FAIR PRICING OF LIFE INSURANCE PARTICIPATING POLICIES WITH A MINIMUM INTEREST RATE GUARANTEED*

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Abstract

In this paper we analyse, in a contingent-claims framework, one of the most common life insurance policies sold in Italy during the last two decades. The policy, of the endowment type, is initially priced as a standard one, given a *risk-neutral mortality probability measure* and a *technical interest rate*. Subsequently, at the end of each policy year, the insurance company grants a *bonus*, which is credited to the mathematical reserve and depends on the performance of a special investment portfolio. More precisely, this bonus is determined in such a way that the total interest rate credited to the insured equals a given percentage (*participation level*) of the annual return on the reference portfolio and anyway does not fall below the technical rate (*minimum interest rate guaranteed*, henceforth). Moreover, if the contract is paid by periodical premiums, it is usually stated that the annual premium is adjusted at the same rate of the bonus, and thus the benefit is also adjusted in the same measure. In such policy the variables controlled by the insurance company (*control-variables*, henceforth) are the technical rate, the participation level and, in some sense, the riskiness of the reference portfolio measured by its volatility. We derive necessary and sufficient conditions under which each control-variable is uniquely determined, given the remaining two ones, by an arbitrage (*fair*) pricing of the contract.

Keywords: *Policies with profits, Minimum guarantee, Fair pricing, Black and Scholes framework*

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* Submitted for publication to *Astin Bulletin*.

1. Introduction

At the end of the seventies a new kind of life insurance product, the so-called *rivalutabile*, was introduced in Italy, together with the index-linked policies[1], in order to match the high level of inflation that led the returns on Treasury Bonds and fixed-income securities up to 20% p.a.. The interest rate of 3% p.a. commonly guaranteed by traditional life insurance products was indeed completely inadequate and seriously jeopardized the marketability of such products.

The term *rivalutabile* identifies the Italian version of the widely known *participating* policy, or policy *with profits* (*Universal Life Insurance*, in the United States). In Italy a special portfolio of investments, covering at least the mathematical reserves of all the policies with profits issued by a same insurance company, is constituted and kept apart from the other assets of the company. Within the end of each calendar year the rate of return on this portfolio (*reference portfolio*, henceforth) in the preceding financial year is computed and certified by a special auditor. The financial year usually begins on November 1st and ends on October 31st. A percentage of this rate of return, that is defined every year and usually cannot be less than a fixed level (e.g. 70%), is granted to the insured. More precisely, if the granted rate of return exceeds the technical interest rate already included in the premium calculation, a bonus computed at the excess rate is credited to the mathematical reserves of all the participating policies when they reach their anniversary (i.e., at the end of the policy year). Observe that, in this way, the technical rate becomes a *minimum interest rate guaranteed.*

Policies with profits are very often paid by annual premiums. If this is the case, it is usually stated that the annual premium increases at the same excess rate credited to the mathematical reserve so that, as like as in the single premium contracts, also the benefits are adjusted in the same measure in order to maintain the actuarial equilibrium with regard to the residual policy period.

Since the pioneering work by Brennan and Schwartz (1976, 1979a, 1979b) and Boyle and Schwartz (1977), a great prominence has been given so far in the financial and actuarial literature to the issues of pricing and hedging equity-linked life insurance contracts with minimum guarantees. In contrast with this, participating policies have not been studied very much in a contingent-claims framework, although they are the most important life insurance products in terms of market size. This is probably due to the fact

[1] Actually, the first index-linked policy traded in Italy dates back to 1968.

that the minimum interest rate guaranteed used to be far lower than the market rates, and therefore the risk associated to the issue of the guarantee seemed to be quite negligible and was not seriously considered a threat to the solvency of a life insurance company. Now that the economic setting has dramatically capsized in most industrial countries and the market interest rates have sunk up to very low levels[2], this threat has become impending. Then an accurate assessment of all the parameters characterizing the guarantees and the bonus mechanism constitutes a crucial problem in the management of a life insurance company.

Some recent contributions in this direction are due to Briys and de Varenne (1997), Miltersen and Persson (1999), Grosen and Jørgensen (1999).

Briys and de Varenne consider a single-period valuation model for the equities and the liabilities of a life insurance company. In particular the policyholders, i.e., the "owners" of the liabilities, earn a minimum interest rate guaranteed plus a bonus. The bonus is given by a percentage (*participation level*) of the difference, if positive, between the final value of the assets times the initial ratio between liabilities and assets, and the minimum guaranteed final value of liabilities. In their valuation model Briys and de Varenne take into account also the risk of default. Under the assumption that the assets follow a lognormal process and the *stochastic* interest rates behave as in Vasicek (1977), they obtain a closed-form solution both for equities and for liabilities. They also derive an equilibrium condition which relates, by an explicit formula, the participation level to the minimum interest rate guaranteed.

Miltersen and Persson consider a multiperiod valuation model in which the "customers" (i.e., the policyholders) are entitled to two different accounts: the "customer's account" and the "bonus account". The customer's account earns, at the end of each year, a minimum interest rate guaranteed plus a percentage of the positive excess between the realized rate of return on a benchmark portfolio and the promised minimum rate. The bonus account, instead, is a sort of buffer that receives, in "good" years, an additional percentage of the positive difference between the above mentioned rates and, in "bad" years, is used for fulfilling the minimum guarantee promise. At maturity, if the bonus account is negative, the deficit is anyway absorbed by the insurance company. Under the Black and Scholes (1973) framework, Miltersen and Persson derive a closed-form solution for the customer's account and use instead the Monte Carlo approach for valuing the bonus account. They also derive an equilibrium condition which relates the

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2 E.g., the return on short-term Italian Treasury zero-coupon-bonds is about 3% p.a..

participation levels, the volatility parameter characterizing the return on the benchmark, and the annual minimum interest rates guaranteed.

