PORTFOLIO SELECTION IN THE PRESENCE OF OPTIONS AND THE DISTRIBUTION OF RETURN OF PORTFOLIOS CONTAINING OPTIONS

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ABSTRACT

The paper starts by developing the algebra for calculating moments of the returns from European option positions, with numerical examples. The author then discusses the search for the Holy Grail of a framework for truly **modern** portfolio selection and puts forward an investigation of percentile returns as an alternative to mean variance optimisation, the latter being shown to be totally inadequate in the presence of options. Finally, the author compares protected put positions with cash/equity portfolios of equivalent expected return, both from the point of view of distribution of returns and from an exponential utility point of view.

1. THE ALGEBRA FOR MOMENTS OF FINANCIAL OPTIONS

Under the commonly used geometric Brownian model of equity market behaviour, given an initial price S, the equity index price S_T at time T is lognormally distributed with the underlying normal distribution being:

$$\ln S_T = N(\ln S + (\mu - 1/2\beta^2)T, \beta T^{1/2})$$

where $(\mu - 1/2\beta^2)$ and β are the mean and standard deviation of the continuously compounded equity return per unit time, and $N()$ is the normal distribution.

Given a price level X (typically an option exercise price), define two series of integrals C_n and P_n (C and P relating to call and put options, n taking integer values from 0 to infinity) as follows:

Put $y = S_T$ to simplify the notation.

$$P_n = \int_0^X (y^n * g(y))dy$$

$$C_n = \int_X (y^n * g(y))dy \qquad \text{(integral from } X \text{ to infinity)}$$

where $g(y)$ is the probability density function of the lognormal distribution.

Thus P_n and C_n are parts of the n-th moment of the (lognormal) distrubution of S_T and indeed the n-th moment is simply the sum of P_n and C_n.

Then, by substituting $w = \ln y$, after some manipulation, the following equations can be obtained (details are available on request - some of the algebra involved can be seen in Wilkie 1992):

$$P_n = S^n \exp\left\{(n\mu + n(n-1)\beta^2/2)T\right\} * N(-d_n)$$

and

$$C_n = S^n \exp\left\{(n\mu + n(n-1)\beta^2/2)T\right\} * N(d_n)$$

$$\text{where } d_n = \left\{\ln(S/X) + (\mu + (2n-1)\beta^2/2)T\right\}/(\beta T^{1/2})$$

and $N()$ here refers to the cumulative normal distribution function.

The reader conversant with the Black-Scholes option pricing formula will already be partly familiar with such expressions, indeed the Black-Scholes formulae for the prices of European put and call options on a non dividend paying stock can be expressed in the above notation (providing one substitutes r, the risk free force of interest, for μ):

$$p = \exp(-rT)\{XP_0 - P_1\}$$
$$p = X\exp(-rT)N(-d_0) - SN(-d_1)$$

and

$$c = \exp(-rT)\{C_1 - XC_0\}$$
$$c = SN(d_1) - X\exp(-rT)N(d_0).$$

Throughout the remainder of this paper I will confine the analysis to European options on a non dividend paying stock. This could be regarded in practice as akin to European options on an index, such as the Euro-FTSE 100 options (although this index is affected from time to time by major stocks going ex dividend) or indeed European options on the Japanese equity indices, where dividend effects are quite small.

2. Means and Standard Deviations of Option Returns

Put Options

The expected value of a put option at expiry can be calculated as follows:

$$E(P) = \int_0^X (X - y)g(y)dy = XP_0 - P_1$$

where now the formulae use μ rather than r (which is only used in the calculation of the prices of the options, not their moments).

Similarly, the expected value of P^2 is:

$$E(P^2) = \int_0^X (X - y)^2 g(y)dy = X^2P_0 - 2XP_1 + P_2$$

and

$$E(P^3) = \int_0^X (X - y)^3 g(y)dy = X^3P_0 - 3X^2P_1 + 3XP_2 - P_3$$

Call Options

Similarly for call options:

$$E(C) = \int_X (y - X)g(y)dy = C_1 - XC_0$$

$$E(C^2) = \int_X (y - X)^2 g(y)dy = C_2 - 2XC_1 + X^2C_0$$

$$E(C^3) = \int_X (y - X)^3 g(y)dy = C_3 - 3XC_2 + 3X^2C_1 - X^3C_0$$

where $\int_X$ denotes an integral from X to infinity.

