Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies*

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Abstract

The paper analyzes one of the most common life insurance products - the so-called *participating* (or *with profits*) policy. This type of contract stands in contrast to Unit-Linked (UL) products in that interest is credited to the policy periodically according to some mechanism which smoothes past returns on the life insurance company's (LIC) assets. As is the case for UL products, the participating policies are typically equipped with an interest rate guarantee and possibly also an option to surrender (sell-back) the policy to the LIC before maturity.

The paper shows that the typical participating policy can be decomposed into a risk free bond element, a bonus option, and a surrender option. A dynamic model is constructed in which these elements can be valued separately using contingent claims analysis. The impact of various bonus policies and various levels of the guaranteed interest rate is analyzed numerically. We find that values of participating policies are highly sensitive to the bonus policy, that surrender options can be quite valuable, and that LIC solvency can be quickly jeopardized if earning opportunities deteriorate in a situation where bonus reserves are low and promised returns are high.

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1 Introduction

Embedded options pervade the wide range of products offered by pension funds and life insurance companies. Interest rate guarantees, bonus distribution schemes, and surrender possibilities are common examples of implicit option elements in standard type policies issued in the United States, Europe, as well as in Japan. Such issued guarantees and written options are liabilities to the issuer. They represent a value and constitute a potential hazard to company solvency and these contract elements should therefore ideally be properly valued and reported separately on the liability side of the balance sheet. But historically this has not been done, to which there are a number of possible explanations. Firstly, it is likely that some companies have failed to realize that their policies in fact comprised multiple components, some of which were shorted options. Secondly, it seems fair to speculate that other companies have simply not cared. The options embedded in their policies may have appeared so *far out of the money*, in particular at the time of issuance, that company actuaries have considered the costs associated with proper assessment of their otherwise neglegible value to far outweigh any benefits. Thirdly, the lack of analytical tools for the valuation of these particular obligations may have played a part. Whatever the reason, we now know that the negligence turned out to be catastrophic for some companies, and as a result shareholders and policyowners have suffered. In the United States, a large number of companies have been unable to meet their obligations and have simply defaulted (see e.g. Briys and de Varenne (1997) and the references cited therein for details), whereas in e.g. the United Kingdom and Denmark, companies have started cutting their bonuses in order to ensure survival.

The main trigger for these unfortunate events is found on the other side of the balance sheet where life insurance companies have experienced significantly lower rates of return on their assets than in the 1970s and 1980s. The lower asset returns in combination with the reluctance of insurance and pension companies to adjust their interest rate guarantees on new policies according to prevailing market conditions have resulted in a dramatic narrowing of the safety margin between the companies' earning power and the level of the promised returns. Stated differently, the issued interest rate guarantees have moved from being far out of the money to being very much *in the money*, and many companies have experienced solvency problems as a result. The reality of this threat has most recently been illustrated in Japan where Nissan Mutual life insurance group collapsed as the company failed to meet interest rate guarantees of 4.7% p.a.[1] Nissan Mutual's uncovered liabilities were estimated to amount to $ 2.56 billion, so in this case policyholders' options indeed expired in the money without the company being able to fulfil its obligations.

Partly as a result of Nissan Mutual Life's collapse, Japanese life insurance companies have been ordered to reduce the interest rate guarantee from 4.5% to 2.5% p.a. In Europe, the EU authorities have also responded to the threat of insolvency from return guarantees. Specifically, Article 18 of the Third EU Life Insurance Directive, which was effective as of November 10, 1992, requires that interest rate guarantees do not exceed 60% of the rate of return on government debt (of unspecified maturity). In relation to this, Table 1 shows the prevalent maximum level of interest rate guarantees as of October 1998 for Japan and the EU member countries. In several of these countries, the maximum guaranteed interest rate has decreased during recent years and further cuts are likely to be seen.[2]

Table 1

Countries	Maximum rate
Japan	2.50%
Denmark and Italy	3.00%
Luxembourg	3.50%
France	3.75%
Austria, Germany, the Netherlands, Portugal, Spain, and Sweden	4.00%
Belgium	4.75%

As a consequence of the problems outlined above, insurance companies have experienced an increased focus on their risk management policy from regulatory authorities, academics, and the financial press. In particular the shortcomings of traditional deterministic actuarial pricing

[1] *The Financial Times*, June 2, 1997.

[2] From personal communication with members of the Insurance Committee of *Groupe Consultatif des Associations d'Actuaires des Pays des Communautés Européennes* (sic!).

principles when it comes to the valuation of option elements are surfacing. Recent years have also revealed an increasing interest in applying financial pricing techniques to the fair valuation of insurance liabilities, see for example Babbel and Merrill (1999), Boyle and Hardy (1997), and Vanderhoof and Altman (1998).[3]

In the literature dealing with the valuation of and to some extent also the reserving for insurance liabilities, several types of contracts and associated guarantees and option elements are recognized. Some of the contracts considered contain option elements of *European type*, meaning that the option(s) can be exercised only at maturity. This contrasts *American type* contracts where the embedded option(s) can be exercised at any time during the life of the contract. Another important distinction must be made between unit-linked contracts[4] and contracts where interest is credited according to some smoothing surplus distribution mechanism. The latter type is generally known as *participating* contracts and the interest rate crediting mechanism applied is often referred to as a *portfolio average method* or an *average interest principle*. Finally, in relation to guarantees it is important to distinguish between maturity guarantees and interest rate guarantees (rate of return guarantees). A maturity guarantee is a promise to repay at least some absolute amount at maturity (75% of the initial deposit, say) whereas an interest rate guarantee promises to credit the account balance with some minimum return every period.[5]

While participating policies are by far the most important in terms of market size, the larger part of the previous literature in this area has been analyzing unit-linked contracts with interest rate or maturity guarantees of the European type (Boyle and Schwartz (1977), Brennan and Schwartz (1976), Brennan and Schwartz (1979), Baccinello and Ortu (1993a), Nielsen and

[3]The concern about traditional deterministic actuarial pricing principles and in particular the *principle of equivalence* is not entirely new. In the United Kingdom, the valuation of *maturity guarantees* (as opposed to the *interest rate guarantees* studied in the present paper) was a concern twenty years ago when the Institute of Actuaries commissioned the Report of the Maturity Guarantees Working Party (1980), in which the valuation of maturity guarantees in life insurance was studied (see also the discussion in Boyle and Hardy (1997)). It was recognized that a guarantee has a cost and that explicit payment for these guarantees is necessary. For an interesting view and a discussion of actuarial vs. financial pricing, the reader is referred to Embrechts (1996).

[4]A policy is unit-linked (equity-linked) if the interest rate credited to the customer's account is linked directly and without lags to the return on some reference (equity) portfolio - the *unit*.

[5]Maturity guarantees and interest rate guarantees are obviously equivalent for single period contracts. However, in a multiple period setting where the interest rate guarantee is on the current account balance and interest is credited according to the principle that "what has once been given, can never be taken away" there is a significant difference between these two types of guarantees.

Sandmann (1995), and Boyle and Hardy (1997)). Some notable exceptions to this are the works by Brennan (1993), Briys and de Varenne (1997), Miltersen and Persson (1998), and Grosen and Jørgensen (1997). Inspired by classic UK with profits policies, Brennan (1993) discusses the efficiency costs of the reversionary bonus mechanism applied to these contracts. In their analysis of the valuation and duration of life insurance liabilities, Briys and de Varenne (1997) explicitly introduce a *participation level* in addition to the guaranteed interest rate attached to policies. However, the model is essentially a single period model where distinctions between interest rate and maturity guarantees and between guarantees of European and American type become less interesting. Miltersen and Persson (1998) present another interesting model in which contracts with an interest rate guarantee and a claim on excess returns can be valued. To model a kind of participation, the authors introduce a bonus account to which a part of the return on assets is distributed in 'good' years and from which funds can be withdrawn and used to fulfil the interest rate guarantee in poor years. The drawbacks of this model are that no averaging or smoothing is built into the distribution mechanism, and that the bonus account, if positive at maturity, is paid out in full to policyholders. This is a somewhat unrealistic assumption. Also the American type *surrender option* is not considered in this model. In Grosen and Jørgensen (1997), arbitrage-free prices of unit-linked contracts with an early exercisable (American) interest rate guarantee are obtained by the application of American option pricing theory. They also point out that the value of the option to exercise prematurely is precisely the value of the the surrender option implicit in many life insurance contracts. The numerical work in Grosen and Jørgensen (1997) demonstrates that this particular option element may have significant value and hence that it must not be overlooked when the risk characteristics of liabilities are analyzed and reserving decisions are made. However, the contracts considered in their paper are unit-linked and bonus mechanisms are not considered.

