An Economic Premium Principle in Multiperiod Time Horizon

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Abstract

This paper considers a multiperiod economic equilibrium model to derive the economic premium principle of Bühlmann (1980, 1983). To do this, we construct a consumption/portfolio model in which each agent characterized by his/her utility function and endowments can invest his/her wealth into insurance market as well as financial market to maximize the expected, discounted total utility from consumption. The state price density in an equilibrium is obtained in terms of the Arrow–Pratt index of absolute risk aversion for the representative agent. As special cases, power and exponential utility functions are examined, and some comparative statics results are derived.

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1 Intorduction

The credit derivatives market has grown rapidly and they are now one of the major tools to hedge credit risk. Credit derivatives are financial instruments whose payoffs are linked to the credit characteristics of reference asset's value. For example, a *credit swap* is an agreement between two parties, A and B say, to exchange risk and cash flows in the future according to a prearranged formula. Typically, Party A pays Party B at the credit events such as default, if before the maturity, the difference between the current value and the market value of an asset. In compensation of the payment, Party B pays Party A an annuity at a rate, called the credit swap premium, until either the maturity of the contract or termination by the designated credit events, whichever happens first.[1] The idea of the credit swap is quite similar to that of insurance products. In fact, we can imagine that Party B pays an insurance premium to cover a sudden loss due to a credit event.

Turning to insurance products, there are frequently traded ART (Alternative Risk Transfer) products in the market. Examples of ART products include insurance linked securities, insurance derivatives and holistic covers that hedge insurance and financial risks simultaneously. Evidently, pricing of such ART products requires not only the theory of insurance but also the theory of finance.

The classic risk theory suggests that insurance premiums are calculated based on the expected loss under the observed probability measure. However, if financial risks are included, as a part, in an insurance product, or the product (or its derivative) is tradable in a financial market, then the premium should be calculated so as to reflect financial risks. One approach to this problem is to use the "non-arbitrage valuation framework" developed in the finance theory directly. For example, Delbaen and Haezendonck (1989) considered, as given, some *risk-neutral* measure under which the underlying price process is a martingale. Under this probability measure, the insurance premium is calculated as the expected, discounted total payoffs.[2] This approach is applicable if we can find the risk-neutral measure. However, for such, the insurance market needs to have enough liquidity.

The other approach to consider financial risks is to use an economic equilibrium model. The pioneering work by Bühlmann (1980) considered a single-period consumption model in which each agent is characterized by his/her utility function and initial wealth, and the state price density is determined so as to achieve an equilibrium.[3] Especially, he showed that the standard premium calculation principle using the Esscher transform falls into a special case

[1] See Duffie (1999) and Kijima (1998) for the credit swap valuation.

[2] See Harrison and Kreps (1979) and Harrison and Pliska (1981) for details.

[3] Another way to tackle this problem may be to use the CAPM (Capital Asset Pricing Model) type models developed in financial economics. See Müller (1987), Cummins (1990) and references therein.

of his result. Since then, several extensions have been made based on his model. Examples of such include Bühlmann (1983), Lienhard (1986), Wyler (1990) and references therein. See Gerber and Shiu (1994) and Bühlmann et al. (1998) for the connection between the Esscher transform and some finance models.

In this paper, we take the line of Bühlmann (1980, 1983) with some modifications and extensions. We consider a multiperiod consumption/portfolio model in which each agent characterized by his/her utility function and endowments can invest his/her wealth into insurance market as well as financial market to maximize the expected, discounted total utility from consumption. It is shown that, even in this standard economic model, a similar result to Bühlmann (1980, 1983) is obtained. Especially, the state price density in an economic equilibrium is derived in terms of the Arrow–Pratt index of absolute risk aversion for the representative agent. For power and exponential utility functions, closed form solutions of the state price density are given and some comparative statics results are obtained accordingly.

This paper is organized as follows. In the next section, we formally state our discrete-time multiperiod model with emphasis on the differences of our model from Bühlmann's model (1980, 1983). It is shown that, under the market completeness assumption together with some technical conditions, an optimal consumption/portfolio process for every agent exists and the state price density in equilibrium can be derived under the market clearance condition. While, in Section 3, the special cases of power and exponential utility functions are examined, Section 4 studies some comparative statics about our economic premium principle. It is shown that the insurance premium for risk is higher for the economy with more risk-averse agents.

Throughout this paper, all the random variables considered are bounded almost surely (a.s.) to avoid unnecessary technical difficulties. Equalities and inequalities for random variables hold in the sense of a.s.; however, we omit the notation a.s. for the sake of notational simplicity.

2 The Discrete-Time Model

In this section, we construct a discrete-time consumption/portfolio model to derive the economic premium principle of Bühlmann (1980, 1983). We consider an economy consisting of a finite number of agents, $i = 1, 2, \cdots, n$ say, who constitute buyers of insurance companies, insurance companies and reinsurance companies.

In the economy, it is assumed that trading dates consist of the set of integers $\mathcal{T} \equiv \{0, 1, \cdots, T\}$ with $T > 0$, i.e. any economic trade occurs at discrete-time points between 0 and T, and that uncertainty is described by a given probability space $(\Omega, \mathcal{F}, P)$. Also, as to resolution of uncertainty of the economy, we are given a P-augmentation of filtration $\mathbb{F} \equiv \{\mathcal{F}_t; t \in \mathcal{T}\}$ such that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$$

with $\mathcal{F}_0 = (\Omega, \emptyset)$. The conditional expectation operator given $\mathcal{F}_t$ is denoted by E_t.

