A SOLVENCY STUDY IN LIFE INSURANCE

RAGNAR NORBERG

ABSTRACT

General expressions are given for first and second moments of present values of stochastic payment streams evaluated by a stochastic discounting function. The results are applied to an authentic portfolio of term life insurance policies assuming mortality of Gompertz-Makeham type and accumulation of interest governed by an Ornstein-Uhlenbeck process. A proposed solvency criterion specifies that the insurer must provide a reserve equal to the mean plus a multiple of the standard deviation.

1. SOLVENCY CONTROL ON BREAK-UP BASIS

A. The break-up scenario.

We consider a life insurance business which is subject to solvency control on a so-called break-up basis, by which only those policies currently in force are considered relevant and the outstanding net liability (benefits less premiums) in respect of those are the object of the control. The solvency requirement specifies that the current reserve must cover, with high probability, the present value of the net liability.

B. Special issues in life insurance.

In assessing the financial strength of an insurance business special problems arise from the uncertainty associated with the future payments. In the life insurance context, where contract terms may extend over several decades, uncertainty emerges at two levels. In the first place there is an element of pure randomness in the development of the individual life histories and, consequently, in the events upon which payments are contingent. In the second place, there is uncertainty associated with the future development of the very laws that govern the individual life histories (intensities of mortality, sickness, recovery, etc.) and also with the future development of the economic environment (administration costs and yields on investments). In short, there is a superimposed risk due to possible adverse development of the technical basis.

2. General description of the solvency assessment

A. Basic model entities – payments and interest.

We adopt the set-up of Norberg (1990). A stream of payments is defined by a finite-valued *payment function* A, which for each time t specifies the total amount $A(t)$ paid in $(-\infty, t]$. Negative payments are allowed for; it is only required that A be of bounded variation in finite intervals and, by convention, right-continuous. This means that $A = B - C$, where B and C are non-negative, non-decreasing, finite-valued, and right-continuous functions representing the outgoes and incomes, respectively, of some business.

In the present context A represents the total payments in respect of a life insurance business; B represents benefits (including administration expenses) and C represents contributed premiums. The total amounts paid are made up by the payments on the individual policies in the portfolio,

$$A = \sum_i A_i \,, \tag{2.1}$$

A_i being payments in respect of policy No. i.

The amount paid in a small interval of extension dt around t is denoted $A(dt)$. We shall be concerned with first and second order moments of discounted values and shall need expressions for $E\,A(dt)$ and $E\,\{A(ds)A(dt)\}$. In the present study we assume that the A_i are stochastically independent. General formulas for $E\,A_i(dt)$ and $E\,\{A_i(ds)A_i(dt)\}$ are given in Norberg (1991) for the case where each A_i is generated by a continuous time Markov process. Formulas for the present context of term insurance are simple and will be displayed in Section 4.

The surplus created by the payments is currently invested and yields interest with spot rate (intensity) $\delta(t)$ at any time t. The corresponding discount function is

$$v(t) = e^{-\Delta(t)} \,, \tag{2.2}$$

where

$$\Delta(t) = \int_0^t \delta(\tau)\, d\tau \tag{2.3}$$

is the accumulated intensity (interpreted as $\int_t^0 \delta(\tau)$ if $t < 0$).

To account for financial risk, Δ is assumed to be a stochastic process. We shall need the first and second moments of the discount function and introduce

$$\phi_1(t) = E\,v(\tau) = E\,e^{-\Delta(t)}\,, \tag{2.4}$$

$$\phi_2(t) = E\,v(t)^2 = E\,e^{-2\Delta(t)}\,, \tag{2.5}$$

$$\phi_2(s,t) = E\,\{v(s)v(t)\} = E\,e^{-\Delta(s)-\Delta(t)}\,, \tag{2.6}$$

thus abbreviating $\phi_2(t) = \phi_2(t,t)$.

It is assumed throughout that the processes A_i and Δ are stochastically independent.

B. Outstanding liabilities – reserves.

For notational convenience, let time be counted from the moment of consideration, that is, the present solvency assessment takes place at time 0. The present value of the future net liabilities is

$$V = \int_0^\infty v(\tau)\,A(d\tau)\,, \tag{2.7}$$

the sum of all payments in small intervals discounted at time 0. (The integral in [2.7] ranges over $(0,\infty)$).

In the present treatment we assume that the discount factor process v is stochastically independent of the process A_i. Then the expected value of V is

$$\begin{aligned} E\,V &= E\int v(\tau)A(d\tau) \\ &= \int \phi_1(\tau)EA(d\tau)\,, \\ &= E_{\phi_1}V\,, \end{aligned} \tag{2.8}$$

where $E_{\phi_1}V$ is the expected value by non-stochastic discount function equal to the expected, ϕ_1.