The present paper attempts to fill a gap in the existing literature by extending the analysis in Grosen and Jørgensen (1997) from unit-linked contracts to traditional participating policies, i.e. to contracts in which some surplus distribution mechanism is employed each period to credit interest at or above the guaranteed rate. The objective is thus to specify a model which encompasses the common characteristics of life insurance contracts discussed above and which can be used for valuation and risk analysis in relation to these particular liabilities. Our work towards this goal will meet a chain of distinct challenges: First, asset returns must be credibly modeled. In this respect we take a completely non-controversial approach and adopt the widely used framework of Black and Scholes (1973). Second, and more importantly, a realistic model for bonus distribution must be specified in a way that integrates the interest rate guarantee. This is where our main contributions lie. The third challenge is primarily of technical nature and concerns the arbitrage-free valuation of the highly path-dependent contract pay-offs resulting from applying the particular bonus distribution mechanism suggested to customer accounts. We will carefully take interest rate guarantees as well as possible surrender options into account and during the course of the analysis we also briefly touch upon the associated problem of reserving for the liabilities. Finally, we provide a variety of illustrative examples. The numerical section of the paper also contains some insights into the effective implementation of numerical algorithms for solving the model.

The paper is organized as follows. Section 2 describes the products which will be analyzed and presents the basic modelling framework. In particular, the bonus policy and the dynamics of assets and liabilities are discussed. In section 3 we present the methodology applied for contract valuation, we demonstrate how contract values can be conveniently decomposed into their basic elements, and computational aspects are addressed. Numerical results are presented in section 4, and section 5 concludes the paper.

2 The Model

In this section we provide a more detailed description of the life insurance contracts and pension plan products which we will analyze. Furthermore, we introduce the basic model to be used in the analysis and valuation of these contracts - especially the valuation of various embedded option elements.

The basic framework is as follows. Agents are assumed to operate in a continuous time frictionless economy with a perfect financial market, so that tax effects, transaction costs, divisibility, liquidity, and short-sales constraints and other imperfections can be ignored. As regards the specific contracts, we also ignore the effects of expense charges, lapses and mortality.[6]

At time zero (the beginning of year one) the *policyholder* makes a single-sum deposit, V_0, with the *insurance company*.[7] He thereby acquires a *policy* or a *contract* of nominal value P_0 which we will treat as a financial asset, or more precisely, as a contingent claim. In general, we will treat P_0 as being exogenous whereas the fair value of the contract, V_0, is to be determined. V_0 may be smaller or larger than P_0 depending on the contractual terms - particularly the various option elements. The policy matures after T years when the account is settled by a single payment from the insurance company to the policyholder. However, in some cases to be further discussed below, we will allow the policy to be terminated at the policyholder's discretion prior to time T.

At the inception of the contract, the insurance company invests the trusted funds in the financial market and commits to crediting interest on the policy's account balance according to some pay-out scheme linked to each year's market return until the contract expires. We will discuss the terms for the market investment and the exact nature of this *interest rate crediting*

[6] Since we ignore the insurance aspects mentioned and focus entirely on financial risks, the reader may also simply think of the products analyzed as a specific form of *guaranteed investment contracts* (GICs). In general, GICs in their various forms have been a significant investment vehicle for pension plans over the last 20 years. In the early 1990s, confidence in GICs was seriously shaken by the financial troubles of some insurance companies that were affected by defaults of junk bonds they purchased in the 1980s and poor mortgage loan results. GICs do not generally enjoy the status of 'insurance', and therefore they are not entitled to state guarantee fund coverage in the event of defaults, (Black and Skipper (1994), p. 814–815.)

[7] The extension to periodic premiums is straightforward, but omitted owing to space considerations.

mechanism in more detail shortly. For now we merely note that the interest rate credited to the policy in year t, i.e. from time $t-1$ to time t, is denoted $r_P(t)$ and is guaranteed never to fall below r_G, the constant, positive, and contractually specified guaranteed annual policy interest rate. Both rates are compounded annually.

The positive difference between the policy interest rate credited in year t and the guaranteed rate is denoted *the bonus interest rate*, $r_B(t)$, and we obviously have

$$r_B(t) = r_P(t) - r_G \geq 0, \quad \forall\, t. \tag{1}$$

The final interest rate to be introduced is the economy's (continuously compounded) riskless rate of interest. We denote it by r and assume that it is a constant.[8]

It is obvious that the interest rate crediting mechanism, i.e. the policy for the determination of each year's $r_P(\cdot)$ (or equivalently $r_B(\cdot)$), is of vital importance for the value of the policyholder's claim. We now turn to the discussion and our modelling of this key issue.

2.1 Bonus Policy and the Dynamics of Assets and Liabilities

Our modelling of the insurance company's bonus policy will use the following simplified time t balance sheet as its point of departure:

Figure 1

Assets	Liabilities
$A(t)$	$P(t)$
	$B(t)$
$A(t)$	$A(t)$

Some comments on Figure 1 are in order. Firstly, note that we use $A(t)$ to denote the time t market value of the assets backing the contract (the *asset base*). The liability side comprises

[8] It is possible and straightforward to include stochastic interest rates in our setting. However, in the interest of simplicity and since we are mainly concerned with studying other effects than those created by a stochastically changing term structure of interest rates, we refrain from making this extension.

two entries: $P(t)$ is the policyholder's account balance or, briefly, the *policy reserve*, whereas $B(t)$ is the ***bonus reserve***. Actuaries also often denote $B(t)$ simply as the ***buffer***. Although added up, these two entities equal the market value of the assets, individually they are not market values. The policy reserve, $P(t)$, is rather a book value, whereas $B(t)$ is a hybrid being residually determined as a difference between a market value and a book value. This construction is applied out of a wish to model actual insurance company behavior rather than in an attempt to describe *the ideal way*. Furthermore, we emphasize that since we have chosen to focus on individual policies (or, alternatively, a cohort of identical policies), Figure 1 is not the company balance sheet but rather a snap-shot of the asset and liability situation at a certain point in time and in relation to some specified policy (cohort of policies). Finally, we note that 'equity' is not missing from the liability side of Figure 1. Since in many cases the owner and the policyholders of the insurance company are the same, it is not essential to distinguish between bonus reserves, i.e. the amount allocated for future distribution, and equity. Hence, we have not included 'equity' as a separate entry on the liability side of the balance sheet.

2.1.1 The Asset Side of the Balance Sheet

The insurance company is assumed to keep the asset base invested in a well-diversified and well-specified *reference portfolio* at all times. Recall that we use $A(t)$ to denote the time t market value of this investment. No assumptions are made regarding the composition of this portfolio with respect to equities, bonds, real estate, or regarding the dynamics of its individual elements. We simply work on an aggregate level and assume that the total market value evolves according to a geometric Brownian motion,[9]

$$dA(t) = \mu A(t)\ dt + \sigma A(t)\ dW(t), \qquad A(0) = A_0. \tag{2}$$

[9]The assumption that assets evolve according to the GBM is not motivated by a wish to obtain *closed form* or analytic solutions to the problems studied. Indeed, due to the complexity of the contract pay-offs, we obtain no such solutions and we resort instead to extensive use of numerical methods. The numerical analysis could just as easily be carried out using another model for the evolution in asset values. Actuaries, for example, may prefer to apply the Wilkie model to describe the evolution of asset values (see e.g. Boyle and Hardy (1997)). Another interesting possibility is to let assets evolve according to a jump-diffusion model.

Here μ, σ, and A_0 are constants and $W(\cdot)$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ on the finite interval $[0, T]$.[10]

For the remaining part of the paper we shall work under the familiar equivalent risk neutral probability measure, Q, (see e.g. Harrison and Kreps (1979)) under which discounted prices are Q-martingales and where we have

$$dA(t) = rA(t)\ dt + \sigma A(t)\ dW^Q(t), \qquad A(0) = A_0, \tag{3}$$

and where $W^Q(\cdot)$ is a standard Brownian motion under Q. The stochastic differential equation in (3) has a well-known solution given by

$$A(t) = A_0 \cdot e^{(r - \frac{1}{2}\sigma^2)t + \sigma W^Q(t)}. \tag{4}$$

In particular, we will need to work with annual (log) returns which are easily established as conditionally normal:

$$\ln \frac{A(t)}{A(t-1)} |\mathcal{F}_{t-1} = r - \frac{1}{2}\sigma^2 + \sigma \left(W^Q(t) - W^Q(t-1)\right) |\mathcal{F}_{t-1} \sim \mathcal{N}\left(r - \frac{1}{2}\sigma^2, \sigma^2\right). \tag{5}$$

2.1.2 The Liability Side of the Balance Sheet

We now direct our attention towards describing the dynamics of the liability side of the balance sheet. Referring to Figure 1, we have previously introduced the two entries $P(t)$ and $B(t)$ as the policy reserve and the bonus reserve, or alternatively, as the policyholder's account balance and the buffer. Regardless of terminology, it is important not to confuse $P(t)$ with the concurrent fair value of the policy.