Agent i is endowed $w_i(t)$ units of a single (perishable) commodity, and he/she encounters *risk* $X_i(t)$ measured in units of the commodity at time $t \in \mathcal{T}$. While the quantities $w_i(t)$ and $X_i(t)$, $t = 1, 2, \cdots, T$, are assumed to be nonnegative random variables, $w_i(0)$ and $X_i(0)$ are nonnegative constants. We call $Z_i(t) \equiv w_i(t) - X_i(t)$ the *net endowment* for agent i at time $t \in \mathcal{T}$. The *aggregated net endowment*

$$Z(t) \equiv \sum_{i=1}^{n} Z_i(t), \qquad t \in \mathcal{T}, \tag{2.1}$$

is assumed to be strictly positive.

Next, we introduce an *insurance* with premium-per-share $p(t)$ at time t, which is strictly positive and satisfies

$$p(t+1) = p(t)(1 + \xi(t+1)), \qquad p(0) = p, \qquad t = 0, 1, \cdots, T-1, \tag{2.2}$$

for some $\mathbb{F}$-adapted, possibly negative process $\xi(t) > -1$, where p is a positive constant.

One generalization of Bühlmann's model (1980, 1983) is to allow the agents to invest their wealth into a financial market consisting of the money market and m risky securities. We denote the time t price of the money market account by $S_0(t)$ whereas the time t price of security j by $S_j(t)$, $j = 1, 2, \cdots, m$. It is assumed that $S_j(t)$ are strictly positive and, while $S_0(t)$ satisfies

$$S_0(t+1) = S_0(t)(1 + r(t+1)), \qquad S_0(0) = 1, \qquad t = 0, 1, \cdots, T-1, \tag{2.3}$$

for some $\mathbb{F}$-adapted, positive process $r(t)$ which represents the *risk-free* interest rate, the risky security prices $S_j(t)$ are defined by

$$S_j(T) = S_j(0)(1 + \xi^{(j)}(t+1)), \qquad S_j(0) = s_j, \qquad t = 0, 1, \cdots, T-1, \tag{2.4}$$

for some $\mathbb{F}$-adapted, possibly negative process $\xi^{(j)} > -1$, where s_j are positive constants. $P(t)$, $S_0(t)$, $S_j(t)$, $j = 1, 2, \cdots, m$, are all assumed to be measured in units of the commodity at time $t \in \mathcal{T}$.

Formally, our discrete-time insurance/financial market is defined as follows. See Pliska (1997) for the basic discrete-time securities market.

Definition 2.1 The discrete-time insurance/financial market consists of

(1) the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$,

(2) the set of $\mathbb{F}$-adapted net endowment processes $Z_i(t)$ with $Z(t) = \sum_{i=1}^{n} Z_i(t) > 0$ for all $t \in \mathcal{T}$,

(3) the positive, $\mathbb{F}$-adapted risk-free interest rate process $r(t)$,

(4) the set of $\mathbb{F}$-adapted processes $\xi^{(j)}(t) > -1$, $j = 1, 2, \cdots, m$,

(5) the $\mathbb{F}$-adapted process $\xi(t) > -1$.

The market is referred to as $\mathcal{M} = (\{Z_i(t)\}_{i=1}^{n}, r(t), \{\xi^{(j)}(t)\}_{j=1}^{m}, \xi(t))$.

Let $Y_i(t)$, $\theta_i^{(0)}(t)$ and $\theta_i^{(j)}(t)$ denote the number of shares of insurance, the number of shares in the money market and the number of shares in security j, respectively, carried by agent i from time t to $t+1$. Hereafter we call $Y_i(t)$ a *risk exchange* and $(\boldsymbol{\theta}_i(t), Y_i(t))$ a *portfolio* of agent i at time t, where $\boldsymbol{\theta}_i(t) \equiv (\theta_i^{(0)}(t), \theta_i^{(1)}(t), \cdots, \theta_i^{(m)}(t))^\top$ and $\top$ denotes the transpose. Once a market $\mathcal{M}$ is given, each agent i chooses a *consumption process* $c_i(t)$ and a portfolio process $(\boldsymbol{\theta}_i(t), Y_i(t))$, $t \in \mathcal{T}$.

