}\)This means that \( \lambda(r, t) \), the market price of interest-rate risk, is already embedded in the parameters.

\(^{(2)}\)The technically inclined read should notice that this expectation is taken with respect to an equivalent martingale measure \( Q \) (see Artzner and Delbaen (1)).
Letting $W_0(v_n)$ and $W_0(D(n))$ be the time 0 market value of $v_n$ and $D(n)$ respectively, we have

$$W_0(v_n) = g_n b(\tau, 0, n) + c(0, n, g_n)$$

$$= W_0(D(n)) + p(0, n, g_n), \quad n = 1, \ldots, N,$$

where $c(0, n, g_n)$ (respectively $p(0, n, g_n)$) denotes the time 0 value of a European call (respectively put) option on $D(n)$ with maturity $n$ and exercise price $g_n$. While it may be natural, under our assumptions, to set $W_0(D(n)) = D(0)$, this is not always the case when the reference fund is an all-fixed-income one. We supply a detailed discussion of this fact in Section 4.

We assume now that sufficient contracts are written in order to eliminate, at least in principle, the necessity for security loadings, so that the following market valuation for the premium results:

$$U = \sum_{n=1}^{N} \alpha_n W_0(v_n) = \sum_{n=1}^{N} \alpha_n g_n b(\tau, 0, n) + \sum_{n=1}^{N} \alpha_n c(0, n, g_n)$$

$$= \sum_{n=1}^{N} \alpha_n W_0(D(n)) + \sum_{n=1}^{N} \alpha_n p(0, n, g_n).$$

We note that the quantity $\sum_{n=1}^{N} \alpha_n p(0, n, g_n)$ represents the part of $U$ charged for providing the guarantee, while $\sum_{n=1}^{N} \alpha_n c(0, n, g_n)$ represents the premium difference between a guaranteed securities-linked endowment policy and a standard one with sums $\{g_n\}_{n=1}^{N}$.

The model introduced so far is general enough to include as subcases the two relevant situations of an all-equity and an all-fixed-income reference fund respectively, that we analyse in the following Sections.

3. THE PRICING PROBLEM WITH AN ALL-EQUITY REFERENCE FUND

We specialize here the general model introduced in the previous Section to the case in which the reference fund is composed all by non-dividend-paying equities. We follow moreover the assumption of Brennan
and Schwartz in describing the dynamics of its value by the following geometric brownian motion:

\[ dD(t) = \mu D(t)dt + \sigma D(t)dZ_2(t), \]

where \( \mu, \sigma \) are positive constants.

In this case clearly \( W_0(D(n)) = D(0) \forall n \), so that the only remaining problem is the valuation of european put and call options when the default free rate is stochastic.

It is now possible, under our assumptions on the stochastic processes governing the spot force of interest and the value of the reference fund, to present a fairly general expression for the value of the options intervening in our problem. This can be achieved by merging together the martingale approaches of Harrison-Kreps (13) for the pricing of contingent claims in a world of interest-rates certainty and of Artzner-Delbaen (1) for the pricing of interest-rate-sensitive contingent claims. This framework would allow us to show that the value of a european option can be expressed as the expected value of the pay-off of the option discounted by the bond with maturity the same as the option to be priced. We outline such a more general approach in Section 6, restricting here our attention to the specific but particularly interesting case in which the force of interest follows an Ornstein-Uhlenbeck process, i.e. \( \beta = 0 \) in [2.1]. Assuming that

\[ dZ_1(t)dZ_2(\tau) = \begin{cases} \rho dt & t = \tau \\ 0 & t \neq \tau \end{cases} \]

(\( \rho \) can then be interpreted as a constant instantaneous coefficient of correlation between the processes \( Z_1(t) \) and \( Z_2(t) \)), it can be shown (see Rabinovitch (15)) that the following formula holds for european call values:

\[ c(D(0), n, g_n, b(r, 0, n)) = D(0)F(A_n) - g_nb(r, 0, n)F(B_n), \]

where
\[ A_n = \frac{\ln(D(0)/g_n) - \ln b(r, 0, n)}{\sqrt{\Gamma_n}} + \frac{1}{2} \sqrt{\Gamma_n}, \]

\[ B_n = A_n - \sqrt{\Gamma_n}, \]

\[ G_n = \int_0^n \left[ \sigma^2 + 2a \eta \left( 1 - e^{-au} \right) + \frac{\eta^2}{a^2} \left( 1 - e^{-au} \right)^2 \right] du \]

and \( F \) is the cumulative distribution function of a standard normal variate.