The distribution of funds to the two liability entries over time is determined by the bonus policy, i.e. by the sequence $r_P(t)$ for $t \in \Upsilon \equiv \{1, 2, \ldots, T\}$. In particular, note that from the

[10]The assumption that μ is constant is made for simplicity and is a bit stronger than necessary. As pointed out by Black and Scholes (1973), standard arbitrage valuation carries through without changes if we allow μ to depend on time and/or the current state.

earlier definition of the policy interest rate we must have

$$P(t) = (1 + r_P(t)) \cdot P(t-1), \quad t \in \Upsilon, \tag{6}$$

from which we deduce

$$P(t) = P_0 \prod_{i=1}^{t} (1 + r_P(i)), \quad t \in \Upsilon. \tag{7}$$

These are obviously two rather empty expressions until we know more about the determinants of the process followed by $r_P(\cdot)$, i.e. the interest rate crediting mechanism. Not until more has been said about this issue can we start to study the dynamics of the liability side of the balance sheet (in particular the policyholder's account) and the valuation of the policyholder's contract.

The question of how $r_P(\cdot)$ is determined in practice is highly subtle involving intangible political, legal, and strategic considerations within the insurance company. There is therefore little hope that we can ever construct simple models which can precisely capture all elements of this process. In particular, the theoretical possibility for the management of the insurance company to change the surplus distribution policy over time – perhaps within specific limits set by law – will be problematic for the task of valuing the policyholder's claim. Put differently, this means that whenever we apply and analyze a certain interest rate crediting mechanism in our models, we should keep in mind that we are not necessarily dealing with strictly (legally) binding agreements between the parties.

Having realized these problems – of which we are unaware of parallels in the standard financial (exotic) options literature – we will proceed with an attempt to specify an interest rate crediting mechanism which is as accurate and realistic an approximation to the true bonus policy as possible. We shall specify the interest rate crediting mechanism in the form of some mathematically well-defined function in order to be able to apply the powerful apparatus of financial mathematics and, in particular, arbitrage pricing.

While problems *do* exist as outlined above, there are fortunately also some well-established principles that can be used as inspiration. Firstly, it is clear that realized returns on the assets in a given year must influence the policy interest rate in the ensuing year(s). Secondly, since one of the main arguments in the marketing of these life insurance policies is that they provide a low-risk, stable, and yet competitive return compared with other marketed assets, our modelling of the interest rate crediting mechanism should also accomplish this. The surplus distribution rule should in other words resemble what is known in the industry as the average interest principle.[11] Thirdly, in order to provide these stable returns to policyholders[12] and to partially protect themselves against insolvency, the life insurance companies aim at building and maintaining a certain level of reserves (the buffer). This could and should also be accounted for in constructing the distribution rule.

In sum, an attempt to model actual interest rate crediting behavior of life insurance companies should involve the specification of a functional relationship from the asset base, $A(\cdot)$, and the bonus reserve, $B(\cdot)$, to the policy interest rate, $r_P(t)$. In this way, the annual interest bonus can be based on the actual investment performance as well as on the current financial position (the degree of solvency) of the life insurance company. Furthermore, the functional relationship should be constructed in such a way that policy interest rates are in effect low-volatility, smoothed market returns.

We thus propose to proceed in the following way. It is first assumed that the insurance company's management has specified a constant target for the ratio of bonus reserves to policy reserves, i.e. the ratio $\frac{B(t)}{P(t)}$. We call this the *target buffer ratio* and denote it by γ. A realistic value would be in the order of 10–15%. Suppose further that it is the insurance company's objective to distribute to policyholders' accounts a positive fraction, α, of any excessive bonus reserve in each period. The company must, however, always credit at least the guaranteed rate, r_G, which it will do if the bonus reserve is insufficient or even negative. Clearly for appropriate values of α this mechanism will ensure a stable smoothing of the surplus. The

[11] The average interest principle is discussed in e.g. Nielsen and Thaning (1996). These authors write

> "... the average interest principle used by most of the pension business, partly to assure the customer a positive yield each year and partly to minimize the companies' risk due to the interest guarantee given - means that the yield given to the insured is very different from the companies' yield on investments each year. The difference is regulated by the so-called dividend equalisation provisions."

[12] Insurance companies sometimes label their bonus reserves 'bonus smoothing reserves' indicating their role in sustaining the average interest principle.

coefficient α is referred to as the *distribution ratio*.[13] A realistic value of α is in the area of 20–30%. Before we proceed, let us briefly recapitulate the most frequently applied notation:

$$\begin{aligned}
P_0 &: \text{ policy account balance at time 0} \\
T &: \text{ maturity date of the contract} \\
r_P(t) &: \text{ policy interest rate in year } t \\
r_G &: \text{ guaranteed interest rate} \\
r_B(t) &: \text{ bonus interest rate in year } t \\
r &: \text{ riskless interest rate} \\
A(t) &: \text{ market value of insurance company's assets at time } t \\
P(t) &: \text{ policy reserve at time } t \\
B(t) &: \text{ bonus reserve at time } t \\
\gamma &: \text{ target buffer ratio} \\
\alpha &: \text{ distribution ratio} \\
\sigma &: \text{ asset volatility}
\end{aligned} \tag{8}$$

The discussion above can now be formalized by setting up the following analytical scheme for the interest rate credited to policyholders' accounts in year t,

$$r_P(t) = \max\left\{r_G,\ \alpha\left(\frac{B(t-1)}{P(t-1)} - \gamma\right)\right\}. \tag{9}$$

This implies a bonus interest rate as stated below,

$$r_B(t) = \max\left\{0,\ \alpha\left(\frac{B(t-1)}{P(t-1)} - \gamma\right) - r_G\right\}. \tag{10}$$

Note first that as in real life contracts, the interest rate credited between time $t-1$ and time t is determined at time $t-1$, i.e. there is a degree of predictability in the interest rate crediting mechanism.[14] Observe also that when $r_G < \alpha\left(\frac{B(t-1)}{P(t-1)} - \gamma\right)$ (reserves are 'high') we can write

$$\begin{aligned}
P(t) &= P(t-1)\left(1 + \alpha\left(\frac{B(t-1)}{P(t-1)} - \gamma\right)\right) \\
&= P(t-1) + \alpha\Big(B(t-1) - \gamma P(t-1)\Big) \\
&= P(t-1) + \alpha\Big(B(t-1) - B^*(t-1)\Big),
\end{aligned} \tag{11}$$

[13] Briys and de Varenne (1997) work with a similar parameter called the *participation coefficient*.

[14] In Denmark, for example, all major insurance companies announce their policy interest rate for the coming year in mid-december the year before.

where $B^*(t-1) \equiv \gamma P(t-1)$ denotes the optimal reserve at time $t-1$. Hence, when reserves are satisfactorily large, a fixed fraction of the excessive reserve is distributed to policyholders.[15]

Returning to the general evolution of the policyholder's account, we have the relation

$$\begin{aligned} P(t) &= P(t-1)\left(1+\max\left\{r_G, \alpha\Big(\frac{B(t-1)}{P(t-1)}-\gamma\Big)\right\}\right) \\ &= P(t-1)\left(1+r_G+\max\left\{0, \alpha\Big(\frac{A(t-1)-P(t-1)}{P(t-1)}-\gamma\Big)-r_G\right\}\right). \end{aligned} \tag{12}$$

This difference equation clarifies a couple of points. Firstly, it is seen that the interest rate guarantee implies a floor under the final payout from the contract. The floor is given as $P_{min}(T) = (1+r_G)^T \cdot P(0)$ and it becomes effective in case bonus is never distributed to the policy. But note also that as soon as bonus has been distributed once, the floor is lifted since the guaranteed interest rate applies to the initial balance, accumulated interest, and bonus. We can therefore conclude that there is a risk free bond element to the contract implied by the interest rate guarantee. Secondly, expression (12) clarifies that the bonus mechanism is indeed an option element of the contract.