Given a portfolio process $(\boldsymbol{\theta}_i(t), Y_i(t))$ and a *cumulative income process* $\sum_{s=0}^{t}(Z_i(s) - c_i(s))$, $t \in \mathcal{T}$, the *wealth process* $\{W_i(t); t \in \mathcal{T}\}$ is defined by $W_i(0) = Z_i(0) - c_i(0)$ and

$$\begin{aligned} W_i(t+1) - W_i(t) &= Z_i(t+1) - c_i(t+1) \\ &+ p(t-1)Y_i(t-1)\xi(t) + \sum_{j=1}^{m} S_j(t-1)\theta_i^{(j)}(t-1)\xi^{(j)}(t) \\ &+ (W(t) - p(t-1)Y_i(t-1) - \sum_{j=1}^{m} S_j(t-1)\theta_i^{(j)}(t-1)\xi^{(j)}(t))r(t) \end{aligned} \tag{2.5}$$

for $t = 1, 2, \cdots, T$. Furthermore, suppose that the *state price density* $\phi(t)$ is given that satisfies $\phi(0) = 1$, $0 < \phi(t) < \infty$, and

$$E_{t-1}[\phi(t)p(t)] = \phi(t-1)p(t-1); \qquad E_{t-1}[\phi(t)S_j(t)] = \phi(t-1)S_j(t-1),\ j = 0, 1, \cdots, m, \tag{2.6}$$

for each $t = 1, 2, \cdots, T$. It is easily seen from (2.5) that

$$\begin{aligned} \phi(t)W_i(t) &= \sum_{s=0}^{t} \phi(s)(Z_i(s) - c_i(s)) + \sum_{s=1}^{t} \phi(s)p(s-1)Y_i(s-1)(\xi(s) - r(s)) \\ &+ \sum_{s=1}^{t}\sum_{j=1}^{m} \phi(s)S_j(s-1)\theta_i^{(j)}(s-1)(\xi^{(j)}(s)) - r(s)), \quad t = 1, 2, \cdots, T. \end{aligned} \tag{2.7}$$

The next definition is similar to the one given in Karatzas and Shreve (1998).

Definition 2.2 A consumption/portfolio process $(c_i(t), (\boldsymbol{\theta}_i(t), Y_i(t)))$ is *admissible* for agent i if the corresponding wealth process satisfies

$$W_i(t) + E_t\left[\sum_{s=t+1}^{T} \phi(s)Z_i(s)\right] \geq 0, \qquad t \in \mathcal{T}. \tag{2.8}$$

The class of admissible process $(c_i(t), (\boldsymbol{\theta}_i(t), Y_i(t)))$ is denoted by $\mathcal{A}_i$.

We note that, from (2.5) – (2.8), if $(c_i(t), (\boldsymbol{\theta}_i(t), Y_i(t)))$ is admissible, then the consumption process $c_i(t)$ must satisfy the *budget constraint*

$$E\left[\sum_{t=0}^{T} \phi(t) c_i(t)\right] \leq E\left[\sum_{t=0}^{T} \phi(t) Z_i(t)\right] \tag{2.9}$$

for each agent i.

Now, suppose that, while all the agents have a common *discount process* $\beta(t)$, agent i has a *utility function* $U_i : \mathbb{R} \to (0, \infty)$ which is strictly increasing, strictly concave and twice continuously differentiable with the properties $U_i'(\infty) \equiv \lim_{x\to\infty} U_i'(x) = 0$ and $U_i'(0+) \equiv \lim_{x\downarrow 0} U_i'(x) = \infty$. The problem that each agent faces in the market $\mathcal{M}$ is as follows.

(MP) Find an optimal consumption/portfolio process $(\hat{c}_i(t), (\hat{\boldsymbol{\theta}}_i(t), \hat{Y}_i(t)))$ to maximize the expected total, discounted utility from consumption

$$E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t} \beta(u)} U_i(c_i(t))\right]$$

over the admissible consumption/portfolio processes $(c_i(t), (\boldsymbol{\theta}_i(t), Y_i(t))) \in \mathcal{A}_i$ that satisfy

$$E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t} \beta(u)} \min\{0, U_i(c_i(t))\}\right] > -\infty. \tag{2.10}$$

For every utility function U_i, we shall denote by I_i the inverse of the derivative U_i'. Under the assumption stated above, the inverse I_i is also continuous, strictly decreasing, and maps $(0, \infty)$ onto itself with the properties $I_i(0+) = U_i'(0+) = \infty$ and $I_i(\infty) = U_i'(\infty) = 0$. The next lemma is easily established (see e.g., Rockafellar (1970)).

Lemma 2.1 *Let $\tilde{U}_i$ be defined by*

$$\tilde{U}_i(y) \equiv \max_{0<x<\infty} \{U_i(x) - xy\}, \qquad 0 < y < \infty. \tag{2.11}$$

Then, $\tilde{U}_i : \mathbb{R} \to (0, \infty)$ is convex, decreasing, continuously differentiable on $(0, \infty)$, and satisfies

$$\begin{aligned} &\tilde{U}_i(y) = U_i(I_i(y)) - yI_i(y), \quad 0 < y < \infty, \\ &\tilde{U}_i(0+) = U_i(\infty), \qquad \tilde{U}_i(\infty) = U_i(0+). \end{aligned} \tag{2.12}$$

The derivative $\tilde{U}_i'$ is well defined, continuous, and increasing on $(0, \infty)$, and given by

$$\tilde{U}_i'(y) = -I_i(y), \qquad 0 < y < \infty. \tag{2.13}$$

Let us define

$$\mathcal{X}_i(y) \equiv E\left[\sum_{t=0}^{T} \phi(t) I_i\left(y\, e^{\sum_{u=0}^{t}\beta(u)}\phi(t)\right)\right], \quad 0 < y < \infty. \tag{2.14}$$