In particular,

\[ b(r, t, T) = \exp \left\{ -\frac{r}{a} \left( 1 - e^{-a(T-t)} \right) + H(T-t) \right\} \]

with

\[ H(\tau) = -\tau \theta + \frac{1}{a} \theta (1 - e^{-a\tau}) + \frac{1}{4} \frac{\eta^2}{a^3} (4e^{-a\tau} - e^{-2a\tau} + 2a\tau - 3) \]

is the solution to the differential problem \([2.2]\) with \( \beta = 0 \).

It is now a simple task to obtain the following close-form solution for the premium:

\[ U = \sum_{n=1}^N \alpha_n g_n b(r, 0, n) \left[ 1 - F(B_n) \right] + D(0) \sum_{n=1}^N \alpha_n F(A_n). \]

For the sake of comparison we recall that in the Brennan-Schwartz model, where the force of interest, \( r \), is deterministic and constant, the single premium is given by \([3.5]\) with \( e^{-rn} \) in place of \( b(r, 0, n) \) and setting \( \eta = 0 \). The main difference between the two formulas is that the quantities \( A_n, B_n \) depend now on \( n \) not only through the price of pure-discount bonds but also through the terms \( \Gamma_n \).

\(^{(3)}\) We notice that, in this case, \((3.3)\) becomes the celebrated Black and Scholes \((3)\) formula for european call options.
4. THE PRICING PROBLEM WITH AN ALL-FIXED-INCOME REFERENCE FUND

Let us consider now the case in which the reference fund is composed all by fixed-income securities. What we have in mind here is the following contractual structure: the insurance company chooses a certain portfolio of fixed-income instruments in which, for simplicity, no interest-rate-sensitive contingent claims such as indexed-bonds, extendible-retractable bonds, callable bonds, and so on, are present. Denoting by \( \{ \gamma_j; j = 1, \ldots, K \} \) the interest and/or capital payments on the fund, that take place at the times \( \{ t_j; j = 1, \ldots, K \} \), no arbitrage implies that

\[
D(0) = \sum_{j=1}^{K} \gamma_j b(r, 0, t_j).
\]

We assume moreover that all payments accruing on the reference fund before the benefit becomes due are not deemed to be reinvested, so that the value of the reference fund for \( t > 0 \) is given by

\[
D(t) = \sum_{j: t_j > t} \gamma_j b(r, t, t_j).
\]

While our framework may appear to be rather ad hoc and somewhat restrictive, nonetheless it allows us to give fairly simple and easily interpretable pricing formulas. Anyway, the no-reinvestment assumption can be justified whenever the reference fund contains mostly coupon-bonds with sufficiently long maturity (i.e. maturity longer than \( N \)). The pricing formula that we obtain, moreover, takes into account in a clear way the effect of the above assumption. A framework in which intermediate payments are deemed to be reinvested is sketched in Section 6.

The valuation of the contract we are now considering requires the valuation of european options on portfolios of pure-discount bonds, as it clearly appears upon recalling that the benefit at time \( n \) is in this case

\[
v_n = g_n + \max \left\{ \sum_{j: t_j > n} \gamma_j b(r, n, t_j) - g_n, 0 \right\}
\]

\[
= D(n) + \max \left\{ g_n - \sum_{j: t_j > n} \gamma_j b(r, n, t_j), 0 \right\}, \quad n = 1, \ldots, N.
\]
Following Jamshidian (14)(4), we argue that a European option on a portfolio of pure-discount bonds can be priced as a portfolio of European options on single-pure-discount bonds, whenever the last ones are strictly monotonic functions of a single state variable. Assume in fact that \( b \) strictly decreases with \( r \) (as it is the case when \( \beta = 0 \) and \( \beta = 1/2 \) in [2.1]) and that it is regular enough to guarantee the existence of a unique solution to the following equations(5):

\[
\sum_{j : t_j > n} \gamma_j b(r, n, t_j) = g_n, \quad n = 1, \ldots, N.
\]

Denoting by \( \{r_n^*\} \) these solutions, one shows that

\[
\max \left\{ \sum_{j : t_j > n} \gamma_j b(r, n, t_j) - g_n, 0 \right\} = \sum_{j : t_j > n} \gamma_j \max \{ b(r, n, t_j) - b(r_n^*, n, t_j), 0 \}, \quad n = 1, \ldots, N.
\]