Expected Returns, Standard Deviations and Skewness

Remembering that prices of p and c must be paid at the beginning of the term for put and call options respectively, the expected returns, standard deviations of return, and skewness of return can be calculated using the following formulae:

Expected Return on Put Option $= E(P)/p$

Expected Return on Call Option $= E(C)/c$

Standard Deviation of Return on Put Option

$$= \{E(P^2) - E(P)^2\}^{1/2}/p$$

Standard Deviation of Return on Call Option

$$= \{E(C^2) - E(C)^2\}^{1/2}/c$$

Skewness of Return on Put Option

$$= \{E(P^3) - 3E(P^2) * E(P) + 2E(P)^3\}/p^3$$

Skewness of Return on Call Option

$$= \{E(C^3) - 3E(C^2) * E(C) + 2E(C)^3\}/c^3$$

Protected Put Positions

Expected Return on Protected Put position (where 1 unit by value of the index is held and a premium of p is paid to purchase a put option on this unit value):

$$\begin{aligned} E(Sh+P)/(1+p) &= \left\{X\int_0^X g(y)dy + \int_X y\,g(y)dy\right\}/(1+p) \\ &= (XP_0 + C_1)/(1+p) \end{aligned}$$

$$\begin{aligned} E(Sh+P)^2 &= X^2\int_0^X g(y)dy + \int_X y^2 g(y)dy \\ &= X^2 P_0 + C_2 \end{aligned}$$

Standard Deviation of Protected Put position:

$$= \left\{E(Sh+P)^2 - \left(E(Sh+P)\right)^2\right\}^{1/2}/(1+p)\,.$$

Covariances and Correlations of Options With The Underlying Security

$$E(Sh * P) = \int_0^X (X - y) y g(y) dy = XP_1 - P_2$$

Covariance Between Share and Put Returns:

Covar $(Sh, P)/p = \{(XP_1 - P_2) - E(Sh) * E(P)\}/p$

$= \{(XP_1 - P_2) - (C_1 + P_1) * E(P)\}/p$.

Correlation Between Share and Put Returns

Corr (Sh, P) returns = Covar $(Sh, P)/\big(p * sd(Sh) * sd(P)\big)$

where $sd(Sh) = (P_2 + C_2 - (P_1 + C_1)^2)^{1/2}$

and $sd(P)$ is the standard deviation of the return on a put option calculated above.

$$E(Sh * C) = \int_X (y - X) y g(y) dy = C_2 - XC_1 .$$

Covariance Between Share and Call Returns:

Covar $(Sh, C)/c = \big\{(C_2 - XC_1) - E(Sh) * E(C)\big\}/c$

$= \big\{(C_2 - XC_1) - (C_1 + P_1) * E(C)\big\}/c$

Correlation Between Share and Call Returns

Corr (Sh, C) returns = Covar $(Sh, C)/\big(c * sd(Sh) * sd(C)\big)$

where $sd(Sh) = (P_2 + C_2 - (P_1 + C_1)^2)^{1/2}$

and $sd(C)$ is the standard deviation of the return on a call option calculated above.

Covariance Between Put and Call with the same term and exercise price = 0, hence:

Correlation Between Put and Call with the same term and exercise price = 0.

Similarly, any higher moments can be calculated and indeed, in the numerical examples that follow, I have used similar formulae to calculate the fourth centralised moments (or **kurtosis** coefficients) for various assets.

3. Practical Examples

Consider the situation where r, the risk free **force** of interest is 8% per annum (pa), μ, the expected **force** of return on the equity index is 11% pa and β, the standard deviation of the underlying normal equity return is 20% pa.

Then, using the formulae set out above, the following can be calculated for options with a one year term:

(Take the starting price of the equity index, S, to be one without loss of generality)

Example 1: At The Money Options

$X = S = T = 1.000$
$r = 0.08$
$\mu = 0.11$
$\beta = 0.20$

n	d_n	$N(d_n)$	$N(-d_n)$	C_n	P_n	d_n^*	$N(d_n^*)$	$N(-d_n^*)$
0	0.45	0.6736	0.3264	0.6736	0.3264	0.3	0.6179	0.3821
1	0.65	0.7422	0.2578	0.8284	0.2878	0.5	0.6915	0.3085
2	0.85	0.8023	0.1977	1.0406	0.2564			
3	1.05	0.8531	0.1469	1.3380	0.2303			
4	1.25	0.8944	1.1056	1.7653	0.2085			

(NB - the relationship $N(-x) = 1 - N(x)$ comes in useful here.)