The function $\mathcal{X}_i$ maps $(0,\infty)$ onto itself, and is continuous and strictly decreasing with the properties $\mathcal{X}_i(0+) \equiv \lim_{y\downarrow 0}\mathcal{X}_i(y) = \infty$ and $\mathcal{X}_i(\infty) \equiv \lim_{y\to\infty}\mathcal{X}_i(y) = 0$. This function has a continuous and strictly decreasing inverse $\mathcal{Y}_i : (0,\infty) \overset{\text{onto}}{\to} (0,\infty)$, which satisfies

$$\mathcal{X}_i(\mathcal{Y}_i(x)) = x, \qquad \forall x \in (0,\infty). \tag{2.15}$$

To proceed further, we need the following assumption regarding the *market completeness*:

(MC) For each t, $t = 1,2,\cdots,T$, and for every $\mathcal{F}_t$ random variable $\mathcal{B}$, there exists a portfolio $(\theta^{(1)}(s),\cdots,\theta^{(m)}(s),Y(s))$, $s = 0,1,\cdots,T-1$, such that

$$\mathcal{B} = \sum_{s=1}^{t} p(s-1)Y(s-1)(\xi(s)-r(s)) + \sum_{s=1}^{t}\sum_{j=1}^{m} S_j(s-1)\theta^{(j)}(s-1)(\xi^{(j)}(s)-r(s)).$$

Remark 2.1 Suppose that Ω consists of a finite number of elements. If the number of securities is no more than the number of the elementary events, then the condition (MC) is satisfied (see e.g. Pliska (1997)).

Theorem 2.1 *Under the conditions stated above, an optimal consumption/portfolio process $(\hat{c}_i(t),(\hat{\boldsymbol{\theta}}_i(t),\hat{Y}_i(t))) \in \mathcal{A}_i$ for agent i and the corresponding wealth process $\hat{W}_i(t)$ are given, respectively, by*

$$\hat{c}_i(t) = I_i\left(y_i e^{\sum_{u=0}^{t}\beta(u)}\phi(t)\right), \qquad t \in \mathcal{T}, \tag{2.16}$$

where

$$y_i = \mathcal{Y}_i\left(E\left[\sum_{t=0}^{T}\phi(t)Z_i(t)\right]\right), \tag{2.17}$$

$(\hat{\boldsymbol{\theta}}_i(t),\hat{Y}_i(t))$ is a solution of

$$\begin{aligned}\sum_{s=1}^{t}\phi(s)p(s-1)Y_i(s-1)(\xi(s)-r(s)) + \sum_{s=1}^{t}\sum_{j=1}^{m}\phi(s)S_j(s-1)\theta_i^{(j)}(s-1)(\xi^{(j)}(s)-r(s))\\ = E_t\left[\sum_{s=0}^{T}\phi(s)(\hat{c}_i(s)-Z_i(s))\right], \qquad t = 1,2,\cdots,T,\end{aligned} \tag{2.18}$$

and

$$\theta_i^{(0)}(t) = \frac{1}{S_0(t)}\left(W_i(t) - p(t)Y_i(t) - \sum_{j=1}^{m} S_j(t)\theta_i^{(j)}(t)\right), \quad t = 0,1,\cdots,T-1, \tag{2.19}$$

and

$$\hat{W}_i(t) = \frac{1}{\phi(t)}E_t\left[\sum_{s=t+1}^{T}\phi(s)(\hat{c}_i(s)-Z_i(s))\right], \qquad t = 0,1,\cdots,T-1, \tag{2.20}$$

where $\hat{W}_i(T) = 0$ for all agents.

Proof. First, we show that $(\hat{c}_i(t), (\hat{\boldsymbol{\theta}}_i(t), \hat{Y}_i(t)))$ is admissible with the property (2.10). From the market completeness assumption (MC), there exists a portfolio $(\hat{\boldsymbol{\theta}}_i(t), \hat{Y}_i(t))$ satisfying (2.18). This leads to (2.20) and $(\hat{c}_i(t), (\hat{\boldsymbol{\theta}}_i(t), \hat{Y}_i(t))) \in \mathcal{A}_i$. From Lemma 2.1 with $y = y_i e^{\sum_{u=0}^{t}\beta(u)}\phi(t)$ and $x = 1$, we obtain

$$e^{-\sum_{u=0}^{t}\beta(u)}U_i(\hat{c}_i(t)) \geq e^{-\sum_{u=0}^{t}\beta(u)}U_i(1) + y_i\phi(t)(\hat{c}_i(t) - 1).$$

It follows that

$$-E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t}\beta(u)} \min\{0, U_i(\hat{c}_i(t))\}\right] \leq \sum_{t=0}^{T} e^{-\sum_{u=0}^{t}\beta(u)}|U_i(1)| + y_i E\left[\sum_{t=0}^{T}\phi(t)\right] < \infty.$$

Next, we show that $(\hat{c}_i(t), (\hat{\boldsymbol{\theta}}_i(t), \hat{Y}_i(t)))$ attains the optimal. To this end, let $(c_i(t), (\boldsymbol{\theta}_i(t), Y_i(t)))$ be any admissible process. Using Lemma 2.1 again, we have

$$e^{-\sum_{u=0}^{t}\beta(u)}U_i(\hat{c}_i(t)) - y_i\phi(t)\hat{c}_i(t) \geq e^{-\sum_{u=0}^{t}\beta(u)}U_i(c_i(t)) - y_i\phi(t)c_i(t),$$

and thus

$$\begin{aligned} E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t}\beta(u)}U(\hat{c}_i(t))\right] &\geq E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t}\beta(u)}U(c_i(t))\right] \\ &\quad + y_i E\left[\sum_{t=0}^{T}\phi(t)\hat{c}_i(t)\right] - y_i E\left[\sum_{t=0}^{T}\phi(t)c_i(t)\right]. \end{aligned}$$