Note that \( b(r_n^*, n, t_j) \) works here as the exercise price of a call option with maturity \( n \) on a pure-discount bond with maturity \( t_j \). Denoting by \( c(n, j) \) the time 0 value of such an option, relation [2.6] becomes:

\[
U = \sum_{n=1}^{N} \alpha_n g_n b(r, 0, n) + \sum_{n=1}^{N} \alpha_n \sum_{j : t_j > n} \gamma_j c(n, j)
\]

or, equivalently,

\[
U = \sum_{n=1}^{N} \alpha_n \sum_{j : t_j > n} \gamma_j b(r, 0, t_j) + \sum_{n=1}^{N} \alpha_n \sum_{j : t_j > n} \gamma_j p(n, j).
\]

(4) See also Turnbull and Milne (17), for the same type of result in a discrete time model.
(5) It is clearly seen that the continuity of \( b \) in \( r \) and the existence, for all \( n \), of \( r', r'' \) s.t.

\[
\sum_{j : t_j > n} \gamma_j b(r', n, t_j) < g_n < \sum_{j : t_j > n} \gamma_j b(r'', n, t_j)
\]

are sufficient conditions for this fact to be true.
In [4.6b] we have, once again, employed the no-arbitrage assumption to obtain \( W_0(D(n)) = \sum_{j: t_j > n} \gamma_j b(r, 0, t_j) \). Moreover, \( p(n, j) \) indicates the time 0 price of a put option maturing in \( n \), with exercise price \( b(r^*_n, n, t_j) \), on a pure-discount bond with maturity \( t_j \).

It is now possible to have a clear account of the way in which our no-reinvestment assumption influences the premium. In fact some simple algebraic manipulations show that

\[
\sum_{n=1}^{N} \alpha_n \sum_{j: t_j > n} \gamma_j b(r, 0, t_j) = D(0) - \sum_{n=1}^{N} \alpha_n \sum_{j: t_j \leq n} \gamma_j b(r, 0, t_j),
\]

so that the premium can be rewritten as follows:

\[
[4.6c] \quad U = D(0) - \sum_{n=1}^{N} \alpha_n \sum_{j: t_j \leq n} \gamma_j b(r, 0, t_j) + \sum_{n=1}^{N} \alpha_n \sum_{j: t_j > n} \gamma_j p(n, j).
\]

The quantity \( \sum_{n=1}^{N} \alpha_n \sum_{j: t_j \leq n} \gamma_j b(r, 0, t_j) \) can easily be interpreted as the market value (with mortality risk taken into account) of the payments that are not reinvested. Since \( \sum_{n=1}^{N} \alpha_n \sum_{j: t_j > n} \gamma_j p(n, j) \) represents the amount charged for providing the guarantee, the remaining amount that enters the premium represents the part of the initial investment \( D(0) \), charged on the insured.

The relations presented so far hold under fairly general assumptions and, in particular, no specific formulas are employed for the intervening options. However expressions that can be easily computed are obtained for the Vasicek and CIR cases.

For the first case (i.e. \( \beta = 0 \)) Jamshidian (14) has developed a close-form solution for options on pure-discount bonds that, applied to our case, yields:

\[
[4.7] \quad c(n, j) = b(r, 0, t_j)F(\hat{A}_{n,j}) - b(r^*_n, n, t_j)b(r, 0, n)F(\hat{B}_{n,j}),
\]

with

\[
\hat{A}_{n,j} = \frac{\ln \left( b(r, 0, t_j)/b(r^*_n, n, t_j) \right) - \ln (b(r, 0, n)) + \hat{\Gamma}_{n,j}}{\hat{\Gamma}_{f,n}},
\]

\[
\hat{B}_{n,j} = \hat{A}_{n,j} - \hat{\Gamma}_{n,j},
\]

\[
\hat{\Gamma}_{n,j} = \frac{\eta}{a} \sqrt{1 - e^{-2\alpha n} \left( 1 - e^{-a(t_j-n)} \right)}.
\]
and $F$ as in [3.5]. The premium is thus given by [4.6a] with $b$ defined in [3.4] and $c(n,j)$ defined in [4.7].