At this point it would have been convenient if from (12) we could establish the probabilistic distribution followed by $P(T)$. However, recursive substitution of the $P(\cdot)$s quickly gets complicated and $P(T)$ is obviously highly dependent on the path followed by $A(\cdot)$. The path dependence of the policyholder's account balance eliminates any hope of finding analytical expressions for the contract values. However, there is a wide range of numerical methods at our disposal to deal with this kind of problem. We will start looking at these in the next section. The present section is rounded off with some plots of simulation runs of the model to give the reader a feel for how things work.

Figure 2 plots the various interest rates over a 20-year period resulting from a typical simulation run of our model with a target buffer ratio (γ) of 10%, an initial buffer of 10%, a distribution rate (α) of 30%, and an asset volatility (σ) of 15%. The plot clearly illustrates

[15] Note that if one prefers to use continuously compounded rates throughout, one should use

$$r_P(t) = \max\left\{r_G, \ln\left(1+\alpha\Big(\frac{B(t-1)}{P(t-1)}-\gamma\Big)\right)\right\}$$

instead of (9) and $P(t) = P(t-1)\cdot e^{r_P(t)}$ instead of (6) and so on.

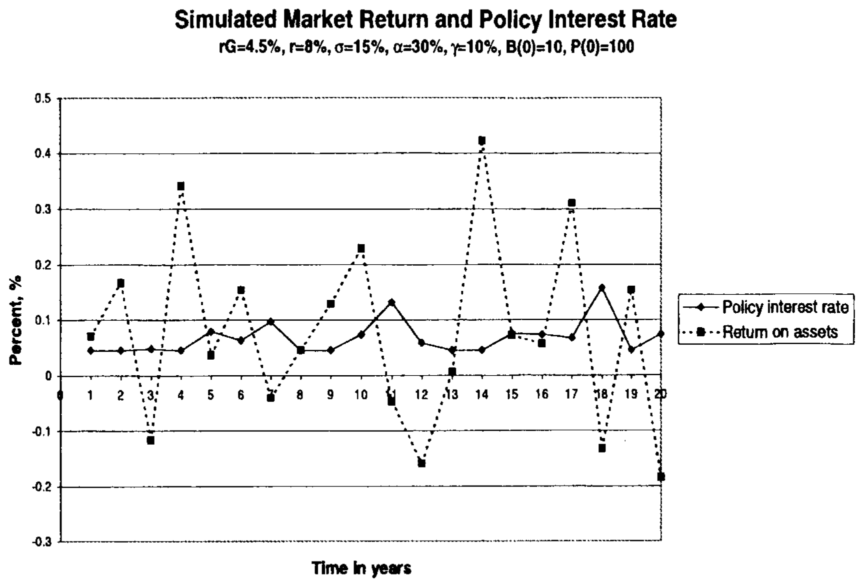

Figure 2: Simulated market return and policy interest rate

the smoothing that takes place in the policy interest rate. These rates vary much less than the market return on assets. Moreover, they are bounded below by the guaranteed rate of interest of 4.5% which comes into effect in certain years. Put differently, in some years there is no bonus whereas in other years the policy interest rate is quite generous compared with the risk free rate which is set at 8% in this example.

Figure 3 shows the evolution of various accounts resulting from the realisation of returns shown in Figure 2. The curve describing the market value of the asset base again clearly represents the more volatile evolution. Contrary to asset values, the policy reserve is strictly increasing due to the interest rate guarantee. It appears significantly less risky than the corresponding asset base, yet with a quite comparable return over the entire contract period. We have also plotted two exponential curves corresponding to the annual interest rate guarantee of 4.5%. The first of these depicts the guaranteed future account balance as seen from time 0 whereas the other curve illustrates the guaranteed future account balance as seen from $t = 15$ where previously distributed bonus has contributed to current account balance. Hence the floor provided by the interest rate guarantee at this point in time is 'lifted'. Finally, the dynamics

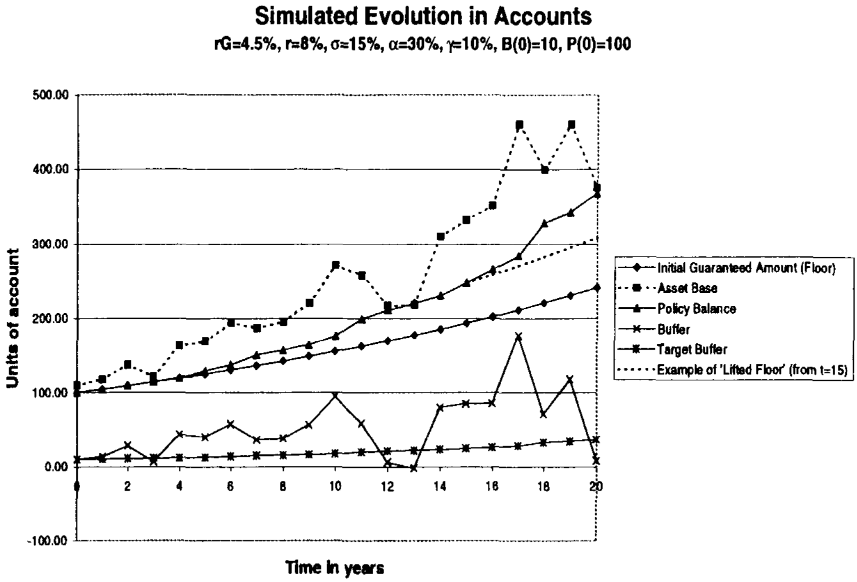

Figure 3: Simulated Evolution in accounts

of the bonus reserve and the target buffer are shown as the two lower curves in the graph. In the present example, $B(\cdot)$ starts out at 10 and continues to fluctuate around the target buffer, $\gamma P(\cdot)$.

Having demonstrated some essential characteristics of the contracts considered we now turn to their valuation.

3 Contract Types and Valuation

It is now time to turn to the valuation of specific contracts. In the present section we discuss two variations – the *European* and the *American* versions – of the contracts described in the previous section. The various numerical valuation techniques employed in the pricing of these contracts will be discussed as we proceed.

3.1 Contract Types

3.1.1 The Participating European Contract

The participating European type of contract is defined as the contract that pays simply $P(T)$ at the maturity date, T. Using the risk-neutral valuation technique (Harrison and Kreps (1979)), the value of this contract at time s, $s \in [0, T]$, can be represented as

$$V^E(s) = E^Q\left\{e^{-r(T-s)}P(T)|\mathcal{F}_s\right\}. \tag{13}$$

As is usual, the expectation in (13) is conditioned on the information set, $\mathcal{F}_s$, but we can substitute this with the triplet $(A(t-1), P(t-1), A(s))$, where $t-1 \leq s < t$ and $t \in \Upsilon = \{1, 2, \ldots, T\}$. In fact, since the pair $(A(t-1), P(t-1))$ uniquely determines $P(t)$, the expectation in (13) can be conditioned merely on $(A(s), P(t))$.[16] This is an important observation and the intuition behind it is the following. On one hand, in order to value the contract at an intermediate date, s, it is not sufficient to know the current value of the reference portfolio (the asset base), i.e. $A(s)$, due to the earlier mentioned path dependence. On the other hand, it is not necessary also to know the entire path followed by $A(\cdot)$ up to time s. However, we *do* need *some* knowledge about this path and the information summarized by $(A(t-1), P(t-1))$, or just $P(t)$, will be adequate. In other words, $P(t)$ contains sufficient information about the status of the reference portfolio at the end of previous years since this in turn has determined the policy interest rates and thus the present account balance.

From a computational point of view, the observation above implies that we have in fact a relatively tractable two-state-variable problem on our hands. This means that finite-difference

[16]Recall that $P(t)$ is $\mathcal{F}_s$-measurable by construction.

and lattice schemes can be relatively easily implemented for numerical valuation of the contracts. We postpone the discussion of these issues a little further and note first that for the European type of contracts, numerical evaluation via Monte Carlo simulation is also a possibility. In particular, the initial value of the participating European contracts is given as

$$V_0 = E^Q \left\{ e^{-rT} P(T) | A(0), P(0) \right\} \tag{14}$$

which will be the point of departure in our Monte Carlo evaluation of the participating European contracts.