Furthermore, (2.14), (2.15), (2.17) and the budget constraint (2.9) together lead to

$$E\left[\sum_{t=0}^{T}\phi(t)\hat{c}_i(t)\right] = \mathcal{X}_i(y_i) = E\left[\sum_{t=0}^{T}\phi(t)Z_i(t)\right] \geq E\left[\sum_{t=0}^{T}\phi(t)c_i(t)\right].$$

It follows that

$$E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t}\beta(u)}U(\hat{c}_i(t))\right] \geq E\left[\sum_{t=0}^{T} e^{-\sum_{u=0}^{t}\beta(u)}U(c_i(t))\right],$$

completing the proof. □

We are now in a position to develop the economic premium principle. To this end, we formally state the notion of an equilibrium market.

Definition 2.3 Given the net endowment processes $Z_i(t)$ and utility functions U_i, $i = 1, 2, \cdots, n$, as well as the discount process $\beta(t)$, we say that $\mathcal{M}$ is an equilibrium market, if the following conditions hold:

(1) Clearing of the commodity market:

$$\sum_{i=1}^{n}\hat{c}_i(t) = Z(t), \qquad t \in \mathcal{T}. \tag{2.21}$$

(2) Clearing of the insurance market:

$$\sum_{i=1}^{n} \hat{Y}_i(t) = 0, \qquad t \in \mathcal{T}. \tag{2.22}$$

(3) Clearing of the securities market:

$$\sum_{i=1}^{n} \hat{\theta}_i^{(j)}(t) = 0, \quad j = 1, 2 \cdots, m, \quad t \in \mathcal{T}. \tag{2.23}$$

(4) Clearing of the money market:

$$\sum_{i=1}^{n} \hat{\theta}_i^{(0)}(t) = 0, \qquad t \in \mathcal{T}. \tag{2.24}$$

Here $\hat{c}_i(t)$, $\hat{\theta}_i^{(j)}(t)$, $\hat{Y}_i(t)$ are the optimal solutions for Problem (MP).

The next theorem characterizes the equilibrium market.

Theorem 2.2 *If $\mathcal{M}$ is an equilibrium market, then*

$$Z(t) = \sum_{i=1}^{n} I_i \left(y_i \, e^{\sum_{u=0}^{t} \beta(u)} \phi(t) \right), \qquad t \in \mathcal{T}, \tag{2.25}$$

where $y_1, \cdots, y_n$ are given by (2.17).

Conversely, if $\mathcal{M}$ is a market for which the state price density ϕ satisfies (2.25) and

$$E\left[\sum_{t=0}^{T} \phi(t) \left\{ I_i \left(y_i \, e^{\sum_{u=0}^{t} \beta(u)} \phi(t) \right) - Z_i(t) \right\} \right] = 0, \qquad i = 1, 2, \cdots, n, \tag{2.26}$$

for some $(y_1, \cdots, y_n) \in (0, \infty)^n$, then $\mathcal{M}$ is an equilibrium market.

Proof. The first statement follows by summing (2.16) over i and using the commodity market clearing condition (2.21). To prove the converse, note that if (2.26) holds, then the optimal consumption/portfolio pair for agent i is given by (2.16), (2.18) and (2.19). Now, summing (2.18) over i and using (2.25), we conclude that

$$\begin{aligned} &\sum_{s=1}^{t} \phi(s) p(s-1)(\xi(s) - r(s)) \sum_{i=1}^{n} Y_i(s-1) \\ &\quad + \sum_{s=1}^{t} \sum_{j=1}^{m} \phi(s) S_j(s-1)(\xi^{(j)}(s) - r(s)) \sum_{i=1}^{n} \theta_i^{(j)}(s-1) = 0, \qquad t = 1, 2, \cdots, T. \end{aligned}$$

It follows that $\sum_{i=1}^{n} \hat{Y}_i(t) = 0$ and $\sum_{i=1}^{n} \hat{\theta}_i^{(j)}(t) = 0$ for $j = 1, \cdots, m$, for all $t \in \mathcal{T}$. The money market clearing condition follows from summing (2.19) over i, proving the theorem. □

Given $\mathbf{\Lambda} = (\lambda_1, \cdots, \lambda_n) \in (0, \infty)^n$, let

$$I(y; \mathbf{\Lambda}) \equiv \sum_{i=1}^{n} I_i\left(\frac{y}{\lambda_i}\right), \qquad 0 < y < \infty. \tag{2.27}$$