An option pricing formula when the spot force of interest follows a mean reverting process with $\beta = 1/2$ has been supplied by Cox-Ingersoll-Ross (8). The Authors first show that, in this case, the solution to the differential problem [2.1] is

\begin{equation}
 b(r,t,T) = \exp \{-rH_1(T-t) + H_2(T-t)\},
\end{equation}

with

\begin{align*}
 H_1(r) &= \frac{2(e^{\gamma r} - 1)}{2\gamma + (a + \gamma)(e^{\gamma r} - 1)}, \\
 H_2(r) &= \frac{2a\theta}{\eta^2} \ln \left( \frac{2\gamma e^{(a+\gamma)\gamma/2}}{2\gamma + (a + \gamma)(e^{\gamma r} - 1)} \right), \\
 \gamma &= \sqrt{a^2 + 2\eta^2}.
\end{align*}

They supply then a formula for call options on pure-discount bonds that, applied to our valuation problem, becomes:

\begin{equation}
 c(n,j) = b(r,0,t_j)G \left( 2r_n^*(\Phi + \Psi + H_1(t_j), \frac{4a\theta}{\eta^2}, \frac{2r\Phi_n^2 e^{\gamma n}}{\Phi_n + \Psi + H_1(t_j)} \right) \\
 - b(r_n^*,n,t_j)b(r,0,n)G \left( 2r_n^*(\Phi + \Psi), \frac{4a\theta}{\eta^2}, \frac{2r\Phi_n^2 e^{\gamma n}}{\Phi_n + \Psi} \right),
\end{equation}

where

\begin{align*}
 \Phi_n &= \frac{2\gamma}{\eta^2 (e^{\gamma n} - 1)}, \\
 \Psi &= \frac{a + \gamma}{\eta^2},
\end{align*}

and $G(x,y,z)$ indicates the value in $x$ of the cumulative distribution function of a non central chi-square variate with $y$ degrees of freedom and noncentrality parameter $z$. All the elements needed to apply relation [4.6a] for the computation of $U$ are then supplied for the CIR case as well.
5. COMPARATIVE STATICS ANALYSIS AND RELATED ISSUES

In the previous Sections we have seen how the introduction of a stochastic spot force of interest both influences the premium of a Brennan-Schwartz type equity-linked policy and allows one to study a broader variety of securities-linked insurance contracts. We devote the present Section to the study of some comparative statics relevant in our extended models. We give moreover some results related to this comparative statics analysis that allow us to better understand our framework and compare it to the standard one.

5.1. THE ALL-EQUITY CASE

The major influence of a stochastic force of interest on the Brennan-Schwartz framework shows up in an increase in the number of parameters with respect to which a comparative statics analysis can be conducted. When the attention, however, is restricted to the key-parameters, i.e. the amount deemed to be invested \( D(0) \), the spot force of interest \( r \) and the guarantees \( \{ g_n \} \), one can show that the signs of the change in the premium due to changes in these parameters are the same as in the standard model (although their absolute values may change). Indeed one shows that \( \frac{\partial \epsilon}{\partial D(0)} > 0 \) and \( \frac{\partial p}{\partial r} < 0 \), \( \frac{\partial p}{\partial g_n} > 0 \) \( \forall n \)\(^{(6)} \), so that \( \frac{\partial U}{\partial D(0)} > 0 \), \( \frac{\partial U}{\partial r} < 0 \), \( \frac{\partial U}{\partial g_n} > 0 \) \( \forall n \). As long as the effects of changes in the other parameters are concerned, one can apply the results given in Rabinovitch (15) to our pricing formulas. We observe however that a clear-cut answer for the sign of some of the remaining comparative statics cannot be given, so that a numerical approach would be called for.

The comparative statics analysis we have briefly presented suggests a natural approach to the following problem: insurance companies, while operating in an environment in which interest-rate risk plays a relevant role in the pricing of their equity-linked products, almost always employ a nonstochastic and constant force of interest, usually referred to as a "technical rate". Under our maintained hypothesis on the behaviour of the reference fund, it is possible to evaluate the technical rate that would give the "correct" premium, i.e. the premium in which interest-rate risk is taken into account. Such a rate \( \delta \) can be determined by

\[^{(6)}\text{The last two derivatives are easily obtained applying put-call parity to [3.3].}\]
simply equating the premium obtained in our model and the one that holds in the standard Brennan-Schwartz world, i.e. solving with respect to \( \delta \) the following equation:

\[
[5.1.1] \quad U = \sum_{n=1}^{N} \alpha_n g_n e^{-\delta n} [1 - F(B_n)] + D(0) \sum_{n=1}^{N} \alpha_n F(A_n),
\]

with

\[
\overline{A}_n = \frac{\ln \left( \frac{D(0)}{g_n} \right) + \left( \delta + \frac{\sigma^2}{2} \right) n}{\sigma \sqrt{n}},
\]

\[
\overline{B}_n = \overline{A}_n - \sigma \sqrt{n},
\]

and \( U \) given by relation [3.5].