The three final columns are calculated using r rather than μ and are used to calculate the prices of at the money put and call options respectively as:

$$p = 0.0442 \quad \text{or } 4.42\% \text{ of the exercise price}$$
$$c = 0.1211 \quad \text{or } 12.11\% \text{ of the exercise price}$$

Using the formulae set out in Section 2, the following calculations can be made:

n	$E(P^n)$	$E(C^n)$	$E(Share^n)$
1	0.038528	0.154806	1.116278
2	0.007047	0.057327	1.296930
3	0.001594	0.027951	1.568312
4	0.000389	0.016708	1.973878

Expected Returns and Standard Deviations, Skewness and Kurtosis:

Asset	Expected (Mean) Return (% pa)	Standard Deviation (% pa)	Skewness	Kurtosis
Cash	8.33	0.0	0.000	0.000
Equities	11.63	22.6	0.007	0.010
At the Money Put	(12.78)	168.8	10.370	52.300
At the Money Call	27.88	150.9	4.930	27.600
Protected Put	10.60	17.5	0.008	0.005

Correlation Matrix:

Asset	Cash	Equities	At the money Put	At the money Call
Cash	1			
Equities	0	1.000		
At the money Put	0	(0.685)	1	
At the money Call	0	0.955	0	1

Comments:

Note that the expected return on the Protected Put Position is slightly lower than the equity expected return, which is to be expected given the negative return on put options and the feeling that, without defining risk precisely, one has somehow reduced risk by holding the put and that this reduction in risk should require a reduction in expected return in an efficient market. However, we can already see that this principle (as currently understood, with risk and return defined as mean and standard deviation of return respectively) requires refinement in the presence of derivatives, otherwise a very high **positive** return would be required on put options. In fact, we see from the above that investors are happy to hold such put options with a **negative** mean return!

I have not included the Protected Put in the correlation matrix, since it is simply a linear combination of the equity and put assets and as such is not a fundamental asset in its own right.

Example 2: Out of the Money Put Option / In the Money Call Option ($S = 1.05X$):

Here, $X = 1/1.05 = 0.9524$ and other parameters are as for example 1.

As for example 1, calculations can be made using the formulae set out in Section 2 to produce the following:

n	d_n	$N(d_n)$	$N(-d_n)$	C_n	P_n	d_n^*	$N(d_n^*)$	$N(-d_n^*)$
0	0.694	0.7561	0.2439	0.7561	0.2439	0.544	0.7068	0.2932
1	0.894	0.8143	0.1857	0.9090	0.2073	0.744	0.7715	0.2285
2	1.094	0.8630	0.1370	1.1193	0.1777			
3	1.294	0.9021	0.0978	1.4149	0.1535			
4	1.494	0.9324	0.0676	1.8405	0.1334			

$p = 0.0294$ or 2.94% of the initial share price (or 3.09% of the exercise price)
$c = 0.1502$ or 15.02% of the initial share price.

n	$E(P^n)$	$E(C^n)$	$E(Share^n)$
1	0.024982	0.188880	1.116278
2	0.004070	0.073636	1.296930
3	0.000851	0.037302	1.568312
4	0.000217	0.022832	1.973878

Expected Returns and Standard Deviations, Skewness and Kurtosis:

Asset	Expected (Mean) Return (% pa)	Standard Deviation (% pa)	Skewness	Kurtosis
Cash	8.33	0.00	0.000	0.00
Equities	11.63	22.60	0.007	0.010
Out of the Money Put	(14.88)	200.00	22.820	197.000
In the Money Call	25.76	129.74	2.670	12.960
Protected Put	10.87	18.93	0.008	0.006

Correlation Matrix:

Asset	Cash	Equities	At the money Put	At the money Call
Cash	1			
Equities	0	1		
Out of the money Put	0	(0.617)	1	
In the money Call	0	0.972	0	1

Comments:

Relative to the figures from Example 1, the expected return on the put option has reduced, while the standard deviation has **increased.** This is because in the case of asymmetric assets which are positively skewed, a high standard deviation is a good thing (because the higher standard deviation means that there is a higher "risk" of a better than expected return and the downside is limited). Indeed, the skewness coefficient has also increased.