Then, we can rewrite (2.25) as

$$Z(t) = I\left(e^{\sum_{u=0}^{t} \beta(u)} \phi(t); \mathbf{\Lambda}^*\right) \tag{2.28}$$

with $\mathbf{\Lambda}^* = (y_1^{-1}, \cdots, y_n^{-1})$, where $y_1, \cdots, y_n$ are given by (2.17). The function $I(y; \mathbf{\Lambda})$ is continuous and strictly decreasing with respect to y, and maps $(0, \infty)$ onto itself with the properties $I(0+; \mathbf{\Lambda}) = \infty$ and $I(\infty; \mathbf{\Lambda}) = 0$. Therefore, it has a continuous, strictly decreasing inverse $\mathcal{H}(\cdot; \mathbf{\Lambda}) : (0, \infty) \overset{\text{onto}}{\to} (0, \infty)$ with the properties $\mathcal{H}(0+; \mathbf{\Lambda}) = \infty$ and $\mathcal{H}(\infty; \mathbf{\Lambda}) = 0$. That is,

$$I(\mathcal{H}(x; \mathbf{\Lambda}); \mathbf{\Lambda}) = x, \qquad \forall\, x \in (0, \infty). \tag{2.29}$$

It follows from (2.28) that, if the market $\mathcal{M}$ is in equilibrium, then the state price density $\phi(t)$ and the aggregated net endowment $Z(t)$ are connected through

$$\phi(t) = e^{-\sum_{u=0}^{t} \beta(u)} \mathcal{H}(Z(t); \mathbf{\Lambda}^*), \qquad t \in \mathcal{T}. \tag{2.30}$$

Also, from (2.16) and (2.30), the optimal consumption process of agent i is given by

$$\hat{c}_i(t) = I_i\left(\frac{\mathcal{H}(Z(t); \mathbf{\Lambda}^*)}{\lambda_i^*}\right), \qquad t \in \mathcal{T}. \tag{2.31}$$

We note from (2.31) that the optimal consumption process depends not on the insurance as well as the securities but only on the aggregated net endowment $Z(t)$, the discount factor $\beta(t)$ and the utility function U_i.

We can characterize the function $\mathcal{H}(y; \mathbf{\Lambda})$ using a utility function of the *representative agent*, which is defined by

$$U(c; \mathbf{\Lambda}) \equiv \max\left\{\sum_{i=1}^{n} \lambda_i U_i(c_i);\; c_i > 0, i = 1, \cdots, n,\; \sum_{i=1}^{n} c_i = c\right\}, \qquad 0 < c < \infty. \tag{2.32}$$

The next theorem is well known. For the proof, see, e.g., Karatzas and Shreve (1998).

Theorem 2.3 *Let $\mathbf{\Lambda} \in (0, \infty)^K$ be given and let U_i be of class $C^3(0, \infty)$. Then the function $U(\cdot; \mathbf{\Lambda})$ is of class $C^3(0, \infty)$, strictly increasing, and strictly concave with*

$$U'(c; \mathbf{\Lambda}) = \mathcal{H}(c; \mathbf{\Lambda}), \qquad 0 < c < \infty. \tag{2.33}$$

Finally, we can obtain the expression for the state price density $\phi(t)$, which is similar to the one given by Bühlmann (1980, 1983). Given a utility function $U(x, \mathbf{\Lambda})$,

$$\rho(x, \mathbf{\Lambda}) \equiv -\frac{U''(x, \mathbf{\Lambda})}{U'(x, \mathbf{\Lambda})} \tag{2.34}$$

is called the Arrow-Pratt index of absolute risk aversion.

Theorem 2.4 *If $\mathcal{M}$ is an equilibrium market, then the state price density $\phi(t)$ is given by*

$$\phi(t) = \frac{\exp\left\{-\sum_{u=0}^{t}\beta(u) - \int_0^{Z(t)}\rho\left(x, \mathbf{\Lambda}^*\right)dx\right\}}{E\left[S_0(t)\exp\left\{-\sum_{u=0}^{t}\beta(u) - \int_0^{Z(t)}\rho\left(x, \mathbf{\Lambda}^*\right)dx\right\}\right]}, \qquad t \in \mathcal{T}. \tag{2.35}$$

Proof. From (2.30) and (2.33), we have

$$\phi(t) = e^{-\sum_{u=0}^{t}\beta(u)}U'\left(Z(t); \mathbf{\Lambda}^*\right).$$

Solving (2.34) with respect to U', we have

$$U'(x, \mathbf{\Lambda}) = K\exp\left\{-\int_0^x \rho\left(x, \mathbf{\Lambda}\right)dx\right\},$$

where K is the normalizing constant. Since $E[\phi(t)S_0(t)] = 1$, the theorem follows. □

If, in particular, $\beta(0) = 0$ and $\beta(t) = \log(1 + r(t))$ so that $S_0(t) = e^{\sum_{u=0}^{t}\beta(u)}$, then we have the following.