Grosen and Jørgensen consider, as Miltersen and Persson, a multiperiod valuation model, and split the Liability Side of the Balance Sheet into two components: the "policy reserve" and the "bonus reserve" (or simply "buffer"). At the end of each policy year the policy reserve earns the maximum between a minimum interest rate guaranteed and a percentage of the (positive) difference between the ratio buffer/policy reserve valued at the end of the preceding year and a target buffer ratio. Grosen and Jørgensen model the assets à la Black and Scholes, and obtain a martingale representation formula for the value of the participating policy, which is computed by means of Monte Carlo simulation. In particular, they decompose the contract into a risk-free bond element, a bonus option and a surrender option. While the bonus option is of European style, the surrender option is of American style.

All the above mentioned authors consider a single-payment contract in which the mortality risk is not taken into account. The object of this paper is the *fair pricing* of a real life insurance participating policy that couples the mortality risk with the financial elements and is paid either by a single premium or by a sequence of periodical premiums.

The policy, of the endowment type, exhibits almost all the features of the Italian products, and in particular the same pricing technique. This technique consists in computing the (initial) net premium, single or annual, as in the case of a standard endowment policy, given the initial sum insured (benefit) and according to a technical interest rate and to death probabilities extracted from a risk-neutral mortality table, hence completely disregarding the financial risk connected to the technical rate guarantee. Then, at the end of each policy year, the benefit and the periodical premium are adjusted according to the bonus mechanism.

By "fair pricing" we mean pricing consistent with no-arbitrage in the financial markets. Therefore, since the rules for computing the premium(s) are anyway fixed, a fair pricing is feasible by suitably choosing the parameters characterizing the contract. The contractual parameters, "controlled" by the insurance company, are the participation level and the technical (or minimum guaranteed) interest rate. Moreover, another parameter which, in some sense, can be also "controlled" by the insurance company is the riskiness of the investments composing the reference portfolio, measured by a volatility coefficient. If, in particular, this volatility is high, there is a good chance of high bonus returns for the insured being the "bad performances" anyway neutralized by the minimum interest rate guarantee. In this case the insured may be satisfied with a lower minimum rate guaranteed and/or a lower participation level. Moreover, it is quite intuitive that there is also a trade-off between the participation level and the minimum rate.

We suggest that the insurance company, instead of keeping together the investments concerning all the participating policies issued, graduates several reference portfolios according to their volatility, and thus offers its customers the choice among different triplets of technical rate, participation level, volatility.

Under the Black and Scholes assumption for the evolution of the reference portfolio and assuming independence between mortality risk and financial risk, we derive a very simple closed-form fairness (or arbitrage) relation, the same both in the case of a single premium and in that of periodical premiums. We then give necessary and sufficient conditions under which each one of the three control-parameters is uniquely (and quasi-explicitly) determined given the remaining two ones and the market instantaneous riskless interest rate. We act in perfectly competitive and frictionless markets, and we do not consider either expenses and connected loadings, or the presence of a surrender option.

The paper is organized as follows. In Section 2 we formalize the structure of the policy and of the bonus mechanism. Section 3 starts with the presentation of our valuation framework and ends with the definition of the arbitrage condition. In Section 4 we derive the fairness relation and give the conditions under which each control-parameter is uniquely determined. Moreover, we present some numerical examples of sets of parameters satisfying this relation. Section 5 concludes the paper.

2. The structure of the policy

Consider a single endowment policy (or a cohort of identical endowment policies) issued at time 0 and maturing at time T. We denote by x the entry age, by C_0 the "initial" sum insured, and by i the annual compounded technical interest rate.

2.1 Single premium contracts

If the policy is paid by a single amount U at the initiation of the contract, and the benefit is assumed to be due at the end of the year of death $t=1,2,...,T$ or, at the latest, at maturity T, the following relation defines U:

$$(1)\quad U = C_0 \overset{(i)}{A_{x:\overline{T}|}} = C_0\left(\sum_{t=1}^{T-1} {}_{t-1/1}q_x\, v^t + {}_{T-1}p_x\, v^T\right),$$

where $v=(1+i)^{-1}$, ${}_{t-1/1}q_x$ represents the probability that the insured dies during the t-th year of contract (i.e., between times t−1 and t), and ${}_{T-1}p_x$ represents the probability that the insured is alive at time T−1 (i.e., he/she dies during the last year of contract or survives the term of the contract).

Note that, as it is standard in actuarial practice, all these probabilities are extracted from a risk-adjusted mortality table, i.e., *they are not "true" probabilities but risk-neutral ones*. This does not mean that the insurance company is risk-neutral with respect to mortality; on the contrary, insurers are always risk-averse. However, in the recent literature on equity-linked policies with minimum guarantees, focusing above all on the management of financial risk, it is usual to assume risk-neutrality with respect to mortality by invoking the *pooling* argument; thus mortality is treated as like as it were deterministic. One of the main concerns of a life insurance company is indeed the possibility of systematic deviations between expected and realized mortality, especially for pure-endowment and annuity contracts ("longevity risk"; see, e.g., Macdonald, Cairns, Gwilt and Miller (1998), Benjamin and Soliman (1993)). Traditionally the insurer protects oneself against this risk by adjusting the mortality probability measure and, in this way, the premiums are implicitly charged by a "safety loading". For instance, in an endowment policy, the risk-adjusted probabilities of death within the term of the contract will be higher than the "true" ones, whereas the probability of survival will be lower. Then market competition should lead to a unique adjusted probability measure for all the insurance companies with respect to the pricing of identical policies, and market completeness to the same mortality measure for "identical" individuals even with respect to different kinds of policies.

Observe, moreover, that relation (1) disregards expense loadings, implicitly assuming the absence of expenses or, alternatively, the perfect matching between expenses and corresponding loadings.

We assume that, at the end of the t-th policy year, if the contract is still in force, the mathematical reserve is adjusted at a rate δ_t ("bonus rate") defined as follows:

$$(2)\quad \delta_t = \max\left\{\frac{\eta g_t - i}{1+i}\ ,\ 0\right\},\quad t=1,2,\dots,T.$$

The parameter η, between 0 and 1, denotes the participation level, for simplicity assumed to be constant in time, and g_t denotes the annual return on the reference portfolio. Relation (2) formally translates the fact that the total interest rate credited to the mathematical reserve during the t-th policy year, $(1+i)(1+\delta_t)-1$, equals the maximum between i and ηg_t, i.e., that i is a minimum rate of return guaranteed to the policyholder.