3.1.2 The Participating American Type Contract

The participating American type of contract differs from the participating European contract only in that it can be terminated (exercised) at the policyholder's discretion at any time during the time interval $[0, T]$. Should the policyholder thus decide to exercise prematurely at time s, he will receive $P(s) \equiv P(t-1)$ where $t - 1 \leq s < t$ and $t \in \Upsilon$.[17] Hence, in addition to the the bonus option and the bond element implied by the interest rate guarantee, the policyholder has an American style option to sell back the policy to the issuing company anytime he likes. This feature is known in the insurance business as a surrender option. The participating American contract thus contains a double option element in addition to the risk free bond element implied by the interest rate guarantee. The total value of the contracts considered can therefore be decomposed as shown in Figure 4:

[17]A possible and reasonable modification of this definition of $P(s)$ would be to define $P(s) \equiv P(t-1) \cdot (1 + r_P(t))^{s-t+1}$ in order to allow for inter-period accumulation of interest to the account. However, in our numerical experiments we have only considered early exercise at dates in the set Υ in which case the extension is irrelevant.

Figure 4

<table>
<tr><td colspan="3">**Participating American Contract Value**</td></tr>
<tr><td>**Risk Free Bond**</td><td>**Bonus Option**</td><td>**Surrender Option**</td></tr>
<tr><td colspan="2">**Participating European Contract Value**</td><td></td></tr>
</table>

The valuation of the risk free part of the contracts is straightforward. At time zero, the value is given by $e^{-rT} \cdot P_0 \cdot (1 + r_G)^T$.

The entire participating American contract can be fairly valued according to the theory for the pricing of American style derivatives (see e.g. Karatzas (1988)). Denoting the fair value at time s by $V^A(s)$, $s \in [0, T]$, we have the following abstract representation of this value

$$V^A(s) = \sup_{\tau \in \mathcal{T}_{s,T}} E^Q \left\{ e^{-r(\tau - s)} P(\tau) | \mathcal{F}_s \right\}, \tag{15}$$

where $\mathcal{T}_{s,T}$ denotes the class of $\mathcal{F}_s$-stopping times taking values in $[s, T]$.[18]

Unfortunately, the introduction of a random expiration date complicates *practical* valuation considerably. Without any knowledge of the distribution of $P(\cdot)$, further manipulation of the probabilistic representation (15) is impossible and Monte Carlo evaluation - being a forward method - is now also problematic. However, as demonstrated in recent works (for example Tilley (1993) and Broadie and Glasserman (1997)) Monte Carlo evaluation *is* feasible for some American style derivatives (low dimension and few exercise dates) but will typically be computationally demanding and we will not follow that avenue. Instead we will handle the

[18]Surrender charges are common in real life contracts. We have not taken this into account.

path-dependence by using the two-state-variable observation made earlier and the associated numerical techniques as discussed next.

3.2 Computational Aspects

3.2.1 Monte Carlo Simulation

As mentioned in the previous section, the participating contracts of European type can be valued by standard Monte Carlo techniques. The way to proceed is to simulate a sequence of asset returns under the risk-neutral measure, Q, (see equation (5)) in order to get a large number (m) of paths for $P(\cdot)$ leading to final values of the policy reserve, $P^i(T)$, for each path indexed by i.

The actual steps involved in simulating a single path are the following:

1st step:	Determine	$r_P(1) = \max\{r_G,\ \alpha\left(\frac{B(0)}{P(0)} - \gamma\right)\}$
	Simulate	$A(1)$
	Calculate	$P(1) = (1 + r_P(1))P(0)$
		$B(1) = A(1) - P(1)$
	Determine	$r_P(2) = \max\{r_G,\ \alpha\left(\frac{B(1)}{P(1)} - \gamma\right)\}$
⋮		
j'th step:	Simulate	$A(j)$
	Calculate	$P(j) = (1 + r_P(j))P(j-1)$
		$B(j) = A(j) - P(j)$
	Determine	$r_P(j+1) = \max\{r_G,\ \alpha\left(\frac{B(j)}{P(j)} - \gamma\right)\}$
⋮		
T'th step:	(Simulate	$A(T)$)
	Calculate	$P(T) = (1 + r_P(T))P(T-1)$
		$(B(T) = A(T) - P(T))$

These steps are repeated m times and the Monte Carlo estimate of the initial contract value,

$\hat{V}^E(0)$, is found by averaging the $P^i(T)'s$ and discounting back to time zero, i.e.

$$\hat{V}^E(0) = \frac{e^{-rT}}{m} \sum_{i=1}^{m} P^i(T). \tag{16}$$

The standard error of the Monte Carlo estimate is obtained as an easy by-product of the simulation run and is given by

$$\sigma(\hat{V}_0^E) = \frac{\sigma\Big(e^{-rT} P^i(T)\Big)}{\sqrt{m}} \tag{17}$$

with obvious notation.

There are a number of methods for enhancing the precision of the Monte Carlo estimates and in our implementation we have employed one of the simplest - the antithetic variable technique, which is described in e.g. Hull (1997). The reader is also referred to the pioneering article by Boyle (1977), the recent survey by Boyle, Broadie, and Glasserman (1997), and the excellent book by Gentle (1998) for further details and more advanced techniques.

The simulation framework is also well-suited for the analysis of issues such as optimal reserving, the distribution of the future solvency degree, and default probabilities at the individual contract level. We take a brief look at some of these issues in section 4 where our numerical work is presented and discussed.

We finally report some technical details in relation to the simulation experiments. Firstly, in generating the necessary sequence of random numbers we employed the multiplicative congruential generator supplied with Borland's Delphi Pascal 4.0 (for details on random number generators, see e.g. Press et al. (1989) and Gentle (1998)). This particular random number generator has a *period* which exceeds 2^{31}. The sequence was initialized and shuffled (to eliminate sequential correlation) according to the Bays and Durham algorithm described in Press et al. (1989), p. 215–216. The uniform variates were transformed to independent normal variates using the Box-Müller transformation (see e.g. Quandt (1983)). All Monte Carlo results reported are based on the same 1,000,000 simulation runs, i.e. on the same sequence of generated random variates. This permits a more direct comparison between the various contracts without having to allow for random differences in simulated conditions. The values of the

European participating contracts reported in Tables 2, 3, and 4 below have been obtained in this way.

3.2.2 The Recursive Binomial Method

In order to be able to price the American participating contracts we have also implemented a binomial tree model á la Cox, Ross, and Rubinstein (1979). The European contracts can also be priced by using this methodology and excluding early exercise so this technique also conveniently provides us with a rough check on the values obtained via Monte Carlo simulation, cf. earlier.

Because of the dependence of the contract values on both $A(\cdot)$ and $P(\cdot)$ we are forced to implement a recursive scheme that keeps track of these variates at all knots in the tree starting from time zero. More specifically, we have used T time steps - typically 20. This allows for $T+1$ final values of $A(T)$, 2^T different paths, and similarly (up to) 2^T different terminal values of $P(T)$.

The prices of the American participating policies reported in Tables 2 and 3 have all been obtained using this methodology.

4 Numerical Results

In this section we present results from the numerical analysis of the model. We first report and discuss the valuation results in relation to the participating European as well as the American contracts. We then perform a numerical study of default probabilities on an individual contract level which is in line with traditional reserving analyses.

The results of the pricing analysis are contained in Tables 2–4 where we study contracts for which $P_0 = 100$, $B_0 = 0$, $T = 20$ years and $r_G = 4.5\%$. In Table 2, the volatility of the reference portfolio is set to 15%[19] whereas $\sigma = 30\%$ in Table 3. The riskless interest rate is chosen from the set $\{8.0\%, 6.0\%, 4.0\%\}$. The distribution rate, α, and the target buffer ratio,

[19]Pennacchi and Lewis (1994) estimate the average volatility of pension assets to about 15%. See also the discussion in Kalra and Jain (1997).

γ, are varied within each panel in the tables.

There are two values in each entry of Tables 2 and 3. The first is the value of the European contract obtained by Monte Carlo simulation as explained earlier. The value of the otherwise equivalent American-type policy is given in parentheses below the European price. These prices are obtained via the recursive binomial method discussed earlier.[20]

In the interpretation of the tables, recall that $\alpha = 0$ corresponds to the extreme situation where surplus is never distributed and the value of γ is redundant. The policyholder receives r_G per year for the entire period of the contract which has thus effectively degenerated into a 20 year zero-coupon bond. The present value of this bond is given as $e^{-rT} \cdot P_0 \cdot (1 + r_G)^T$ in the top row of each of the three panels in Tables 2 and 3. This value is 'clean' in the sense that it does not contain any contribution from bonus or surrender options, and it will obviously be below par when the market interest rate is high relative to the guaranteed interest rate and vice versa. Note that $\alpha = 0$ and European contract values below par imply that American contract values are precisely at par indicating that these contracts should be terminated immediately.