Corollary 1 *Suppose that $\beta(0) = 0$ and $\beta(t) = \log(1 + r(t))$, $t = 1, 2, \cdots, T$. Then, under the conditions of Theorem 2.4, we have*

$$\phi(t) = \frac{\exp\left\{-\int_0^{Z(t)}\rho\left(x, \mathbf{\Lambda}^*\right)dx\right\}}{S_0(t)E\left[\exp\left\{-\int_0^{Z(t)}\rho\left(x, \mathbf{\Lambda}^*\right)dx\right\}\right]}. \tag{2.36}$$

We note that if $S_0(t) = 1$, $t \in \mathcal{T}$, then Corollay 1 agrees with the result of Bühlmann (1983). Also, the expression (2.35) or (2.36) suggests us consider $\phi(t)S_0(t)$ rather than the state price density $\phi(t)$ itself. Namely, we can define a new probability measure P^* whose conditional expectation, given $\mathcal{F}_t$, is defined by

$$E_t^*[X] = \frac{1}{\phi(t)S_0(t)}E_t\left[\phi(T)S_0(T)X\right], \qquad t \in \mathcal{T}, \tag{2.37}$$

for any random variable X. For any price $S(t)$, we define the *relative* price with respect to the money market account $S_0(t)$ by

$$S^*(t) = \frac{S(t)}{S_0(t)}.$$

The next result can be easily proved using (2.6). The new probability measure P^* may be called a *risk-neutral* measure.

Theorem 2.5 *The relative insurance premium $p^*(t)$ as well as the relative security prices $S_j^*(t)$ is a martingale under P^*.*

For a risk $X(t)$ at future time t, the insurance premium $\mathcal{P}(X(t))$ at time 0 is given by the economic premium principle

$$\mathcal{P}(X(t)) \equiv E[X(t)\phi(t)] = E_0^*[X^*(t)]. \tag{2.38}$$

Here the equality follows from (2.37) with $t = 0$ and $T = t$. That is, the economic premium principle agrees with the risk-neutral valuation in finance, which calculates the expectation of relative price under the risk-neutral measure.

Now, as a special case, suppose that the utility functions of agents are all exponential. This case will be examined in the next section. Then, as will be shown later, the utility function U of the representative agent is also exponential, and the Arrow-Pratt index of absolute risk aversion is constant, ρ say. It follows from (2.36) that

$$\phi(t)S_0(t) = \frac{e^{-\rho Z(t)}}{E\left[e^{-\rho Z(t)}\right]},$$

the *Esscher transform*. The connection between the Esscher transform and the martingale measure has been discussed by Bühlmann el al. (1998).

Remark 2.2 Although we do not present here, the results obtained so far can be transferred to the continuous-time setting as they stand. The advantage of the continuous-time model is that an optimal consumption/portfolio process can be obtained as a closed form solution for some particular cases such as power and exponential utility functions. We will report such results elsewhere.

3 Some Special Cases

In this section, we consider an equilibrium market $\mathcal{M}$ with some special utility functions for the agents. Namely, we study the cases of power and exponential utility functions, and show that the state price density $\phi(t)$ can be expressed in terms of the aggregated net endowment $Z(t)$, the discount function $\beta(t)$ and the parameter of the utility function. Recall that the general form (2.35) in Theorem 2.4 includes the unknown values λ_i^*. However, in the special cases, these parameters are expressed by the initial aggregated net endowment in equilibrium.

3.1 Power Utility Functions

First, we consider the case in which each agent has a utility function defined by

$$U_i(x) = \gamma_i^{1-\alpha}\frac{x^\alpha}{\alpha}, \quad 0 < x < \infty,\ \gamma_i > 0,\ \alpha \in (-\infty, 1). \tag{3.1}$$

Note that every agent has the common shape parameter α. Also, the case $\alpha = 0$ corresponds to the logarithmic utility function.

Now, it is easily seen that the inverse of the marginal utility is given by

$$I_i(y) = \gamma_i y^{-1/(1-\alpha)}, \qquad y \in (0, \infty).$$

It follows from (2.27) that

$$I(y; \mathbf{\Lambda}) = \sum_{i=1}^{n} I_i \left(\frac{y}{\lambda_i} \right) = \left(\sum_{i=1}^{n} \gamma_i \lambda_i^{1/(1-\alpha)} \right) y^{-1/(1-\alpha)}.$$

Note that, in equilibrium, we have from (2.28) with $t = 0$ that

$$Z(0) = I(1; \mathbf{\Lambda}^*) = \sum_{i=1}^{n} \gamma_i (\lambda_i^*)^{1/(1-\alpha)}.$$

Since $\mathcal{H}(x; \mathbf{\Lambda})$ is the inverse of $I(y; \mathbf{\Lambda})$, we obtain

$$\mathcal{H}(x; \mathbf{\Lambda}^*) = \left(\frac{x}{Z(0)} \right)^{-(1-\alpha)}, \qquad x \in (0, \infty).$$

It follows from (2.30) that

$$\phi(t) = e^{-\sum_{u=0}^{t} \beta(u)} \left(\frac{Z(t)}{Z(0)} \right)^{-(1-\alpha)}, \qquad t \in \mathcal{T}.$$

Especially, in the case that $S_0(t) = e^{\sum_{u=0}^{t} \beta(u)}$ as in Corollary 1, we have

$$\phi(t) S_0(t) = \left(\frac{Z(t)}{Z(0)} \right)^{-(1-\alpha)}, \qquad t \in \mathcal{T}.$$

Since $E[\phi(t) S_0(t)] = 1$, it follows that

$$\phi(t) S_0(t) = \frac{Z^{-(1-\alpha)}(t)}{E[Z^{-(1-\alpha)}(t)]}, \qquad t \in \mathcal{T}.$$