The variance of V is

$$\operatorname{Var} V = E\left(\int v(\tau)A(dt)\right)^2 - (EV)^2$$

$$= E\left(\int\int v(\vartheta)v(\tau)A(d\vartheta)A(d\tau)\right) - (EV)^2$$

[2.9] $$= \int\int E\{v(\vartheta)v(\tau)\}E\{A(d\vartheta)A(d\tau)\} - (EV)^2$$

$$= 2\int\int_{\vartheta<\tau} \phi_2(\vartheta,\tau)E\{A(d\vartheta)A(d\tau)\}$$

[2.10] $$+ \int \phi_2(\tau)E(A(d\tau))^2 - (EV)^2 .$$

Upon inserting $E\{v(\vartheta)v(\tau)\} = \operatorname{Cov}\{v(\vartheta), v(\tau)\} + \phi_1(\vartheta)\phi_1(\tau)$ in [2.9] and recalling [2.8], it is seen that

[2.11] $$\operatorname{Var} V = \int\int \operatorname{Cov}\{v(\vartheta), v(\tau)\}E\{A(d\vartheta)A(d\tau)\} + \operatorname{Var}_{\phi_1} V ,$$

where $\operatorname{Var}_{\phi_1} V$ is the variance based on non-stochastic discounting by ϕ_1.

C. Required reserve.

By tradition, the reserve provided in life insurance is the expected value of the net liabilities. To meet possible claims in excess of the expected, an additional fluctuation reserve should be provided. A simple rule is to require provision of the reserve

[2.12] $$R = EV + 2\sqrt{\operatorname{Var} V},$$

which, by Tchebycheff's inequality, is sufficient to cover the discounted future liabilities with a probability no less than 0.75. If the distribution of V is symmetric, the probability is at least 0.875, and if the distribution is approximately normal, the probability is about 0.975.

3. THE INTEREST PROCESS

A. A stochastic interest model.

The uncertainty associated with the future yields on investments is a major issue in the solvency assessment. This uncertainty can be accommodated in the analysis by modelling the discount function as a stochastic process. We shall assume that the spot rate δ is an Ornstein-Uhlenbeck given by the stochastic differential equation

$$d\delta(t) = \kappa(\delta_0 - \delta(t))dt + \sqrt{\lambda}dW(t)\,, \tag{3.1}$$

where δ_0 and $\kappa > 0$ are constants and W is a standard Brownian motion. The first term on the right serves to always pull the process to the mean level δ_0, and the second term represents pure noise. The process is Markov with continuous (but non-differentiable) paths and fluctuates with a certain inertness around its typical value δ_0. It can be shown by discrete approximation that the cumulative spot rate process in this model is Gaussian and, more specifically, that

$$\Delta(t)|_{\delta(0)} \sim N\left(\mu_t, \sigma_t^2\right)\,, \tag{3.2}$$

with

$$\mu_t = t\delta_0 + \frac{1}{\kappa}(1 - e^{-\kappa t})(\delta(0) - \delta_0)\,, \tag{3.3}$$

$$\begin{aligned}\sigma_t^2 &= \frac{\lambda}{\kappa^3}\left(t\kappa - 2(1 - e^{-\kappa t}) + \frac{1}{2}(1 - e^{-2\kappa t})\right) \\ &= \frac{\lambda}{\kappa^3}\left(t\kappa + \frac{1}{2}\left(1 - (2 - e^{-\kappa t})^2\right)\right)\,.\end{aligned} \tag{3.4}$$

It can also be shown that

$$\Delta(s) + \Delta(t)|_{\delta(0)} \sim N(\mu_{s,t}, \sigma_{s,t}^2)\,, \tag{3.5}$$

where

$$\mu_{s,t} = (s + t)\delta_0 + \frac{1}{\kappa}\left(2 - e^{-\kappa s} - e^{-\kappa t}\right)(\delta(0) - \delta_0)\,, \tag{3.6}$$

and for $0 < s \leq t$, putting $r = t - s$,

$$\begin{aligned}\sigma_{s,t}^2 = \frac{\lambda}{\kappa^3}\Big[&(r + 4s)\kappa + \frac{1}{2}(1 + e^{-\kappa r})^2(1 - e^{-2\kappa s}) + \\ &-4(1 + e^{-\kappa r})(1 - e^{-\kappa s}) + \frac{1}{2}\left(1 - (2 - e^{-\kappa r})^2\right)\Big]\,.\end{aligned} \tag{3.7}$$

4. Term insurance – a worked example

A. The term insurance contract.

As a simple example, we shall carry out the solvency analysis for a portfolio of standard term insurance policies. The contract specifies that the sum insured falls due immediately upon death within the term of the contract and that premiums fall due continuously at level intensity as long as the contract is in force.