Since we are dealing with a single premium contract, the bonus credited to the mathematical reserve implies a proportional adjustment, at the rate δ_t, also of the sum insured. In particular, if the insured dies within the term of the contract, we assume that the benefit profits by an additional (last) adjustment just before being paid at the end of the year of death. This is in contrast with what happens in Italy for participating policies, where the amount of the benefit due in a given policy year is fixed at the beginning of the year and therefore there is a sort of predictability with respect to the relevant information characterizing the financial uncertainty. We point out that our assumption is not motivated by the wish of obtaining closed-form solutions since, under the valuation framework depicted in the next section, the market value of the policy would anyway be expressible in closed-form. However, as we will see in the sequel of the paper, it is just this assumption that allows us to derive a very simple and explicit fairness relation, depending only on four variables: the participation level, the technical interest rate, the volatility of the reference portfolio, and the market interest rate.

Denoting by C_t, t=1,2,...,T, the benefit paid at time t if the insured dies between ages x+t−1 and x+t or, for t=T, in case of survival, the following recursive relation links then the benefits of successive years:

(3) $$C_t = C_{t-1}(1+\delta_t), \qquad t=1,2,\ldots,T.$$

The iterative expression for them is instead:

(4) $$C_t = C_0 \prod_{j=1}^{t}(1+\delta_j), \quad t=1,2,\ldots,T.$$

2.2 Periodic premium contracts

Assume now that the policy is paid by a sequence of periodical premiums, due at the beginning of each year of contract, if the insured is alive. The initial premium, P_0, paid at time 0, is given by

$$(5)\quad P_0 = C_0\, \overset{(i)}{P}_{x:\overline{T}|} = C_0 \frac{\overset{(i)}{A}_{x:\overline{T}|}}{{}_{/T}\ddot{a}^{(i)}_x} = C_0 \frac{\sum_{t=1}^{T-1} {}_{t-1/1}q_x\, v^t + {}_{T-1}p_x\, v^T}{\sum_{t=0}^{T-1} {}_tp_x\, v^t},$$

where the death probabilities ${}_{t-1/1}q_x$ and the survival probabilities ${}_tp_x$ are extracted from the same risk-adjusted table introduced in the previous subsection. Moreover, most of the considerations and assumptions made in that subsection are still valid, in particular the bonus mechanism described by relation (2).

In Italy it is usual that the periodical premium of a participating policy is annually adjusted at the same bonus rate δ_t credited to the mathematical reserve. In this case, denoting by P_t, $t=1,2,...,T-1$, the (t+1)-th premium paid at time t, if the insured is alive, one has

$$(6)\quad P_t = P_{t-1}(1+\delta_t), \qquad t=1,2,...,T-1$$

or, alternatively,

$$(7)\quad P_t = \begin{cases} P_0 & t=0 \\ P_0 \prod_{j=1}^{t}(1+\delta_j) & t=1,2,...,T-1 \end{cases} .$$

If this is the case, the benefit C_t is also adjusted in the same measure, so that relation (3) or, alternatively, (4), still holds.

In this paper we also make this assumption of identical adjustment rates for the mathematical reserve, the premium and the benefit. However, we observe that sometimes it could be instead stated that the adjustment rate of the periodical premium is only a fraction, for instance one half, of δ_t, or even 0 (i.e., the premiums are constant). In these cases an actuarial equilibrium relation concerning the residual policy period imposes that the adjustment rate of the benefit is a suitable mean of the remaining two adjustment rates (see, e.g., Pentikäinen (1968)). Unfortunately this mean turns out to be path-dependent, and therefore it is hard to obtain closed-form relations for the market value of the contract.

3. The valuation model

In this section we describe, first of all, the basic assumptions concerning the financial set-up. Then, observing that both the periodical premiums and the benefit are typical contingent-claims, we apply the martingale approach put forward by Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983) to obtain a valuation formula for them. Finally, the mortality risk comes into play in order to establish a fairness condition in the pricing of the contract.

3.1 Assumptions

Assume that markets are populated by rational and non-satiated agents, aiming at maximizing their profits. Moreover, let markets be perfectly competitive and frictionless (in particular, arbitrage opportunities are ruled out of them), and let trading take place continuously.

We assume that the continuously compounded riskless interest rate in the economy is deterministic and constant, and denote it by r. Therefore, in our framework, there is a unique source of financial uncertainty, reflected by a stochastic evolution of the reference portfolio whose performance determines the bonus mechanism. Assume that this uncertainty is generated by a standard brownian motion W, defined on a filtered probability space $(\Omega, \mathcal{F}, Q)$ in the time interval [0,T]. In particular, Q represents the equivalent martingale measure, under which the continuously discounted price of any financial security is a martingale (see Harrison and Kreps (1979)), and $(\mathcal{F}_t, 0 \leq t \leq T)$ is a filtration, satisfying the usual conditions and representing the revelation of information.

We assume that the reference portfolio is a well-diversified one, and that dividends, coupons or whatever else yielded by the assets composing it are immediately reinvested in the same portfolio and thus contribute to increase its unit price. We assume in fact that this portfolio is split into shares, or *units*. Therefore its annual returns are completely determined by the evolution of its unit price and not by that of its total value, which reflects also new investments (corresponding, for instance, to the payment of periodical premiums or to the entry of new policies into the portfolio) and withdrawals (when some policy expires). We denote by G_t the unit price at time t of the reference portfolio and model it, under Q, as a geometric brownian motion:

(8) $\frac{dG_t}{G_t} = r\,dt + \sigma\, dW_t, \qquad t\in[0,T],$

with the constant σ representing the volatility parameter and G_0 given. As it is well known, the solution to the stochastic differential equation (8) is given by

(9) $G_t = G_0 \exp\left\{\left(r-\frac{\sigma^2}{2}\right)t + \sigma\, W_t\right\}, \qquad t\in[0,T].$

We assume that the annual compounded rates of return g_t introduced in the previous section are defined as

(10) $g_t = \frac{G_t}{G_{t-1}} - 1, \qquad t=1,2,...,T$ [3],

so that $1+g_t=\exp\left\{r-\sigma^2/2+\sigma(W_t-W_{t-1})\right\}$ are independent and identically distributed (i.i.d.) for t=1,2,...,T and their logarithms, representing continuously compounded rates of return, are all independent and normally distributed with mean $r-\sigma^2/2$ and variance σ^2. Therefore also the bonus rates δ_t defined by relation (2) of Section 2 turn out to be i.i.d..