The economic premium principle for this case is given, from (2.38), by

$$\mathcal{P}(X(t)) = \frac{E[X^*(t) Z^{-(1-\alpha)}(t)]}{E[Z^{-(1-\alpha)}(t)]}, \qquad t \in \mathcal{T}. \tag{3.2}$$

3.2 Exponential Utility Functions

Next, we consider the case in which each agent has a utility function defined by

$$U_i(x) = \frac{1 - e^{-\gamma_i x}}{\gamma_i}, \quad 0 < x < \infty, \quad \gamma_i > 0. \tag{3.3}$$

Using similar arguments to the above, it is not difficult to derive

$$I(y;\mathbf{\Lambda}^*) = -\sum_{i=1}^{n} \frac{\log y}{\gamma_i} + Z(0)$$

Letting γ be such that

$$\frac{1}{\gamma} = \sum_{i=1}^{n} \frac{1}{\gamma_i},$$

we then obtain

$$\mathcal{H}(x;\mathbf{\Lambda}^*) = e^{-\gamma(x-Z(0))}, \qquad x \in (0,\infty).$$

It follows that

$$\phi(t) = \exp\left\{-\sum_{u=0}^{t} \beta(u) - \gamma(Z(t) - Z(0))\right\}, \qquad t \in \mathcal{T}.$$

Especially, in the case that $S_0(t) = e^{\sum_{u=0}^{t}\beta(u)}$, we have

$$\phi(t)S_0(t) = \frac{e^{-\gamma Z(t)}}{E[e^{-\gamma Z(t)}]}, \qquad t \in \mathcal{T}.$$

The economic premium principle for this case is given by

$$\mathcal{P}(X(t)) = \frac{E[X^*(t)e^{-\gamma Z(t)}]}{E[e^{-\gamma Z(t)}]}, \qquad t \in \mathcal{T}.$$

Especially, if $X^*(t) = Z(t)$, then

$$\mathcal{P}(S_0(t)Z(t)) = \frac{E[Z(t)e^{-\gamma Z(t)}]}{E[e^{-\gamma Z(t)}]}, \qquad t \in \mathcal{T},$$

which is called the *Esscher principle*.

4 Some Comparative Statics

We have seen that the present value of risk $X(t)$ is calculated as (2.38). In this section, we assume that the conditions of Corollary 1 hold and consider a risk given by $X^*(t) = f(Z(t))$ for a certain function f. If the utility function of the representative agent is given by U, then the present value of this risk is given by[4]

$$\mathcal{P}_t(f;U) = \frac{E[f(Z(t))U'(Z(t);\mathbf{\Lambda}^*)]}{E[U'(Z(t);\mathbf{\Lambda}^*)]}. \tag{4.1}$$

Now, consider two economies, E_1 and E_2 say. We denote the utility function of the representative agent of economy E_k by U_k, $k = 1,2$. In this section, we compare the present value of risk $X(t)$ with respect to their marginal utilities.

First, according to Kijima and Ohnishi (1996), we have the following.

[4] If $f(x) = x$, the right hand side of (4.1) is called the generalized harmonic mean (GHM), which has been studied extensively in the context of portfolio selection problems. See Kijima and Ohnishi (1999) for details.

Lemma 4.1 *Let U_i, $i = 1, 2$, be utility functions such that $\frac{U_2'(x)}{U_1'(x)}$ is nonincreasing in x. If f is nonincreasing (nondecreasing, respectively) then, for any risk $X(t)$ such that $X^*(t) = f(Z(t))$, we have*

$$\mathcal{P}_t(f;U_1) \leq \ (\geq)\ \mathcal{P}_t(f;U_2).$$

Let $\rho(x;U)$ denote the Arrow–Pratt index of risk-aversion for the representative agent whose utility function is U. The agent with utility function U_2 is more risk averse than the agent with U_1, if $\rho(x;U_1) \leq \rho(x;U_2)$. According to Jewitt (1989), $\rho(x;U_1) \leq \rho(x;U_2)$ if and only if $\frac{U_2'(x)}{U_1'(x)}$ is nonincreasing in x, whence we have the following.

Theorem 4.1 *Suppose that the representative agent in economy E_2 is more risk averse than the one in E_1. If f is nonincreasing (nondecreasing, respectively) then, for any risk $X(t)$ such that $X^*(t) = f(Z(t))$, we have*

$$\mathcal{P}_t(f;U_1) \leq \ (\geq)\ \mathcal{P}_t(f;U_2).$$

In our model, the aggregated net endowment $Z(t)$ is the difference between the aggregated endowments and risk. Hence, of interest is the case that f is nonincreasing. The next results are intuitively appealing.

Corollary 2 *Suppose that f is nonincreasing and that risk $X(t)$ is given by $X^*(t) = f(Z(t))$.*

1. *Suppose that each agent in economy E_k has the power utility with the shape parameter α_k. If $\alpha_1 \leq \alpha_2$, then $\mathcal{P}_t(f;U_1) \leq \mathcal{P}_t(f;U_2)$.*

2. *Suppose that agent i in economy j has the exponential utility with parameter γ_i^k. If $\sum_{i=1}^{n_1} \frac{1}{\gamma_i^1} \leq \sum_{i=1}^{n_2} \frac{1}{\gamma_i^2}$, then $\mathcal{P}_t(f;U_1) \leq \mathcal{P}_t(f;U_2)$.*

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