The expected return and standard deviation of the call option have reduced, and this is simply because the more a call option is in the money, the more it behaves like the underlying share. Since the at the money call option has an expected return and standard deviation which are both higher than the underlying equities, both will reduce as the option becomes in the money.

4. Portfolio Selection in the 1990s: is Mean/Variance Obsolete?

Let us assume that we are trying to select portfolios that are optimal in some sense, for a risk averse client with a single period time horizon, with the following choice of possible asset sectors:

- Cash
- Equities
- At the Money Put
- At the Money Call

each asset having the characteristics calculated in Example 1 of Section 3.

Now, it is well known that the original Markowitz mean variance portfolio selection criterion only produces optimal portfolios if either:

- portfolio returns are normally distributed (which is patently untrue over periods much longer than one year)

or:

- investors have quadratic utility functions (which is again unlikely and has the undesirable feature that risk tolerance reduces with increasing wealth).

In addition, as pointed out in the comments to Section 3, the standard deviation is particularly inappropriate as a risk measure when considering highly asymmetric assets, such as options. The very high positive skewness coefficients shown in Section 3 indicate that a high standard deviation of return is not necessarily undesirable for such assets.

Nevertheless, the number of possible portfolios that could be selected is infinitely large, and mean variance optimisation is widely used as a sifting process to help narrow the field and produce a shortlist of suitable candidate portfolios for consideration. However, the weaknesses of mean variance optimisation (as mentioned above, together with the ever present one of sensitivity of the results to the input assumptions) should not be forgotten and in my own work with my colleagues in the field of asset liability modelling, I have found that:

- it is quite often necessary to **refine the inputs** to the mean variance optimisation process in order to take account of the skewness and non normality of unadjusted asset values/returns over periods of typically 10 years or more (for example, by analysing annualised returns or the rate of growth of surplus relative to the liabilities)
- the **outputs** should not be taken as carved in tablets of stone and indeed it is often the case that portfolios that lie slightly off the "efficient frontier" may be the best practical solution for the client.

This last point is particularly valid of course in the present context: because of the way that mean variance optimisers assume that asset returns are distributed symmetrically, it is quite likely that such optimisers will be biased against options because of their high standard deviations.

However, let us look at the results of running a conventional mean variance optimiser (where I show below five portfolios chosen to have minimum standard deviation for a given mean return) for the above problem.

Traditional "Efficient Frontier" (Assets as in Example 1)

Portfolio:	A	B	C	D	E
Cash	100	31.3	0	0	0
Equities	0	68.7	100	52	0
At the money Put	0	0	0	0	0
At the money Call	0	0	0	48	100
Total	100%	100%	100%	100%	100%
Expected (mean) return (% pa)	8.33	10.59	11.63	19.16	27.88
Standard Deviation (% pa)	0	15.5	22.6	83.7	150.9

Note that from A to C, the optimiser simply ignores the existence of the options and chooses the same range of portfolios that it would have selected had there been only two assets available, cash and equities. The maximum return portfolio is of course that which is invested entirely in the call option and portfolios from C to E consist of mixtures of equities and call option positions.

I selected portfolio B to have an expected (mean) return close to that of the Protected Put portfolio from Section 3. B has a lower standard deviation (15.5% as opposed to 17.5%- see Section 3) and so

the optimiser ignores the Protected Put portfolio, even though once the downside risk has been limited, the higher the standard deviation the better!

To encapsulate all information about risk into a single figure, the standard deviation, is obviously throwing away the baby with the bathwater in the case of derivatives and for many clients where more weight is attached to avoiding adverse performance than to achieving outperformance, ie the attitude to risk is almost invariably asymmetrical. Thus, I show below the kind of analysis that we find more helpful in making comparisons between portfolios for a range of portfolios, including B and the Protected Put:

Portfolio Analysis (Assets as in Example 1) ($X = 1.000$):