Finally, we assume independence between mortality and the financial elements, so that the valuation of the contract can be performed in two separate stages: in the first stage premiums and benefits defined by relations (7) and (4) of Section 2 are priced as like as they were (purely-financial) contingent-claims due with certainty at a fixed (future) date; in the second stage their time 0 prices are "weighted" with the risk-neutral life and mortality probabilities introduced in Section 2 in order to get a "fair" price of the contract.

3.2 Fair valuation of single premium contracts

To value these contracts, we first need to compute, for any t=1,2,...,T, the market value of the contingent-claim C_t, defined by relation (4) of the previous section and assumed to be due with certainty at time t. To this end we exploit the martingale approach put forward by Harrison and Kreps (1979) and Harrison and Pliska (1981,

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3 As described in the Introduction, the annual rate of return on the reference portfolio for Italian participating policies is actually referred to a financial year, that generally ends at least two months before a policy year. Here, for simplicity, we have instead assumed that g_t is referred to a policy year.

1983) and express the time 0 price of C_t, denoted by $\pi(C_t)$, as the following expectation under the risk-neutral measure Q:

$$(11)\quad \pi(C_t) = E^Q\left[\exp\{-rt\}C_t\right], \qquad t=1,2,...,T.$$

Exploiting relations (4) and (2) of Section 2 together with the stochastic independence of the bonus rates δ_j for j=1,2,...,T, and after some algebraic manipulations, we get then

$$(12)\quad \pi(C_t) = C_0 \prod_{j=1}^{t}\left(\exp\{-r\}+\frac{\eta}{1+i}\, E^Q\left[\exp\{-r\}\max\left\{(1+g_j)-(1+i/\eta),\, 0\right\}\right]\right),$$

$$t=1,...,T.$$

Recalling that $1+g_j$ are, for any j, identically and lognormally distributed with, in particular, the same distribution as the time 1 stock price in the classical Black and Scholes (1973) model (given a time 0 price of the stock equal to 1), it is immediate to realize that the Q-expectation into the round brackets in the RHS of relation (12) represents the time 0 value of a European call option on a non dividend paying stock with initial price equal to 1, option with maturity 1 and strike price equal to $1+i/\eta$. Denoting this value by c, we have then

$$(13)\quad \pi(C_t) = C_0\left(\exp\{-r\}+\frac{\eta}{1+i}\, c\right)^t, \qquad t=1,2,...,T,$$

with c given by the classical Black and Scholes (1973) formula:

$$(14)\quad c = F(d_1) - \left(1+\frac{i}{\eta}\right)\exp\{-r\}F(d_2),$$

where $d_1 = \dfrac{r + \frac{\sigma^2}{2} - \ln\left(1 + \frac{i}{\eta}\right)}{\sigma}$, $d_2 = d_1 - \sigma$, and F denotes the cumulative distribution function of a standard normal variate.

The fair price of the single premium contract analysed in this paper, FVB, can be obtained by summing up, for t=1,2,...,T, the time 0 values of C_t weighted with the risk-neutral probabilities introduced in Section 2 that they are exactly due at time t:

$$(15)\quad FVB = C_0\left(\sum_{t=1}^{T-1} {}_{t-1/1}q_x\, v_*^t + {}_{T-1}p_x\, v_*^T\right) = C_0\, A_{x:\overline{T}|}^{(i_*)},$$

where $v_* = \exp\{-r\} + \frac{\eta}{1+i} c$ and $i_* = v_*^{-1} - 1$.

Then the contract is fair if and only if the single premium U equals FVB, i.e., recalling relation (1) of Section 2, if and only if the following condition is satisfied:

$$(16)\quad \overset{(i)}{A}_{x:\overline{T}|} = \overset{(i_*)}{A}_{x:\overline{T}|}.$$

3.3 Fair valuation of periodic premium contracts

Most of what said in the previous subsection for single premium contracts is still valid in the case of periodical premiums. In particular the fair value of the benefit is still given by relation (15), while the fair value of the sequence of periodical premiums, FVP, is given by

$$(17)\quad FVP = \sum_{t=0}^{T-1} {}_tp_x \pi(P_t),$$

where $\pi(P_t) = E^Q[\exp\{-rt\}P_t]$ represents the time 0 price of the contingent-claim P_t, defined by relation (7) of Section 2 and assumed to be paid with certainty at time t. Exploiting the same arguments employed in the previous subsection, we have then

$$(18)\quad \pi(P_t) = \begin{cases} P_0 & t=0 \\ P_0 v_*^t & t=1,2,\dots,T-1 \end{cases},$$

so that

$$(19)\quad FVP = P_0 \sum_{t=0}^{T-1} {}_tp_x v_*^t = P_0\, {}_{/T}\ddot{a}_x^{(i_*)}.$$

The fairness requirement implies now that the fair value of the benefit, FVB, equals the fair value of the premiums, FVP, i.e., that

$$(20)\quad C_0 \overset{(i_*)}{A}_{x:\overline{T}|} = P_0\, {}_{/T}\ddot{a}_x^{(i_*)}.$$

Recalling the definition of P_0 given in relation (5) of Section 2, we conclude this subsection by stating that the contract is fair if and only if the following condition holds:

$$(21)\quad \overset{(i)}{P}_{x:\overline{T}|} = \overset{(i_*)}{P}_{x:\overline{T}|},$$

being $\overset{(i_*)}{P}_{x:\overline{T}|} = \dfrac{\overset{(i_*)}{A}_{x:\overline{T}|}}{{}_{/T}\overset{(i_*)}{\ddot{a}}_{x}}$.