The fact that equation [5.1.1] has a unique solution follows upon observing that its RHS is a continuous, strictly decreasing function of \( \delta \), that diverges positively when \( \delta \to -\infty \) and converges to \( D(0) \) when \( \delta \to +\infty \), and \( U > D(0) \).

In order to have a feeling for the initial average rate of interest profit, one can compare the solution \( \delta^* \) to the level of technical rate usually employed by insurance companies (ranging in the Italian market from 0 to 4\% in a per-annum basis).

As it is clearly expected, \( \delta^* \) can be shown to be a strictly increasing function of the level of the market force of interest. The effect on \( \delta^* \) of changes in the other relevant parameters does not yield to clear-cut answers, so that once again some numerical valuation would be helpful.

5.2. THE ALL-FIXED-INCOME CASE

We have highlighted throughout the paper the fundamental importance of the market force of interest in the pricing of securities-linked policies. Since in the all-fixed-income case such a variable is indeed the only one relevant in the pricing problem, the most important comparative static is the derivative of the single premium \( U \) with respect to the current force of interest, \( r \). In the Vasicek case (\( \beta = 0 \)) we have

\[
[5.2.1] \quad \frac{\partial U}{\partial r} = \sum_{n=1}^{N} \alpha_n \frac{\partial b(r,0,n)}{\partial r} \left[ g_n - \sum_{j:t_j > n} \gamma_j b(r^*_n,n,t_j) F(\overline{B}_{n,j}) \right]
\]

\[
\quad + \sum_{n=1}^{N} \alpha_n \sum_{j:t_j > n} \gamma_j \frac{\partial b(r,0,t_j)}{\partial r} F(\overline{A}_{n,j}).
\]
Observing that \( 0 < F(\tilde{B}, n, j) < 1 \) \( \forall n, j \) and \( \sum_{j:t_j>n} \gamma_j b(r^n, n, t_j) = g_n \) \( \forall n, \)
\( \frac{\partial U}{\partial r} < 0 \) follows from the fact that \( \frac{\partial b(r, t, T)}{\partial r} = -\frac{1}{a} b(r, t, T)(1-e^{-a(T-t)}) < 0 \) \( \forall t < T. \)

While the negative sign of \( \frac{\partial U}{\partial r} \) is clearly to be expected, relation [5.2.1] is nonetheless important since it allows us to present a close-form formula for the stochastic duration of the portfolio of interest-rate-sensitive securities whose value gives the premium. We recall briefly that the stochastic duration of any interest-rate-sensitive securities as defined by Cox-Ingersoll-Ross (7) is given by the time to maturity of a pure-discount bond with the same basis risk, where basis risk is minus the logarithmic derivative of the price with respect to \( r \). The stochastic duration, \( \Delta(U) \), is thus defined implicitly by

\[
[5.2.2] \quad -\frac{\partial \ln b(r, 0, \Delta(U))}{\partial r} = -\frac{\partial \ln U}{\partial r}.
\]

For our single premium we obtain, after some simple algebraic manipulations,

\[
[5.2.3] \quad \Delta(U) = -\frac{1}{a} \ln \left\{ \left( \sum_{n=1}^{N} w_n e^{-an} + \sum_{n=1}^{N} \sum_{j:t_j>n} q_{n,j} e^{-at_j} \right) / \left( \sum_{n=1}^{N} w_n + \sum_{n=1}^{N} \sum_{j:t_j>n} q_{n,j} \right) \right\}
\]

where

\[
w_n = \alpha_n b(r, 0, n) \left[ g_n - \sum_{j:t_j>n} \gamma_j b(r^n, n, t_j) F(\tilde{B}, n, j) \right],
\]

\[
q_{n,j} = \alpha_n \gamma_j b(r^n, n, t_j) F(\tilde{B}, n, j).
\]