Portfolio:	100% Equity	B	Protected Put	100% Put	100% Call=E
Cash	0	31.3	0	0	0
Equities	100	68.7	95.8	0	0
At the money Put	0	0	4.2	100	0
At the money Call	0	0	0	0	100
Total	100%	100%	100%	100%	100%
Absolute Worst Case Return (% pa)	(100)	(66.1)	(4.2)	(100)	(100)
Worst Case (5th percentile)	(21.3)	(12)	-4.2	(100)	(100)
Lower Quartile	(4.4)	(0.4)	(4.2)	(100)	(100)
40th percentile	4	5.4	(0.4)	(100)	(67)
Most Likely Return (Median)	9.4	9.1	4.8	(100)	(22.2)
60th percentile	15.1	13	10.2	(100)	24.9
Upper Quartile	25.2	19.9	19.9	(0.8)	108.2
Best Case (95th percentile)	52	38.4	45.6	381.3	329.9
Probability of Total Loss	<1%	zero	zero	67%	33%
Probability of a negative return	33%	26%	41%	75%	55%

A comparison of the protected put portfolio with portfolio B (which was chosen to have the same mean return as the protected put portfolio) shows that the protected put portfolio does better when large equity price movements (either positive or negative) are experienced, but in the event of small price moves, B does better. In fact, the probability of B outperforming the protected put is more than 50%.

Nevertheless, given the fact that the downside of the protected put is limited to -4.2%, it is possible that some investors (pension fund trustees, for example) may prefer the protected put to B. (B is not very much removed from the typical asset distribution of UK pension funds over the last five years or so).

I show in the following pages similar tables for options with different exercise prices (again with B chosen to have the same mean return as the protected put portfolio), ranging from $X = 0.8$ to $X = 1.2$. Different investors may prefer different exercise prices for a protected put portfolio, depending on their own risk return preferences. For example, using a put option that is slightly out of the money provides only slightly worse downside protection while reducing the cost in the event of a small equity price move, whereas choosing a put option that is slightly in the money can remove the possibility of capital loss altogether (this is the basis of some unit trust and personal pension products currently on the market) at the expense of increased underperformance in the event of small equity price movements.

Portfolio Analysis (Assets as in Example 2) ($X = 0.95238$):

Portfolio:	100% Equity	B	Protected Put	100% Put	100% Call=E
Cash	0	22.9	0	0	0
Equities	100	77.1	97.1	0	0
Put	0	0	2.9	100	0
Call	0	0	0	0	100
Total	100%	100%	100%	100%	100%
Absolute Worst Case Return (% pa)	(100)	(75.2)	(7.5)	(100)	(100)
Worst Case (5th percentile)	(21.3)	(14.5)	-7.5	(100)	(100)
Lower Quartile	(4.4)	(1.5)	(7.1)	(100)	(97.5)
40th percentile	4	5	1.0	(100)	(41.7)
Most Likely Return (Median)	9.4	9.2	6.3	(100)	(5.6)
60th percentile	15.1	13.6	11.8	(100)	32.4
Upper Quartile	25.2	21.3	21.6	(100)	99.5
Best Case (95th percentile)	52	42	47.7	462.1	278.2
Probability of Total Loss	<1%	zero	zero	76%	24%
Probability of a negative return	33%	28%	38%	80%	52%

Portfolio Analysis ($X = 0.800$):

Portfolio:	**100% Equity**	B	**Protected Put**	**100% Put**	**100% Call=E**
Cash	0	5.0	0	0	0
Equities	100	95	99,5	0	0
Put	0	0	0.5	100	0
Call	0	0	0	0	100
Total	100%	100%	100%	100%	100%
Absolute Worst Case Return (% pa)	(100)	(94.6)	(20.4)	(100)	(100)
Worst Case (5th percentile)	(21.3)	(19.8)	(20.4)	(100)	(100)
Lower Quartile	(4.4)	(3.7)	(4.8)	(100)	(41.4)
40th percentile	4	4.2	3.5	(100)	(9.9)
Most Likely Return (Median)	9.4	9.4	8.9	(100)	10.4
60th percentile	15.1	14.8	14.6	(100)	31.9
Upper Quartile	25.2	24.4	24.6	(100)	69.7
Best Case (95th percentile)	52	49.9	51.3	160.2	170.5
Probability of Total Loss	<1%	zero	zero	94%	6%
Probability of a negative return	33%	32%	33%	94%	45%

Portfolio Analysis ($X = 0.900$):