4. The fairness relation

In the previous section we have seen that a participating policy is fairly priced if and only if $K(i)=K(i_*)$, being

$$K(y) = \overset{(y)}{A}_{x:\overline{T}|} = \sum_{t=1}^{T-1} {}_{t-1/1}q_x\,(1+y)^{-t} + {}_{T-1}p_x\,(1+y)^{-T}$$

for single premium contracts, and

$$K(y) = \overset{(y)}{P}_{x:\overline{T}|} = \frac{\sum_{t=1}^{T-1} {}_{t-1/1}q_x\,(1+y)^{-t} + {}_{T-1}p_x\,(1+y)^{-T}}{\sum_{t=0}^{T-1} {}_{t}p_x\,(1+y)^{-t}}$$

for periodic premium ones (see relations (16) and (21) respectively). Since, in both cases, K is a strictly decreasing function of y, then both conditions (16) and (21) are satisfied if and only if $i=i_*$, i.e., if and only if the following simple relation holds:

$$(22)\quad \exp\{-r\}(1+i) + \eta c - 1 = 0.$$

Note that relation (22) depends only on four parameters: the market instantaneous interest rate r, the annual compounded technical rate i, the participation level η, and the volatility coefficient σ. While the rate r is exogenously given, the remaining parameters can be chosen by the insurance company, hence they are *control-variables*. In particular, i and η are directly fixed by the insurer, whereas σ can be indirectly determined by a suitable choice of the assets that compose the reference portfolio.

It is quite intuitive that relation (22) defines a trade-off between any pair of control-parameters, given the third one and r. If the minimum interest rate guaranteed i is high, then the insurance company cannot afford to fix a great participation level since, in "good" years (i.e., when $g_t > i$), it has to put aside a sufficient amount of non-distributed funds in order to be able to fulfil the minimum guarantee promise in "bad" years (when $g_t < i$). Similarly, a highly volatile reference portfolio can produce high returns as like as heavy losses. The losses, however, are entirely suffered by the insurer since the policyholder benefits of the minimum interest rate guarantee. Therefore in this case, to protect itself, the insurance company must keep the technical interest rate and/or the participation level down. In what follows this trade-off will formally turn out from the fact that all the partial derivatives with respect to the control-parameters i, η, σ of the function

$$(23) \quad g(r,i,\eta,\sigma):=\exp\{-r\}(1+i) + \eta c(r,i,\eta,\sigma) - 1,$$

with $c(r,i,\eta,\sigma):=c$ defined by relation (14), are of the same sign (in particular, positive).

In the remaining part of this section we will analyse, separately for each one of the three control-parameters, necessary and sufficient conditions under which there exists a unique solution to the equation (22) for any given positive value of r and once the insurance company has "fixed" the values of the other two control-parameters. Before doing this, however, observe that relation (22) is equivalent to

$$c = \frac{1 - \exp\{-r\}(1+i)}{\eta} .$$

Since the Black-Scholes value c is always strictly positive, a necessary (and indeed quite obvious) condition for a fair pricing of the contract is

$$(24) \quad i < \exp\{r\}-1$$

or, equivalently,

$$(25) \quad \ln(1+i) < r.$$

This condition states that the technical interest rate i must be strictly less than the annual compounded market rate $\exp\{r\}-1$ or, equivalently, that the continuously compounded technical rate, $\ln(1+i)$, must be less than r.

4.1 Solutions with respect to the technical rate i

Given a market rate $r>0$, imagine that the insurance company has already fixed the participation level η, between 0 and 1, and chosen the assets composing the reference portfolio, so that also $\sigma>0$ is given. We are now going to analyse if there exists a technical interest rate i, non negative and less than the annual compounded market rate $\exp\{r\}-1$, such that the fairness relation (22) holds, or, equivalently, such that the function g defined by relation (23) equals 0.

To this end observe, first of all, that

$$(26)\quad \frac{\partial g}{\partial i} = \exp\{-r\}\left[1-F(d_2)\right] > 0,$$

i.e., that g is strictly increasing with respect to i. Moreover, since

$$(27)\quad \sup_{i<\exp\{r\}-1} g(r,i,\eta,\sigma) = \lim_{i\to\exp\{r\}-1} g(r,i,\eta,\sigma) = \eta c(r,\exp\{r\}-1,\eta,\sigma) > 0,$$

then a necessary and sufficient condition under which there exists a unique solution to the equation $g(r,i,\eta,\sigma)=0$, is

$$(28)\quad \left(\min_{i\geq 0} g(r,i,\eta,\sigma) = g(r,0,\eta,\sigma) =\right) \exp\{-r\} + \eta c(r,0,\eta,\sigma) - 1 \leq 0.$$

Substituting relation (14) of Section 3 for the Black-Scholes price, condition (28) becomes

$$(29)\quad \eta \leq \frac{1-\exp\{-r\}}{F\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)-\exp\{-r\}F\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)}.$$

Observe that relation (29) defines an actual upper bound for η, i.e., that

$$(30)\quad h(r,\sigma) := \frac{1-\exp\{-r\}}{F\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)-\exp\{-r\}F\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)} < 1.$$

This is due to the facts that

$$(31)\quad \frac{\partial h}{\partial \sigma} = \frac{\left[\exp\{-r\}-1\right]\exp\left\{-\frac{1}{2}\left(\frac{r^2}{\sigma^2}+\frac{\sigma^2}{4}+r\right)\right\}}{\sqrt{2\pi}\left[F\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)-\exp\{-r\}F\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\right]^2} < 0,$$

i.e., h is strictly decreasing with respect to σ, and

$$(32)\quad \sup_{\sigma>0} h(r,\sigma) = \lim_{\sigma\to 0} h(r,\sigma) = 1.$$

Summing up, given $r>0$, $\sigma>0$, and $\eta\in(0,h(r,\sigma)]$, there exists a unique $i\in[0,\exp\{r\}-1)$ such that the fairness relation holds.

To get a numerical insight, in Tables 1 to 5 we provide some examples of solutions to equation (22) with respect to i for given values of r, η, σ. To this end we have fixed for r either a value very close to the actual Italian rates (3% p.a.), or a value very close to the Italian rates when the first policies with profits were launched (20% p.a.), and of course we have also considered some intermediate values between these two extremes.