The above-mentioned bond element is present in all prices but when $\alpha > 0$ there will also be a contribution to the contract value from the bonus option. A large α obviously implies a more favorable bonus option, ceteris paribus, so contract values are rising in α. Conversely, an increase in the target buffer ratio, γ, means less favorable terms for the option elements, so contract values decrease as γ increases. Hence, bonus policies with relatively low αs and high γs can be classified as *conservative* whereas contracts with the opposite characteristics can be labelled *aggressive*. In this connection it should be recalled that $\gamma = 0.00$ corresponds to the situation where the life insurance company does not aim at building reserves.

Another conclusion is immediate from Tables 2 and 3. It can be seen that constructing par contracts ($V_0 = 100$) is a matter of extreme sensitivity to the distribution policy parameters γ and α. This aspect is further illustrated in Table 4, where the contract values from Tables 2 and 3 have been decomposed into their basic components, i.e. the risk free bond element, the

[20]In a few cases where early exercise is never optimal (bottom panel of Table 2) we obtained values of the American contracts which were slightly lower (less than 1%) than those of their European counterparts. This is due to discretization error in the binomial approach so in these cases we have relied upon and reported the values obtained by Monte Carlo simulation also for the American contracts.

bonus option and the surrender option.

In the tables we see that the American contracts are always more valuable than their European counterparts, the difference being the value of the surrender option. Also as expected, contract values increase as the bonus policy moves from conservative towards more aggressive. However, note that the value of the surrender option decreases with a more aggressive bonus policy. This is quite intuitive since a more aggressive bonus policy is purely to the advantage of the policyholder. Consequently, his incentive to exercise prematurely may be partly or fully removed in this way.

When the risk free interest rate drops towards and perhaps below the guaranteed interest rate, the contract value increases although the value of the option elements decreases. This is explained by the fact that the interest rate guarantee itself moves from being in a sense *out of the money* to being *at* or *in the money.* Being in effect a riskless bond, this contract element of course rises in value when the market interest rate drops. Note at the same time that the value of the surrender option decreases as staying in the company becomes more attractive the closer the riskless rate is to the guaranteed interest rate. In other words, the incentive to exercise prematurely will gradually disappear as the market interest rate drops.

The effect of the level of volatility of the underlying portfolio can be seen from comparing Tables 2 and 3. As is usual, option elements become more valuable with increasing uncertainty.

We do not report valuation results for non-zero initial values of the buffer. It is quite clear, however, that if for example $B_0 > 0$, the contract values shown in the tables would increase since surplus obtained over the contracts' life would not have to be partly used to build up and maintain the target buffer. Finally, in each panel we indicate the *average relative standard error* of the 24 prices obtained by Monte Carlo simulation. The individual standard errors (not reported) are calculated as the standard error of the Monte Carlo estimate divided by the price estimate itself.

Table 2

Values of European and American participating contracts
for different levels of interest rates and bonus policy parameters, γ and α.

Other parameters:
$P_0 = 100$, $B_0 = 0$, $T = 20$ years, $\sigma = 15\%$, and $r_G = 4.5\%$.

	γ = 0.00	0.05	0.10	0.15	0.20	0.25
$r = 8\%$						
α 0.00	48.69 (100.00)	48.69 (100.00)	48.69 (100.00)	48.69 (100.00)	48.69 (100.00)	48.69 (100.00)
α 0.25	83.57 (101.26)	81.20 (100.00)	79.03 (100.00)	77.04 (100.00)	75.21 (100.00)	73.53 (100.00)
α 0.50	97.68 (112.46)	94.23 (109.49)	91.09 (106.91)	88.23 (104.61)	85.60 (102.82)	83.20 (101.15)
α 0.75	104.94 (119.95)	100.96 (115.69)	97.35 (112.10)	94.05 (109.23)	91.04 (106.63)	88.29 (104.50)
α 1.00	109.73 (124.51)	105.44 (119.79)	101.54 (115.65)	97.99 (112.05)	94.75 (109.16)	91.79 (106.65)
Average relative standard error: 0.00029						
$r = 6\%$						
α 0.00	72.64 (100.00)	72.64 (100.00)	72.64 (100.00)	72.64 (100.00)	72.64 (100.00)	72.64 (100.00)
α 0.25	99.95 (108.67)	97.59 (106.66)	95.47 (105.17)	93.57 (103.85)	91.86 (102.71)	90.31 (101.84)
α 0.50	113.18 (121.30)	109.59 (117.79)	106.37 (114.78)	103.49 (112.09)	100.91 (109.82)	98.58 (107.81)
α 0.75	120.34 (128.85)	116.11 (124.31)	112.33 (120.54)	108.94 (117.18)	105.91 (114.19)	103.18 (111.71)
α 1.00	125.19 (133.50)	120.57 (128.48)	116.44 (124.09)	112.74 (120.34)	109.42 (116.97)	106.44 (114.11)
Average relative standard error: 0.00026						
$r = 4\%$						
α 0.00	108.37 (108.37)	108.37 (108.37)	108.37 (108.37)	108.37 (108.37)	108.37 (108.37)	108.37 (108.37)
α 0.25	128.07 (128.07)	125.88 (125.88)	123.98 (123.98)	122.31 (122.31)	120.86 (120.86)	119.58 (119.58)
α 0.50	140.09 (140.09)	136.46 (136.46)	133.31 (133.31)	130.57 (130.57)	128.18 (128.18)	126.10 (126.10)
α 0.75	147.08 (147.37)	142.64 (142.64)	138.79 (138.79)	135.44 (135.44)	132.52 (132.52)	129.98 (129.98)
α 1.00	152.02 (152.15)	147.04 (147.04)	142.72 (142.72)	138.96 (138.96)	135.69 (135.69)	132.84 (132.84)
Average relative standard error: 0.00021						

Table 3

Values of European and American participating contracts
for different levels of interest rates and bonus policy parameters, γ and α.

Other parameters:
$P_0 = 100$, $B_0 = 0$, $T = 20$ years, $\sigma = 30\%$, and $r_G = 4.5\%$.

$r = 8\%$

α \ γ	0.00	0.05	0.10	0.15	0.20	0.25
0.00	48.69 (100.00)	48.69 (100.00)	48.69 (100.00)	48.69 (100.00)	48.69 (100.00)	48.69 (100.00)
0.25	104.07 (126.76)	101.63 (124.32)	99.35 (122.00)	97.24 (119.99)	95.26 (118.24)	93.41 (116.60)
0.50	127.26 (152.62)	123.42 (148.48)	119.89 (144.61)	116.62 (141.02)	113.60 (137.72)	110.79 (134.91)
0.75	140.83 (168.76)	136.19 (163.63)	131.94 (158.83)	128.02 (154.39)	124.41 (150.27)	121.07 (146.50)
1.00	150.56 (180.46)	145.40 (174.43)	140.67 (168.84)	136.34 (163.63)	132.34 (158.82)	128.66 (154.35)

Average relative standard error: 0.00089

$r = 6\%$

α \ γ	0.00	0.05	0.10	0.15	0.20	0.25
0.00	72.64 (100.00)	72.64 (100.00)	72.64 (100.00)	72.64 (100.00)	72.64 (100.00)	72.64 (100.00)
0.25	127.53 (139.13)	124.81 (136.34)	122.29 (133.71)	119.95 (131.46)	117.79 (129.40)	115.78 (127.50)
0.50	152.15 (166.44)	147.82 (161.91)	143.86 (157.68)	140.22 (153.79)	136.87 (150.12)	133.77 (147.02)
0.75	166.85 (183.29)	161.60 (177.73)	156.80 (172.55)	152.40 (167.74)	148.37 (163.27)	144.65 (159.20)
1.00	177.55 (195.54)	171.66 (189.04)	166.30 (183.00)	161.40 (177.38)	156.91 (172.21)	152.79 (167.41)

Average relative standard error: 0.00078

$r = 4\%$

α \ γ	0.00	0.05	0.10	0.15	0.20	0.25
0.00	108.37 (108.37)	108.37 (108.37)	108.37 (108.37)	108.37 (108.37)	108.37 (108.37)	108.37 (108.37)
0.25	162.49 (163.10)	159.45 (159.93)	156.66 (156.93)	154.09 (154.37)	151.73 (152.02)	149.55 (149.83)
0.50	188.91 (192.09)	183.98 (187.01)	179.50 (182.25)	175.41 (177.87)	171.67 (173.74)	168.24 (170.30)
0.75	205.11 (209.96)	199.06 (203.75)	193.57 (197.95)	188.57 (192.59)	184.01 (187.60)	179.84 (183.09)
1.00	217.09 (223.14)	210.27 (215.88)	204.08 (209.16)	198.46 (202.91)	193.34 (197.17)	188.66 (191.81)