We notice that \( \Delta(U) \) is (obviously) strictly less than \( N \). What is more interesting to analyse is the relationship between \( \Delta(U) \) and the stochastic duration of both a standard endowment policy with sums \( \{g_n\}_{n=1}^{N} \) and a securities-linked one without minimum guarantees. Recalling that the premiums for these last two types of policies are given by \( U^S = \sum_{n=1}^{N} \alpha_n g_n b(r, 0, n) \) and \( U^{WG} = \sum_{n=1}^{N} \sum_{j:t_j>n} \alpha_n \gamma_j b(r, 0, t_j) \) respectively, we have

\[
[5.2.4] \quad \Delta(U^S) = -\frac{1}{a} \ln \left\{ \sum_{n=1}^{N} \alpha_n g_n b(r, 0, n)e^{-an} / \sum_{n=1}^{N} \alpha_n g_n b(r, 0, n) \right\},
\]

\[
[5.2.5] \quad \Delta(U^{WG}) = -\frac{1}{a} \ln \left\{ \sum_{n=1}^{N} \sum_{j:t_j>n} \alpha_n \gamma_j b(r, 0, t_j) e^{-at_j} / \sum_{n=1}^{N} \sum_{j:t_j>n} \alpha_n \gamma_j b(r, 0, t_j) \right\}.
\]
\[ \Delta(U^{WG}) = \frac{1}{a} \ln \left\{ \frac{\sum_{n=1}^{N} \alpha_n \sum_{j: t_j > n} \gamma_j b(r, 0, t_j) e^{-at_j}}{\sum_{n=1}^{N} \alpha_n \sum_{j: t_j > n} \gamma_j b(r, 0, t_j)} \right\}. \]

We believe that the measures of duration we present in [5.2.3] to [5.2.5] may improve the analysis of securities-linked policies in two ways. First, since many practitioners single out securities-linked policies with minimum-amount-guaranteed as a particularly risky line of business, the above measures allow one to ascertain quantitatively to which extent they are so, especially when compared to more traditional products. Second, when the increase in risk due to the particular features of the policy under scrutiny is documented, stochastic duration can be used as the basis on which to build an efficient investment management of the reserves, thus allowing to hedge against the increased risk\(^7\).

We conclude this Section by noting that the same considerations we have presented when the spot force of interest follows an Ornstein-Uhlenbeck process can be carried out when the CIR model is employed. In this case however we are not able to present a close-form for the derivative of the premium with respect to the spot force \(r\) since no close-form derivatives with respect to \(r\) are available for the options intervening in \(U\). A numerical approach is granted to give interesting insights and answers to these problems.

6. Extensions

The pricing formulas we have derived so far allow us to have a fairly good description of the characteristics of securities-linked contracts. However we have already pointed out in the previous Sections that our formulas are obtained under assumptions that are somewhat restrictive and rather ad hoc. In particular, when discussing the all-equity case we have restricted our attention to situations in which a Vasicek model for the spot force of interest is employed. Moreover, when discussing the all-fixed-income case, we have explicitly required that interest and/or capital payments made on the fund when the insured is still alive before maturity are not reinvested. In what follows we sketch some possible extensions that allow us to discuss the pricing

\(^7\)See De Felice and Moriconi (9) for a detailed survey of the modern theory of stochastic immunization.
problem in frameworks less restrictive than the ones outlined in Sections 3 and 4.

Let us consider first the pricing problem with an all-equity reference fund when no particular assumptions are made on $b$ in [2.1]. The best way to proceed in this case is to apply the martingale approach to contingent claims valuation introduced by Harrison and Kreps (13) in their seminal paper. However, since their framework requires the presence of a nonstochastic spot force of interest, it cannot be applied wholesale to our problem. We give here a brief, non-technical outline of how their results can be extended in order to account for interest-rate stochasticity(8).

To this end, we first recall an important result from Artzner and Delbaen (1). These Authors show that, when $r$ follows a diffusion process with time-independent drift and volatility, and no arbitrage opportunities are present on the market, then the price process $b_T$ of a pure-discount bond with maturity $T^{(9)}$ follows itself a diffusion process, i.e.

\[ db_T = m_T(b_T, t)b_T dt + s_T(b_T, t)b_T dZ_1, \]

where $Z_1(t)$ is the standard brownian motion introduced in [2.1].