Portfolio:	**100% Equity**	B	**Protected Put**	**100% Put**	**100% Call=E**
Cash	0	15	0	0	0
Equities	100	85	98,3	0	0
Put	0	0	1.7	100	0
Call	0	0	0	0	100
Total	100%	100%	100%	100%	100%
Absolute Worst Case Return (% pa)	(100)	(83.7)	(11.5)	(100)	(100)
Worst Case (5th percentile)	(21.3)	(16.8)	(11.5)	(100)	(100)
Lower Quartile	(4.4)	(2.5)	(6)	(100)	(69.9)
40th percentile	4	4.6	2.2	(100)	(25)
Most Likely Return (Median)	9.4	9.3	7.5	(100)	(4.1)
60th percentile	15.1	14.1	13.2	(100)	34.6
Upper Quartile	25.2	22.7	23.1	(100)	88.7
Best Case (95th percentile)	52	45.5	49.4	547.9	232.5
Probability of Total Loss	<1%	zero	zero	84%	16%
Probability of a negative return	33%	30%	36%	86%	49%

Portfolio Analysis ($X = 1.100$):

Portfolio:	**100% Equity**	B	**Protected Put**	**100% Put**	**100% Call=E**
Cash	0	50.4	0	0	0
Equities	100	49.6	91.2	0	0
Put	0	0	8.8	100	0
Call	0	0	0	0	100
Total	100%	100%	100%	100%	100%
Absolute Worst Case Return (% pa)	(100)	(45.4)	1.1	(100)	(100)
Worst Case (5th percentile)	(21.3)	(6.3)	1.1	(100)	(100)
Lower Quartile	(4.4)	2	1.1	(100)	(100)
40th percentile	4	6.2	1.1	(100)	(100)
Most Likely Return (Median)	9.4	8.9	1.1	(93.3)	(100)
60th percentile	15.1	11.7	5.8	(32)	(29.7)
Upper Quartile	25.2	16.7	15.1	63	108.9
Best Case (95th percentile)	52	30	39.7	254.3	477.6
Probability of Total Loss	<1%	zero	zero	49%	51%
Probability of a negative return	33%	19%	zero	65%	64%

Portfolio Analysis ($X = 1.200$):

Portfolio:	**100% Equity**	B	**Protected Put**	**100% Put**	**100% Call=E**
Cash	0	67.9	0	0	0
Equities	100	32.1	85.1	0	0
Put	0	0	14.9	100	0
Call	0	0	0	0	100
Total	100%	100%	100%	100%	100%
Absolute Worst Case Return (% pa)	(100)	(26.4)	4.5	(100)	(100)
Worst Case (5th percentile)	(21.3)	-1.2	+4.5	(100)	(100)
Lower Quartile	(4.4)	4.3	4.5	(100)	(100)
40th percentile	4	6.9	4.5	(67.1)	(100)
Most Likely Return (Median)	9.4	8.8	4.5	(28.7)	(100)
60th percentile	15.1	10.5	4.5	7.7	(100)
Upper Quartile	25.2	13.7	9	64.2	27.7
Best Case (95th percentile)	52	22.4	32.4	177.8	685.8
Probability of Total Loss	<1%	zero	zero	32%	68%
Probability of a negative return	33%	8%	zero	58%	74%

5. Investors Preferences: Utility Functions

The mention of investors' risk return preferences in Section 4 leads naturally on to the use of utility functions. According to utility theory, investors solve the portfolio selection problem simply by choosing the (probably but not necessarily unique) portfolio that maximises their expected utility. The remarkable fact about utility functions is that the first moment of the utility function encapsulates the totality of the investor's views about risk and return and one therefore does not need to consider higher moments.

Of course, this is a theoretical ideal, because human beings probably do not behave logically or consistently enough to have a consistent utility function, even if they did, it may not fall within the types described below, and finally it does not appear to be possible to combine **individual members' utility functions** (in the case of a group decision making body such as the trustees of a UK pension fund) to form a **group utility function.**

Nevertheless, this paper provides a framework for calculating the expected utility of a portfolio containing options, and hence to rank such portfolios against more traditional ones in the scale of investors' preferences.

For risk averse investors, utility functions are usually convex, and include the following types:

- exponential utility functions (for which I shall show some results below)
- quadratic utility functions (which I discussed briefly at the start of Section 4 and which are unrealistic in my view)
- power utility functions (of the form $u(x) = x^a$ where $0 < a < 1$)
- they may even be discontinuous (Clarkson in the UK has referred to "kinked utility functions")

Examples: Exponential Utility Functions

If an investor has a utility function of the form $u(x) = -\exp(-ax)$, where $a > 0$, then expected utility $Eu(X) = E(-\exp(-aX))$ uses the moment generating function of the distribution of X. (X is a random variable representing the amount of wealth at the end of the period arising from an investment of 1 at the beginning of the period in the investment policy under consideration).