Average relative standard error: 0.00066

Table 4

Interest Rate Scenario	Bonus Policy Scenario	Bond Element (1)	Bonus Option (2)	Surrender Option (3)	European Contract (1)+(2)	American Contract (1)+(2)+(3)
	Conservative [a]	48.69	0.00	51.31	48.69	100.00
$r = 8\%$	Neutral [b]	48.69	28.35	22.96	77.04	100.00
	Aggressive [c]	48.69	61.04	14.78	109.73	124.51
	Conservative	72.64	0.00	27.36	72.64	100.00
$r = 6\%$	Neutral	72.64	20.93	10.28	93.57	103.85
	Aggressive	72.64	52.55	8.31	125.19	133.50
	Conservative	108.37	0.00	0.00	108.37	108.37
$r = 4\%$	Neutral	108.37	13.94	0.00	122.31	122.31
	Aggressive	108.37	43.65	0.13	152.02	152.15

Decomposition of Participating European and American Contract Values
$P_0 = 100$, $B_0 = 0$, $T = 20$ years, $\sigma = 15\%$, and $r_G = 4.5\%$.

[a] Conservative scenario with $\alpha = 0.00$.

[b] Neutral scenario with $\alpha = 0.25$ and $\gamma = 0.15$.

[c] Aggressive scenario with $\alpha = 1.00$ and $\gamma = 0.00$.

In order to provide a different perspective on the implications of the model, we have extended the simulation analysis to cover an issue of relevance to the reserving process. Specifically, we have estimated by simulation the probability of default on the individual contract for various bonus policy parameters. The probabilities reported are *risk neutral*, i.e. they are Q-probabilities. These differ from the *true* probabilities (P-probabilities) to the extent that the expected return on assets differ from the risk free rate of interest. We note that to obtain default probabilities under the P-measure would be computationally equivalent but would require a subjective choice of the value of the parameter μ in equation (2).

We consider European contracts only and Table 5 below contains the estimated default probabilities where default is defined as the event that $B(T) < 0$. The top panel represents the base case where we have taken $r_G = 4.5\%, r = 8\%, B_0 = 0$, and $\sigma = 15\%$. Different bonus policies are studied by varying α and γ as in our earlier pricing analysis. These parameters are seen to represent a rather risky regime for the insurance company: default probabilities vary between 0.23 and 0.68. The following three panels each depicts a situation where we have changed one, and only one, parameter from the base case in a favorable direction as seen from the side of the company. First, the volatility of assets was lowered to 10%. Second, the guaranteed interest rate was lowered to 2.5%, and third, the initial reserve, B_0, was set to 20. In the fifth panel all three changes were applied simultaneously.

Table 5

Default probabilities for Participating European Contracts

$P_0 = 100$, $T = 20$ years, and $r = 8\%$.

			γ					
			0.00	0.05	0.10	0.15	0.20	0.25
$\sigma = 15\%$, $r_G = 4.5\%$, $B_0 = 0$	α	0.00	0.23	0.23	0.23	0.23	0.23	0.23
		0.25	0.37	0.34	0.32	0.31	0.29	0.28
		0.50	0.52	0.47	0.43	0.40	0.37	0.35
		0.75	0.62	0.56	0.51	0.46	0.43	0.39
		1.00	0.68	0.62	0.57	0.51	0.47	0.43
$\sigma = 10\%$, $r_G = 4.5\%$, $B_0 = 0$	α	0.00	0.08	0.08	0.08	0.08	0.08	0.08
		0.25	0.14	0.12	0.11	0.10	0.10	0.09
		0.50	0.29	0.23	0.18	0.15	0.13	0.12
		0.75	0.41	0.32	0.25	0.20	0.17	0.14
		1.00	0.50	0.39	0.31	0.24	0.20	0.16
$\sigma = 15\%$, $r_G = 2.5\%$, $B_0 = 0$	α	0.00	0.09	0.09	0.09	0.09	0.09	0.09
		0.25	0.25	0.22	0.20	0.18	0.16	0.15
		0.50	0.41	0.35	0.31	0.27	0.23	0.21
		0.75	0.52	0.45	0.38	0.33	0.29	0.25
		1.00	0.59	0.52	0.45	0.39	0.34	0.29
$\sigma = 15\%$, $r_G = 4.5\%$, $B_0 = 20$	α	0.00	0.16	0.16	0.16	0.16	0.16	0.16
		0.25	0.33	0.31	0.28	0.26	0.24	0.23
		0.50	0.51	0.46	0.41	0.37	0.34	0.31
		0.75	0.62	0.55	0.50	0.45	0.40	0.37
		1.00	0.68	0.62	0.56	0.50	0.45	0.41
$\sigma = 10\%$, $r_G = 2.5\%$, $B_0 = 20$	α	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		0.25	0.06	0.04	0.03	0.02	0.01	0.01
		0.50	0.18	0.12	0.08	0.05	0.04	0.03
		0.75	0.30	0.20	0.13	0.09	0.06	0.04
		1.00	0.39	0.27	0.18	0.12	0.08	0.06

The change in volatility from 15% to 10% in the second panel is seen to be quite effectful. For what we consider to be realistic values of the bonus policy parameters, the default probability is more than cut in half. As seen from the third panel, the effect on default probabilities of a lowering of the guaranteed interest rate can also be dramatic, whereas for this particular choice of parameters the impact of an initial buffer of substantial size is not that significant (fourth panel). Of course, when the effects are combined as in the fifth panel, the default probabilities can be made quite small. This is particularly outspoken when bonus policies are in the conservative to moderate area.

An alternative view on default probabilities is given in Figures 5 and 6 where we have drawn *iso-default probability curves*, i.e. curves depicting combinations of α and γ which result in the same default probabilities for an otherwise given set of parameters. Figure 5 uses parameters as in the base case in Table 5 and thus represents a problematic situation where the company must set a very conservative bonus policy to ensure default probabilities below 25%. Figure 6, on the other hand, represents a relatively safe scenario corresponding to the fifth panel in Table 5 where the company would have to specify a very aggressive bonus policy in order to experience default probabilities in excess of 25%.

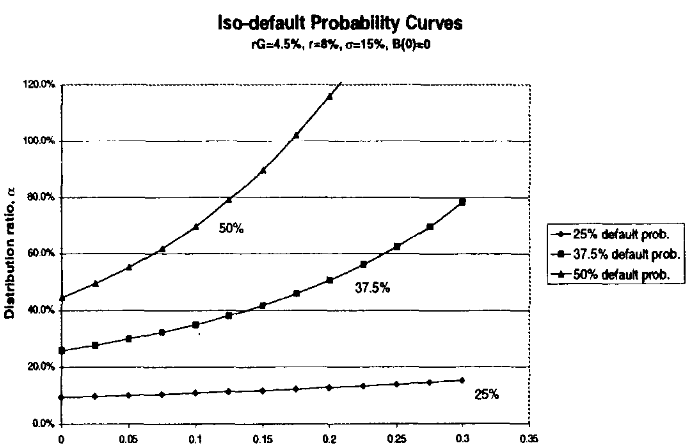

Figure 5: Iso-default probability curves

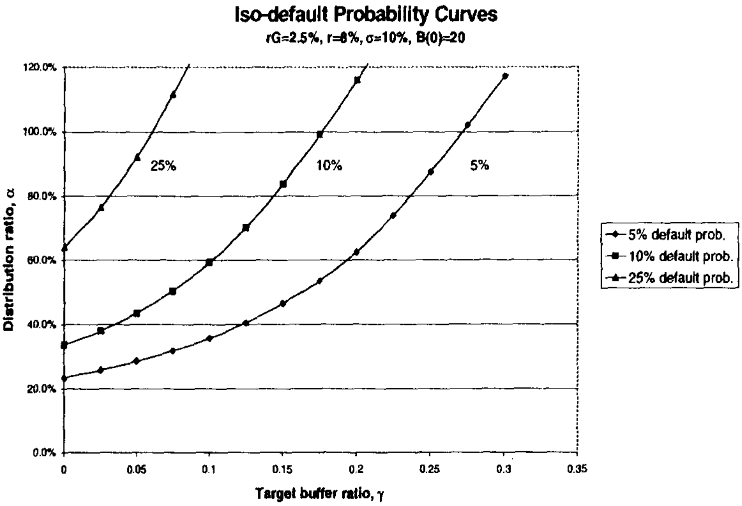

Figure 6: Iso-default probability curves

5 Conclusion

As the general level of interest rates has dropped in the 1990s, life insurance companies have experienced a dramatic narrowing in the safety margin between their earning power on the asset side and the level of promised returns on the liability side. As a consequence, the interest rate guarantees issued with traditional participating life insurance policies have become particularly valuable to policyholders and threatened the solvency of the issuing companies. The situation has in many cases been worsened by established bonus practices and by the fact that some contracts have been issued with an option to surrender the policy before maturity.