We restrict now our attention to the (arbitrage-free) economy in which the pure-discount bond with price process $b_T$ and the reference fund with price process $D$ (satisfying [3.1]) are traded. To obtain a pricing formula for european options on the reference fund, it is useful to consider a "normalized" economy in which a pure-discount bond with constant unit price and a reference fund with price $D'(t) = D(t)/b_T(t)$ are traded. The rationale behind such a change in units is that our "normalized" economy satisfies Harrison and Kreps' hypothesis of a nonstochastic interest rate constantly equal to zero. We observe, moreover, that applying Ito's Lemma to $D'(t)$ one gets:

\[ dD' = \left[ \mu - m_T(b_T, t) - \rho \sigma s_T(b_T, t) + s_T^2(b_T, t) \right] D' dt + \sqrt{\sigma^2 - 2\rho \sigma s_T(b_T, t) + s_T^2(b_T, t)} D'dZ, \]

(8) We refer the reader interested in further details to Cheng (6) and Frigerio-Ortu-Pressacco (12).

(9) To simplify the notation $T$ is used here as a subscript.
where the standard brownian motion \( Z \) satisfies
\[
\sigma^2 - 2\rho \sigma s_T(b_T, t) + s_T^2(b_T, t) \, dZ = \sigma dZ_2 - s_T(b_T, t) \, dZ_1.
\]

Under suitable assumptions on the drift and volatility terms in [6.2], one can apply Theorem 3 in Harrison and Kreps (13) (page 396) in order to show that in the "normalized" economy every contingent claim \( x' \) can be priced by arbitrage and the Corollary to their Theorem 3 shows how to obtain such a price. In particular, if \( T' = n \) and \( x' = \max\{D'(n) - g_n, 0\} \), we have
\[
[6.4] \quad W_0'(x') = E\left[ \max\{D^*(n) - g_n, 0\}\right]^{(10)},
\]
where the process \( D^* \) satisfies
\[
[6.5] \quad dD^* = \sqrt{\sigma^2 - 2\rho \sigma s_n(b_n, t) + s_n^2(b_n, t)} \, D^* dZ
\]
with \( D^*(0) = D'(0) \).

We are however interested in pricing \( x = \max\{D(n) - g_n, 0\} \), which is a european call option in the "original" economy, where the pure-discount bond with price \( b \) and the reference fund with price \( D \) are traded. Letting \( W_0(x) \) be its time 0 value, observing that \( x = x' \) since \( D'(n) = D(n)/b_n(n) = D(n) \) and "denormalizing" we obtain:
\[
[6.6] \quad W_0(x) = b_n(0)W_0'(x') = b_n(0)E\left[ \max\{D^*(n) - g_n, 0\}\right].
\]

It is now interesting to obtain the pricing formula [3.3] for european call options as a particular case of [6.6]. In this case, applying Ito's Lemma to the bond price in [3.4] one gets:
\[
[6.7] \quad s_T(b_T, t) = -\frac{\eta}{a}(1 - e^{-a(T-t)}),
\]
so that the volatility term in [6.5] becomes
\[
\sqrt{\sigma^2 + \frac{2\rho \eta}{a}(1 - e^{-a(n-t)}) + \frac{\eta^2}{a^2}(1 - e^{-a(n-t)})^2}.
\]

\(^{(10)}\) Note that we have "primed" \( W_0 \) since here the price is normalized, i.e. in units of the pure-discount bond.
Exploiting now a well-known theorem regarding driftless geometric brownian motions (see, for instance, Smith (16), page 16, for the case in which the volatility is constant) one gets:

\[ E \left[ \max \{ D^*(n) - g_n, 0 \} \right] = D^*(0) F \left( \frac{\ln(D^*(0)/g_n)}{\sqrt{\Gamma_n}} + \frac{1}{2} \sqrt{\Gamma_n} \right) - g_n F \left( \frac{\ln(D^*(0)/g_n)}{\sqrt{\Gamma_n}} - \frac{1}{2} \sqrt{\Gamma_n} \right) \]

[6.8]

where \( \Gamma_n \) is as defined in [3.3]. Recalling that \( D^*(0) = D'(0) = D(0)/b_n(0) \), we have indeed

\[ W_0(x) = c(D(0), n, g_n, b(r, 0, n)). \]

In cases more general than the Vasicek one, no close-form solutions are known for the expectation in [6.6], hence for the option price. However, a numerical approach based on Montecarlo simulation is likely to yield a satisfactory approximation for the required value.

Consider next the all-fixed-income case. If one wishes to avoid the no-reinvestment assumption, a strategy for the reinvestment of the payments due on the fund before maturity when the insured is still alive is required. Since there is a wide variety of possible reinvestment strategies, each one characterizing a particular type of contract, we choose to single out a specific one, whose rationale can be traced back to the basic principle of diversification.