For an individual policy we denote, dropping the subscript i for the time being,

x, the age at entry,
t', the date (calendar time) of entry,
n, the term of contract (maximal duration of coverage),
$t'' = t' + n$, date of expiry of the contract,
b, the sum insured,
c^*, the premium per year (an intensity).

Assume that premiums are calculated on a first order technical basis with the following elements:

μ_x^*, the mortality intensity at age x, with corresponding survival function ${}_xp_0^*$,
δ^*, the interest intensity, with corresponding discount function $v^*(t) = e^{-\delta^* t}$,
α^*, commission and other costs per unit sum insured, incurring upon issue of the policy,
β^*, premium encashment costs per unit premium paid,
γ^*, general administration costs per unit sum insured, incurring per time unit as long as the policy is in force,

all independent of calendar time.

First, the following basic quantities are calculated by a numerical integration procedure (Laplace or Simpson) and tabulated:

$$g_1^*(x) = {}_x p_0^* = e^{-\int_0^x \mu_\tau^* d\tau} ,$$
$$g_2^*(x) = {}_x p_0^* \mu_x^* ,$$
$$g_3^*(x) = v^*(x)\, {}_x p_0^* ,$$
$$g_4^*(x) = \int_0^x g_3^*(y) dy .$$

The premium c^* is determined by the principle of equivalence,

$$c^* \bar{a}_{x,n} = b\alpha^* + b\bar{A}^1_{x,n} + \beta^* c^* \bar{a}_{x,n} + \gamma^* b\bar{a}_{x,n}$$

where

$$\bar{a}_{x,n} = \frac{g_4^*(x+n) - g_4^*(x)}{g_3^*(x)},$$

$$\bar{A}^1_{x,n} = 1 - \delta^* \frac{g_4^*(x+n) - g_4^*(x)}{g_3^*(x)} - \frac{g_3^*(x+n)}{g_3^*(x)}.$$

We find

$$c^* = \frac{b}{1-\beta^*} \frac{g_3^*(x)(\alpha^*+1) - g_3^*(x+n)}{g_4^*(x+n) - g_4^*(x)} - (\delta^* - \gamma^*).$$

B. The first and second order moments of the total liability.

Considering now the entire portfolio of policies currently in force at time 0, we reintroduce the subscript for individual policies. The elements of a realistic second order technical basis are denoted by dropping the asterisk $*$ from the corresponding symbols of the first order basis. Future payments in respect of policy No. i are given by

[4.1] $$A_i(dt) = (b_i N_i(dt) - c_i I_i(t)dt)\, 1_{(0,t_i'')}(t),$$

where $I_i(t)$ is 1 or 0 according as the insured is alive or dead at time t, $N_i(t) = 1 - I_i(t)$ is the number of deaths (0 or 1) of the insured by time t, and

[4.2] $$c_i = (1-\beta)c_i^* - b_i\gamma.$$

Denote by $p_i(t)$ and $\mu_i(t)$, respectively, the (conditional) probability of survival to t and the mortality intensity at time t for the insured, $t \in (0, t_i'')$. It is convenient to define them as 0 for $t \geq t_i''$ (confer [4.1]), hence

[4.3] $$p_i(t) = \frac{g_1(x_i + t - t_i')}{g_1(x_i - t_i')} 1_{(t_i', t_i'')}(t),$$

[4.4] $$\mu_i(t) = \mu(x_i + t - t_i') 1_{(t_i', t_i'')}(t).$$

The basic quantities involved in [2.8] – [2.10]are the moments ϕ_1 and ϕ_2 and the following moments of the increments $A(d\tau)$. First, for $\tau > 0$,

[4.5] $$E\,A(d\tau) = Q(\tau)d\tau,$$

with

$$Q(\tau) = \sum_i Q_i(\tau), \quad Q_i(\tau) = (b_i\mu_i(\tau) - c_i)p_i(\tau), \tag{4.6}$$

and

$$E\,(A(d\tau))^2 = S(\tau)d\tau, \tag{4.7}$$

with

$$S(\tau) = \sum_i S_i(\tau), \qquad S_i(\tau) = b_i^2 p_i(\tau)\mu_i(\tau). \tag{4.8}$$