TABLE 1

Solutions with respect to the technical rate i (basis points) when r=0.03

σ \ η	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**
0.05	305	304	297	276	238	180	95
0.10	304	288	239	156	41		
0.15	300	248	143				
0.20	289	193	28				
0.25	273	128					
0.30	252	57					
0.35	227						
0.40	200						
0.45	171						
0.50	141						
0.55	111						
0.60	80						
0.65	49						
0.70	18						

TABLE 2

Solutions with respect to the technical rate i (basis points) when r=0.05

σ \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	513	513	512	504	483	441	373	262	69
0.10	513	509	485	430	341	213	36		
0.15	512	491	422	305	141				
0.20	509	456	335	153					
0.25	502	409	233						
0.30	490	354	123						
0.35	474	294	8						
0.40	455	229							
0.45	433	162							
0.50	410	94							
0.55	384	26							
0.60	357								
0.65	330								
0.70	302								
0.75	274								
0.80	245								
0.85	217								
0.90	189								

TABLE 3

Solutions with respect to the technical rate i (basis points) when r=0.1

σ \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	1052	1052	1052	1051	1048	1031	990	905	726
0.10	1052	1052	1049	1030	982	893	750	524	138
0.15	1052	1049	1027	963	849	676	428	70	
0.20	1052	1039	981	862	677	419	71		
0.25	1051	1019	916	737	482	143			
0.30	1048	989	837	598	274				
0.35	1043	950	748	450	61				
0.40	1035	904	652	297					
0.45	1024	853	551	143					
0.50	1010	799	449						
0.55	994	741	345						
0.60	975	682	241						
0.65	955	621	138						
0.70	933	560	36						
0.75	909	499							
0.80	885	438							
0.85	860	378							
0.90	834	318							

TABLE 4

Solutions with respect to the technical rate i (basis points) when r=0.15

σ \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	1618	1618	1618	1618	1618	1613	1589	1523	1358
0.10	1618	1618	1618	1613	1588	1526	1408	1201	815
0.15	1618	1618	1611	1577	1497	1355	1131	786	206
0.20	1618	1616	1587	1507	1361	1135	808	335	
0.25	1618	1607	1545	1411	1194	886	462		
0.30	1618	1590	1487	1295	1009	621	108		
0.35	1616	1565	1416	1165	813	351			
0.40	1613	1532	1335	1027	611	79			
0.45	1607	1493	1247	883	406				
0.50	1599	1448	1154	736	202				
0.55	1588	1399	1057	588	1				
0.60	1575	1346	958	440					
0.65	1560	1291	858	295					
0.70	1542	1234	758	152					
0.75	1522	1176	659	12					
0.80	1501	1116	560						
0.85	1479	1057	464						
0.90	1456	997	369						

TABLE 5

Solutions with respect to the technical rate i (basis points) when r=0.2

σ \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	2214	2214	2214	2214	2214	2213	2200	2150	1997
0.10	2214	2214	2214	2213	2200	2158	2061	1870	1485
0.15	2214	2214	2212	2194	2138	2021	1818	1483	890
0.20	2214	2213	2200	2146	2028	1828	1518	1049	274
0.25	2214	2210	2172	2070	1884	1599	1189	597	
0.30	2214	2201	2129	1972	1716	1349	844	142	
0.35	2213	2184	2072	1857	1533	1088	495		
0.40	2212	2160	2003	1730	1340	821	149		
0.45	2209	2130	1925	1594	1141	554			
0.50	2204	2093	1839	1453	939	289			
0.55	2197	2050	1748	1309	738	30			
0.60	2187	2003	1654	1163	539				
0.65	2175	1953	1557	1017	344				
0.70	2161	1899	1458	872	153				
0.75	2145	1843	1359	729					
0.80	2127	1786	1260	589					
0.85	2107	1727	1161	452					
0.90	2085	1668	1064	318					

The results reported in Tables 1 to 5 do not require many comments. We only point out that, when the volatility parameter σ and/or the participation level η are low, the price c of the call option defined by relation (14) of Section 3 practically vanishes and then the rounded values of i and $\exp\{r\}-1$ coincide, in terms of basis points. Moreover, observe that with the actual Italian market rates (about 3%) and a volatility coefficient of 15-20%, there are non negative solutions for i only when $\eta \leq 30\%$ (see Table 1), whilst, for instance, when r=20% and σ=30%, a participation level between 70% and 80% leads to a fair technical rate between 8.44% and 1.42% (see Table 5).

4.2 Solutions with respect to the participation level η

Assume now that, given $r>0$, the insurance company has already fixed a technical interest rate $i \in [0, \exp\{r\}-1)$, and chosen a reference portfolio with volatility coefficient $\sigma>0$. We are then concerned with the determination of a participation level η, between 0 and 1, such that the contract is fair.

As in the case analysed in the previous subsection, we observe first of all that

$$(33)\quad \frac{\partial g}{\partial \eta} = c(r,i,\eta,\sigma) + \frac{i}{\eta}\exp\{-r\}F(d_2) > 0,$$

i.e., that g is strictly increasing also with respect to η. Moreover:

$$(34)\quad \inf_{\eta>0} g(r,i,\eta,\sigma) = \lim_{\eta\to 0} g(r,i,\eta,\sigma) = \exp\{-r\}(1+i) - 1 < 0,$$

$$(35)\quad \sup_{\eta<1} g(r,i,\eta,\sigma) = \lim_{\eta\to 1} g(r,i,\eta,\sigma) = \exp\{-r\}(1+i) + c(r,i,1,\sigma) - 1 > 0.$$

The first inequality follows immediately from the fact that $i<\exp\{r\}-1$. To establish the second one define

$$(36)\quad z(r,i,\sigma) := \exp\{-r\}(1+i) + c(r,i,1,\sigma) - 1,$$

$$(37)\quad f(y) := F'(y) = \frac{1}{\sqrt{2\pi}}\exp\{-y^2/2\},$$

and observe that

$$(38)\quad \frac{\partial z}{\partial \sigma} = \exp\{-r\}(1+i) f\left(\frac{r-\ln(1+i)}{\sigma} - \frac{\sigma}{2}\right) > 0,$$

i.e., z is strictly increasing with respect to σ, and

(39) $\inf_{\sigma>0} z(r,i,\sigma) = \lim_{\sigma\to 0} z(r,i,\sigma) = 0.$

Therefore, given $r>0$, $\sigma>0$, and $i\in[0,\exp\{r\}-1)$, there is a unique $\eta\in(0,1)$ such that $g(r,i,\eta,\sigma)=0$.