Life insurance companies have traditionally not given much attention to the proper valuation of the various option elements with which their policies have been issued, and this has undoubtedly contributed to the problems now experienced in the life insurance business. In the present paper we have presented a dynamic model for use in valuing the common family of life insurance products known as participating contracts. The model was based on the well-developed theory of contingent claims valuation. We have shown that the typical contract can be decomposed into three basic elements: a risk free bond, an option to receive bonus, and a surrender option.

The properties of the model were explored numerically. The analysis showed that contract values are highly dependent on the assumed bonus policy and on the spread between the market interest rate and the guaranteed rate of interest built into the contract. In another application of the model, we estimated by simulation the relation between bonus policies and the probability of default at the individual contract level. This analysis showed default probabilities of substantial size for realistic parameter choices indicating that the management of the life insurance company should take solvency problems very seriously. In this respect, initiatives on both the asset and the liability sides could be considered. An obvious first choice could be to reduce asset volatility by changing the asset composition towards less risky assets. But another potential problem enters here. The return offered by less risky assets (bonds) might be so low that such a move would only make it certain that interest guarantees could not be honored in the future. A classical asset substitution problem where the company management is forced into taking risks might thus exist. Alternatively, to reduce the probability of default, the company could consider more conservative bonus policies to the extent that this is permitted by law and the contractual terms. In fact, the possibility for the management to change the bonus policy parameters in a sense constitutes a *counter option* which we have not explicitly taken into account in this paper. This is an interesting subject for future research as would be the analysis of how the asset mix could possibly be optimally described as a function of the liability situation of the life insurance company.

Some other natural paths for further research emerge. In the present article, a model was constructed in order to describe actual insurance company behavior with particular respect to applied accounting principles and bonus policies. Future research should try to establish the ideal way in the sense that systematic market value accounting should constitute the basis of the balance sheet as well as the bonus policy. Furthermore, realism could obviously be added to the model by considering periodic premiums and taking into account mortality, lapses and various expense charges. Lastly, an interesting issue would be to incorporate the possibility of default of the life insurance company and to analyze how this would affect contract values and possibly also the optimal surrender strategy.

References

Albizzati, M.-O. and H. Geman (1994): "Interest Rate Risk Management and Valuation of the Surrender Option in Life Insurance Policies," *Journal of Risk and Insurance*, 61(4):616–637.

Babbel, D. F. and C. Merrill (1999): "Economic Valuation Models for Insurers," *North American Actuarial Journal*. Forthcoming.

Baccinello, A. R. and F. Ortu (1993a): "Pricing Equity-Linked Life Insurance with Endogenous Minimum Guarantees," *Insurance: Mathematics and Economics*, 12(3):245–258.

———— (1993b): "Pricing Equity-Linked Life Insurance with Endogenous Minimum Guarantees: A Corrigendum," *Insurance: Mathematics and Economics*, 13:303–304.

Bilodeau, C. (1997): "Better Late than Never: The Case of the Rollover Option," *Insurance: Mathematics and Economics*, 21:103–111.

Black, F. and M. Scholes (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81(3):637–654.

Black, K. and H. D. Skipper (1994): *Life Insurance*, Prentice Hall, Englewood Cliffs, NJ, USA, twelfth edition.

Boyle, P. P. (1977): "Options: A Monte Carlo Approach," *Journal of Financial Economics*, 4:323–338.

Boyle, P. P., M. Broadie, and P. Glasserman (1997): "Monte Carlo Methods for Security Pricing," *Journal of Economic Dynamics and Control*, 21:1267–1321.

Boyle, P. P. and M. R. Hardy (1997): "Reserving for Maturity Guarantees: Two Approaches," *Insurance: Mathematics and Economics*, 21:113–127.

Boyle, P. P. and E. S. Schwartz (1977): "Equilibrium Prices of Guarantees under Equity-Linked Contracts," *Journal of Risk and Insurance*, 44:639–660.

Brennan, M. J. (1993): "Aspects of Insurance, Intermediation and Finance," *The Geneva Papers on Risk and Insurance Theory*, 18(1):7–30.

Brennan, M. J. and E. S. Schwartz (1976): "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee," *Journal of Financial Economics*, 3:195–213.

———— (1979): "Alternative Investment Strategies for the Issuers of Equity Linked Life Insurance Policies with an Asset Value Guarantee," *Journal of Business*, 52(1):63–93.

Briys, E. and F. de Varenne (1997): "On the Risk of Life Insurance Liabilities: Debunking Some Common Pitfalls," *Journal of Risk and Insurance*, 64(4):673–694.

Broadie, M. and P. Glasserman (1997): "Pricing American-style Securities Using Simulation," *Journal of Economic Dynamics and Control*, 21:1323–1352.

Cox, J. C., S. A. Ross, and M. Rubinstein (1979): "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7:229–263.

Embrechts, P. (1996): "Actuarial versus Financial Pricing of Insurance," Working Paper, Wharton Financial Institutions Center, WP 96-17.

Gentle, J. E. (1998): *Random Number Generation and Monte Carlo Methods*, Springer-Verlag, New York.

Grosen, A. and P. L. Jørgensen (1997): "Valuation of Early Exercisable Interest Rate Guarantees," *Journal of Risk and Insurance*, 64(3):481–503.

Harrison, M. J. and D. M. Kreps (1979): "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20:381–408.

Hull, J. C. (1997): *Options, Futures, and other Derivatives*, Prentice-Hall, Inc., 3rd edition.

Kalra, R. and G. Jain (1997): "A Continuous-time Model to Determine the Intervention Policy for PBGC," *Journal of Banking and Finance*, 21:1159–1177.

Karatzas, I. (1988): "On the Pricing of the American Option," *Applied Mathematics and Optimization*, 17:37–60.

Lewis, C. M. and G. G. Pennacchi (1997): "Valuing Insurance for Defined-Benefit Pension Plans," Working Paper.

Maturity Guarantees Working Party (1980): "Report," *Journal of the Institute of Actuaries*, 107:103–209.

Miltersen, K. R. and S.-A. Persson (1997): "Pricing Rate of Return Guarantees in a Heath-Jarrow-Morton Framework," Working Paper, Odense University.

———— (1998): "Guaranteed Investment Contracts: Distributed and Undistributed Excess Return," Working Paper, Odense University.

Nielsen, J. A. and K. Sandmann (1995): "Equity-Linked Life Insurance: A Model with Stochastic Interest Rates," *Insurance: Mathematics and Economics*, 16:225–253.

Nielsen, J. P. and H. L. Thaning (1996): "Matching of Assets and Liabilities in Danish Pension Companies," Working Paper, PFA Pension and Copenhagen Business School.

Pennacchi, G. and C. Lewis (1994): "The Value of Pension Benefit Guaranty Corporation Insurance," *Journal of Money, Credit and Banking*, 26:735–753.

Persson, S.-A. and K. K. Aase (1997): "Valuation of the Minimum Guaranteed Return Embedded in Life Insurance Products," *Journal of Risk and Insurance*, 64(4):599–617.

Press, W. H., B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling (1989): *Numerical Recipes in Pascal,* Cambridge University Press.

Quandt, R. (1983): "Computational Problems and Methods," in Griliches, Z. and M. Intriligator, editors, *Handbook of Econometrics*, volume 1, chapter 12. North-Holland.

The Danish Financial Supervisory Authority (1998): *Rapport om betaling for rentegaranti (Report on the Payment for Interest Rate Guarantees),* Finanstilsynet/The Danish Financial

Supervisory Authority. The report (in Danish) and an English summary are available at the internet address 'www.ftnet.dk'.

Tilley, J. (1993): "Valuing American Options in a Path Simulation Model," *Transactions of the Society of Actuaries*, 45:83–104.

Vanderhoof, I. T. and E. I. Altman, editors (1998): *The Fair Value of Insurance Liabilities,* Kluwer Academic Publishers. The New York University Salomon Center Series on Financial Markets and Institutions.