In fact, providing that the distribution of X is sufficiently stable to make this expansion converge,

$$Eu(X) = -1 + aEX - a^2EX^2/2! + a^3EX^3/3! + \ldots \text{ etc}$$

and thus, for a portfolio containing options, uses the moments of such portfolios developed in Section 1.

For numerical examples, it would have been possible (but unfortunately quite tedious due to the binomial expansion of C_n and P_n terms involved!) to use a spreadsheet or mainframe computer program to calculate the above expansion to thirty of forty terms say, and check whether convergence was occurring and thus calculate exact theoretical values of $Eu(X)$.

Thankfully, it is also possible to obtain these values to a high degree of accuracy by simulation. (Indeed, this may be the only practical method in the case of some of the other types of utility functions mentioned above.)

Numerical Examples Using Exponential Utility Functions

In the examples which follow, I show the results of 10,000 random simulations for a 1 year time horizon (using the parameters for cash and equities under the geometric Brownian motion model as set out in Sections 1 and 3), which should provide a sufficiently large sample to give accurate estimates of expected utility.

An interesting first question is:

- what value of the parameter **a** is consistent with an investor being sufficiently risk averse to prefer a 70% equity/30% cash policy (which I shall take as a slightly simplistic representation of the sort of investment policy favoured by UK pension fund trustees) to a 100% equity policy, but not to the extent that a 100% cash policy is preferred to a 70/30 policy?

(To be precise, one should really be answering the question as to what value of **a** is such as to lead to the 70/30 policy as having maximum expected utility amongst all portfolios containing just equities and cash, but the above will suffice for illustration purposes). The values of expected utility for each of the three policies are given below under different values of **a**:

Value of parameter a	Eu(X) for 100% Cash	Eu(X) for 100% equity	Eu(X) for 70/30 policy
1	(0.3385)	(0.3355)	(0.3347)
2	(0.114569)	(0.1177)	(0.114612)
3	(0.03878)	(0.04297)	(0.04007)
5	(0.004443)	(0.006349)	(0.005182)

It can be seen from the above (bearing in mind that the exponential utility function produces negative values, so policies are preferred the closer their expected utility gets to zero) that the higher the value of **a**, the more risk averse the investor is. Certainly for values of **a** greater or equal to 2, the investor would prefer 100% cash to either of the two alternatives. The table shows that **a** = 1 is a suitable value for the type of investor who prefers the 70/30 policy to either 100% cash or 100% equities.

A natural question that then arises is:

- would such an investor (with **a** = 1) find any protected put portfolio (with a suitable exercise price X) preferable to the 70/30 portfolio? ie can one find an exercise price X such that the expected utility of the protected put portfolio is higher than -0.3347?

The expected utilities (with **a** = 1) are shown on the next page for various values of the exercise price X:

Exercise Price of Put Option (Share Price = 1)	Expected Utility of Protected Put Portfolio
0.8	(0.3355)
0.9	(0.3354)
0.952	(0.3355)
1	(0.3356)
1.05	(0.3357)
1.1	(0.3359)
1.2	(0.3365)
1.5	(0.3381)

I must admit to finding the above results disappointing: my intuitive feeling is that a risk averse investor would find a protected put portfolio (with exercise price in the range 0.95 to 1.05) more appealing, on the basis of the percentile return figures given in Section 4, than a 70/30 equity/ cash portfolio. However, the above results do not lend support to this view if investors have exponential utility functions.

Further research needs to be carried out to investigate:

- whether the results change with a more accurate value for $\mathbf{a}$ than $\mathbf{a} = 1$
- what the results would be under different utility functions or a different investment model (perhaps the Wilkie model or else a geometric Brownian model extended to incorporate Poisson jumps)
- what the results would be for a ten year time horizon (although the only long term options likely to be available for the foreseeable future will be over the counter options).

In addition, because of the theoretical limitations of utility functions, perhaps we also need simply to ask investors (such as those in insurance companies responsible for investment or pension fund trustees) what their preferences are, given the sort of percentile information given in Section 4.

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