Tables 6 to 10 report some examples of solutions to the fairness condition with respect to η for given values of r, i, σ.

TABLE 6

Solutions with respect to the participation level η (b.p.) when r=0.03

i \ σ	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
0.000	5295	3140	2225	1724	1410	1195	1038	920	828	754
0.005	4929	2883	2029	1564	1273	1074	930	821	736	668
0.010	4522	2606	1819	1394	1128	947	816	717	640	578
0.015	4061	2299	1589	1208	971	810	694	606	538	483
0.020	3516	1948	1330	1000	797	659	559	484	426	380
0.025	2818	1514	1013	750	588	479	401	342	297	261
0.030	1494	737	463	324	240	185	146	118	97	81

TABLE 7

Solutions with respect to the participation level η (b.p.) when r=0.05

i \ σ	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
0.000	7167	4667	3427	2706	2238	1911	1670	1486	1341	1224
0.005	6930	4461	3259	2564	2114	1800	1569	1393	1254	1143
0.010	6672	4245	3083	2416	1985	1686	1466	1298	1166	1059
0.015	6392	4016	2898	2261	1851	1567	1358	1199	1074	974
0.020	6084	3771	2704	2099	1711	1443	1246	1097	979	885
0.025	5742	3507	2496	1926	1563	1312	1129	990	880	793
0.030	5358	3219	2271	1742	1405	1173	1005	876	776	696
0.035	4916	2898	2023	1539	1233	1023	870	755	664	592
0.040	4387	2527	1741	1311	1040	855	721	620	542	479
0.045	3702	2066	1396	1034	809	656	546	463	399	348
0.050	2525	1318	851	606	456	357	286	234	195	164

TABLE 8

Solutions with respect to the participation level η (b.p.) when r=0.1

i \ σ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	9232	7171	5687	4684	3977	3458	3062	2752	2503	2299
0.01	9069	6927	5449	4465	3777	3273	2890	2590	2350	2154
0.02	8878	6664	5198	4235	3567	3081	2712	2423	2193	2005
0.03	8655	6377	4930	3993	3348	2880	2526	2251	2030	1851
0.04	8392	6064	4643	3737	3117	2670	2333	2071	1862	1692
0.05	8082	5718	4333	3462	2872	2447	2129	1882	1685	1525
0.06	7712	5333	3994	3165	2608	2210	1912	1682	1499	1351
0.07	7262	4894	3615	2838	2320	1952	1678	1466	1299	1164
0.08	6698	4377	3181	2467	1996	1664	1418	1229	1081	961
0.09	5938	3731	2651	2022	1612	1326	1115	955	830	730
0.10	4699	2763	1887	1393	1079	864	708	591	500	429

TABLE 9

Solutions with respect to the participation level η (b.p.) when r=0.15

i \ σ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	9809	8516	7179	6130	5330	4711	4223	3831	3509	3243
0.01	9760	8375	7014	5964	5170	4558	4077	3691	3375	3114
0.02	9701	8224	6840	5791	5004	4400	3927	3548	3239	2982
0.03	9630	8061	6658	5612	4832	4237	3772	3401	3099	2848
0.04	9546	7884	6466	5424	4654	4069	3614	3251	2955	2711
0.05	9446	7694	6263	5228	4469	3895	3450	3096	2808	2571
0.06	9329	7488	6048	5022	4276	3715	3281	2936	2657	2427
0.07	9190	7264	5819	4806	4074	3527	3105	2771	2501	2278
0.08	9027	7020	5575	4577	3863	3331	2922	2599	2339	2125
0.09	8835	6752	5312	4334	3639	3125	2731	2420	2171	1966
0.10	8607	6456	5028	4073	3401	2906	2529	2233	1995	1800
0.11	8334	6124	4717	3791	3146	2673	2314	2034	1809	1626
0.12	8002	5748	4371	3482	2867	2421	2083	1820	1611	1440
0.13	7590	5309	3977	3134	2558	2142	1830	1587	1395	1239
0.14	7051	4775	3511	2727	2200	1822	1541	1324	1153	1015
0.15	6277	4063	2908	2212	1752	1427	1188	1005	862	748
0.16	4635	2711	1817	1309	987	767	610	493	405	336

TABLE 10

Solutions with respect to the participation level η (b.p.) when r=0.2

i \ σ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.00	9958	9234	8164	7189	6383	5727	5192	4750	4382	4072
0.01	9945	9155	8050	7063	6254	5600	5067	4629	4264	3957
0.02	9929	9068	7930	6932	6122	5469	4940	4506	4144	3841
0.03	9909	8975	7804	6797	5985	5336	4810	4379	4022	3722
0.04	9884	8875	7672	6656	5845	5198	4677	4251	3898	3602
0.05	9854	8767	7534	6510	5700	5057	4540	4119	3771	3480
0.06	9818	8650	7388	6358	5550	4912	4400	3985	3642	3355
0.07	9774	8524	7235	6200	5395	4762	4257	3847	3509	3228
0.08	9722	8388	7074	6036	5234	4608	4109	3706	3374	3097
0.09	9661	8241	6904	5864	5067	4448	3957	3561	3235	2964
0.10	9588	8083	6724	5684	4894	4283	3800	3411	3093	2828
0.11	9501	7910	6533	5495	4713	4112	3638	3257	2946	2688
0.12	9400	7723	6330	5296	4524	3933	3469	3098	2795	2544
0.13	9281	7519	6113	5086	4326	3747	3294	2932	2638	2395
0.14	9140	7295	5880	4864	4117	3551	3111	2760	2475	2240
0.15	8973	7048	5629	4625	3895	3345	2918	2580	2305	2080
0.16	8775	6773	5355	4369	3657	3125	2714	2389	2127	1911
0.17	8537	6464	5053	4089	3401	2889	2496	2187	1937	1734
0.18	8246	6108	4714	3779	3119	2632	2260	1968	1734	1543
0.19	7879	5690	4325	3428	2803	2345	1998	1727	1511	1335
0.20	7394	5172	3857	3012	2432	2012	1696	1452	1258	1101
0.21	6679	4466	3237	2474	1959	1592	1320	1111	948	817
0.22	5011	2999	2019	1450	1086	837	658	527	427	350