Next, for $0 < \vartheta < \tau$,

$$\begin{aligned} E\{A(d\vartheta)A(d\tau)\} &= E\,A(d\vartheta)E\,A(d\tau) + \mathrm{Cov}\,\{A(d\vartheta),\, A(d\tau)\} \\ &= Q(\vartheta)Q(\tau)d\vartheta d\tau + \sum_i \mathrm{Cov}\,\{A_i(d\vartheta),\, A_i(d\tau)\} \\ &= Q(\vartheta)Q(\tau)d\vartheta d\tau - \sum_i (Q_i(\vartheta) + c_i)Q_i(\tau)d\vartheta d\tau, \end{aligned} \tag{4.9}$$

having used

$$\begin{aligned} \mathrm{Cov}\,\{A_i(d\vartheta),\, A_i(d\tau)\} &= E\{(b_iN_i(d\vartheta) - c_iI_i(\vartheta)d\vartheta)(b_iN_i(d\tau) - c_iI_i(\tau)d\tau)\} \\ &\quad - Q_i(\vartheta)Q_i(\tau)d\vartheta d\tau \\ &= E\{-c_i d\vartheta(b_iN_i(d\tau) - c_iI_i(\tau)d\tau)\} + \\ &\quad - Q_i(\vartheta)Q_i(\tau)d\vartheta d\tau \\ &= -c_iQ_i(\tau)d\vartheta d\tau - Q_i(\vartheta)Q_i(\tau)d\vartheta d\tau. \end{aligned}$$

On combining these results with [2.8]- [2.10], we gather the following formulas, which form a basis for numerical algorithms:

$$E\,V = \int_0^\infty \phi_1(\tau)Q(\tau)d\tau, \tag{4.10}$$

$$\begin{aligned} \mathrm{Var}\;V &= 2\int_0^\infty R(\tau)Q(\tau)d\tau - 2\int_0^\infty \sum_i T_i(\tau)Q_i(\tau)d\tau \\ &\quad + \int_0^\infty \phi_2(\tau)S(\tau)d\tau\; - (E\,V)^2, \end{aligned} \tag{4.11}$$

where

$$[4.12] \qquad R(\tau) = \int_0^T \phi_2(\vartheta, \tau) Q(\vartheta) d\vartheta \,,$$

$$[4.13] \qquad T_i(\tau) = \int_0^T \phi_2(\vartheta, \tau)(Q_i(\vartheta) + c_i) d\vartheta \,.$$

C. Numerical results for an authentic portfolio.

Data from a Norwegian portfolio of 7073 term insurance policies has been put at our disposal. For a typical policy the age of the insured is between 30 and 50, the sum insured is between NOK 100.000 and 1.000.000 (NOK 1 = \$ 0.17), and the remaining contract period is between 0 and 10 years.

In our example the first and second order technical bases are assumed to be the same and equal to the standard Norwegian first order technical basis, by which $\mu^*(x) = 0.0009 + 0.000044 \cdot 10^{0.042x}$, $\delta^* = \ln(1.04)$, $\alpha^* = 0.03$, $\beta^* = 0.05$, and $\gamma^* = 0.00275$. The results turned out as follows: $EV =$ -59.718.706 (negative due to the substantial expenses incurring upon issue of policies), $\sqrt{\text{Var } V} =$ 8.375.241, the reserve for the entire portfolio by formula [2.12] is -42.968.225, and the fluctuation reserve per policy ($2\sqrt{\text{Var } V}$/(number of policies)) is 2.368.

The fluctuation reserve per policy will of course be roughly proportional to the inverse square root of the number of policies. One important conclusion that can be drawn from the present analysis is that it is of considerable size even for a fairly large business. This is due to the nature of the term insurance: deaths are rare events with serious economic consequences for the insurer.

The numerical results are not substantially affected by letting the interest rate be truly stochastic and generated by the Ornstein-Uhlenbeck process of Paragraph 3 as long as $\delta(0) = \delta_0 = \ln(1.04)$ and λ is given realistic values between 0 and 0.001: the reserve turns out to deviate from the value above by less than one per cent. This experience is presumably due to the overall short durations of the contracts in term insurance and may well come out differently for long term pension contracts.

5. Issues of further studies

The present paper is an excerpt of a preliminary report prepared for a working party commissioned by the Norwegian Insurance Association to propose a system for solvency control of life insurance companies. A comprehensive report is in preparation. It will include further investigation of appropriate models for stochastic interest, formulas for moments up to third order, computation of approximate upper percentiles based on the three first moments, and analyses extended to pension plans and more complex forms of insurance.

Bibliography

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