Suppose to consider a certain set \( \{1, \ldots, K\} \) of times to maturity and a set \( \{ \gamma_1, \ldots, \gamma_K \} \) of fractions invested, in the sense that \( \gamma_j(\geq 0) \) is the fraction invested in a pure-discount bond with maturity \( j \), and \( \sum_{j=1}^{K} \gamma_j = 1 \). We define the unit price of the reference fund, \( S(n) \), by

\[ S(n) = \sum_{j=1}^{K} \gamma_j b(r, n, n + j), \quad n = 0, 1, \ldots, N, \]

[6.10]

and we fix our attention on an all-fixed-income contract for which all the payments are reinvested in such a way that at any time \( n \) the composition of the reference fund consists of pure-discount bonds with times to maturity ranging from 1 to \( K \), each bond with time to maturity \( j \) being held in the proportion \( \gamma_j \). In order to exemplify, we let \( m_n \) be
the number of units held at time $n$ if the insured is alive and we show how $m_n$ can be obtained recursively once $m_0$ is given. Letting

\begin{equation}
D(n) = m_n S(n), \quad n = 0, 1, \ldots, N
\end{equation}

be the value at time $n$ of the reference fund, since

\begin{equation}
D(n) = m_{n-1} \left( \gamma_1 + \sum_{j=2}^{K} \gamma_j b(r, n, n - 1 + j) \right), \quad n = 1, \ldots, N
\end{equation}

is also the time $n$ value of the portfolio established at time $n - 1$, we can solve recursively to obtain

\begin{equation}
m_n = \begin{cases} 
m_0 & n = 0 \\
gamma_1 + \sum_{j=2}^{K} \gamma_j b(r, n, n - 1 + j) \\
m_{n-1} \frac{S(n)}{S(n-1)} & n = 1, \ldots, N
\end{cases}
\end{equation}

Two things may be remarked now: first the equality between the RHS's of [6.11] and [6.12] clearly formalizes the self-financing constraint that characterizes all reinvestment strategies. In our case, moreover, the self-financing constraint together with the requirement that the fractions $\{\gamma_j\}$ remain constant through time imply that the company may need to implement a fairly dynamic trading strategy on the bond market.

As long as the pricing of the single premium for this type of contract is concerned, the general model outlined in Section 2 clearly still works. The problem here is to value the options intervening in [2.6], since the underlying asset is in this case

\begin{equation}
D(n) = m_0 \frac{\prod_{h=1}^{n} \left( \gamma_1 + \sum_{j=2}^{K} \gamma_j b(r, h, h - 1 + j) \right)}{\prod_{h=1}^{n-1} \left( \gamma_1 + \sum_{j=1}^{K} \gamma_j b(r, h, h + j) \right)}, \quad n = 1, \ldots, N
\end{equation}

as some fairly simple algebra shows. The complexity of [6.14] makes clear that in general no close-form solutions would be available for the premium of this type of contract. Our guess however is that the framework of Artzner-Delbaen could serve as an excellent basis on which a sound numerical approach via simulation can be implemented, once the dynamic behaviour of $r$ is described by [2.1].
7. CONCLUDING REMARKS

We believe that the models of Sections 3 and 4 for pricing securities-linked life insurance policies with the minimum-amount guaranteed feature give, along with the extensions of Section 6, a sufficiently good insight into the problems arising when the stochasticity of interest rates has to be taken into account. We also believe, nonetheless, that the approaches suggested both leave some important questions unanswered and rise other issues to be scrutinized.

On the first side, the problem of pricing this type of contracts when periodic premiums are paid has not been touched at all. This is a mostly significant issue, since almost always this form of payment is the one preferred by the buyers. We notice that, while in both the all-equity and the all-fixed-income cases no close-form solutions for the periodic premium are to be expected, the martingale approaches of Harrison-Kreps and Artzner-Delbaen are likely to successfully prepare the ground for obtaining a solution via simulation. This is clearly a central objective of our future research. However, we point out that the pricing framework introduced here becomes very useful when a single recurrent premiums scheme is adopted. In this case, in fact, a contract in which periodic premiums (not necessarily constant through time) are paid can be priced as a "sum" of N single premium policies, issued in case of survival, each one with entry-age $x + n$ and term $N - n$ ($n = 0, 1, \ldots, N - 1$).

On the other side, the discussion concerning the comparative statics of our approaches requires to be completed. Indeed, such an analysis constitutes a relevant first step towards the study of an efficient investment management of the reserves that an insurance company holds to hedge against the risk implied by the contracts sold. This topic however requires itself consistent future research.

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