As far as the results reported in Tables 6 to 10 are concerned, we observe that, when r=3%, a reference portfolio with a medium or a high volatility produces a very low fair participation level. For instance, if σ=30%, a technical rate between 0 and 3% gives rise to a fair participation level between 22.25% and 4.63% (see Table 6). When instead r=20%, the fair participation levels are rather high. For instance, a 3%-value of the technical rate, very common in Italy at the end of the seventies, when the policies with profits were introduced, leads to fair participation levels between 99.09% and 67.97%, corresponding to volatility coefficients between 10% and 40% (see Table 10). This explains why the first participation contracts usually provided a minimum participation level of about 70%.

4.3 Solutions with respect to the volatility coefficient σ

We analyse now the problem of finding a volatility coefficient $\sigma > 0$ in order to satisfy the fairness relation, given a market rate $r > 0$ and once the insurance company has fixed a participation level $\eta \in (0,1)$ and a technical rate $i \in [0, \exp\{r\}-1)$.

Once again, we exploit the strict monotonicity of g with respect to the third control-parameter σ. Observe, in fact, that

$$(40)\quad \frac{\partial g}{\partial \sigma} = \eta\left(1+\frac{i}{\eta}\right)\exp\{-r\} f(d_2) > 0.$$

Moreover,

$$(41)\quad \inf_{\sigma>0} g(r,i,\eta,\sigma) = \lim_{\sigma\to 0} g(r,i,\eta,\sigma) = \begin{cases} [1-\exp\{-r\}](\eta-1) < 0 & \text{if } \frac{i}{\eta} < \exp\{r\}-1 \\ \exp\{-r\}(1+i)-1 \quad < 0 & \text{if } \frac{i}{\eta} \geq \exp\{r\}-1 \end{cases}$$

and

$$(42)\quad \sup_{\sigma>0} g(r,i,\eta,\sigma) = \lim_{\sigma\to+\infty} g(r,i,\eta,\sigma) = \exp\{-r\}(1+i) + \eta - 1.$$

Then a necessary and sufficient condition for the existence of a unique solution in σ to the equation $g(r,i,\eta,\sigma)=0$ is $\exp\{-r\}(1+i) + \eta - 1 > 0$. This condition produces the following (strictly positive) lower bound for η:

$$(43)\quad \eta > 1-\exp\{-r\}(1+i).$$

Summing up, given $r>0$, $i \in [0, \exp\{r\}-1)$ and $\eta \in (1-\exp\{-r\}(1+i), 1)$, there exists a unique $\sigma > 0$ such that the contract is fair.

Some numerical solutions with respect to σ for given values of r, i, η are reported in Tables 11 to 15.

TABLE 11

Solutions with respect to the volatility coefficient σ (b.p.) when r=0.03

i \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.000	7296	3387	2113	1475	1087	823	628	472	333
0.005	6478	3049	1907	1331	979	740	562	420	293
0.010	5670	2702	1693	1180	867	652	493	366	253
0.015	4855	2338	1466	1020	747	559	420	309	210
0.020	4002	1943	1218	845	615	457	340	247	164
0.025	3039	1484	927	639	461	338	248	175	112
0.030	1501	727	446	300	210	149	103	67	37

TABLE 12

Solutions with respect to the volatility coefficient σ (b.p.) when r=0.05

i \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.000	12700	5694	3534	2462	1814	1373	1047	786	555
0.005	11651	5325	3318	2314	1704	1289	981	734	515
0.010	10661	4958	3100	2164	1593	1203	913	681	476
0.015	9717	4590	2879	2010	1478	1114	844	627	435
0.020	8806	4219	2653	1853	1361	1024	773	572	393
0.025	7916	3840	2421	1690	1239	930	699	514	350
0.030	7031	3449	2178	1519	1111	831	622	454	305
0.035	6133	3038	1920	1336	975	725	539	390	258
0.040	5189	2591	1637	1136	825	610	449	321	208
0.045	4126	2071	1305	901	649	475	345	241	151
0.050	2596	1301	812	552	390	278	195	130	74

TABLE 13

Solutions with respect to the volatility coefficient σ (b.p.) when r=0.1

i \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	39052	11902	7183	4962	3643	2753	2097	1574	1110
0.01	29577	10939	6685	4638	3410	2576	1960	1468	1030
0.02	24522	10027	6197	4315	3175	2399	1822	1361	950
0.03	20928	9156	5714	3991	2939	2219	1682	1252	869
0.04	18076	8314	5234	3665	2700	2036	1540	1141	786
0.05	15663	7491	4751	3334	2455	1848	1394	1028	702
0.06	13526	6675	4260	2994	2203	1654	1242	910	615
0.07	11554	5849	3753	2638	1938	1450	1083	787	524
0.08	9652	4990	3213	2257	1653	1230	912	655	428
0.09	7694	4044	2608	1826	1330	981	719	507	321
0.10	5327	2829	1818	1262	907	658	470	320	190

TABLE 14

Solutions with respect to the volatility coefficient σ (b.p.) when r=0.15

i \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00		19576	11090	7543	5503	4146	3153	2364	1666
0.01		18123	10497	7183	5253	3961	3012	2255	1585
0.02		16814	9926	6829	5005	3777	2871	2147	1505
0.03		15618	9373	6481	4759	3593	2729	2038	1424
0.04		14511	8835	6138	4515	3409	2588	1929	1343
0.05	42644	13478	8311	5797	4270	3225	2446	1820	1262
0.06	32366	12503	7797	5459	4026	3040	2303	1709	1180
0.07	27169	11575	7291	5121	3780	2853	2158	1597	1097
0.08	23504	10684	6790	4782	3532	2664	2012	1484	1013
0.09	20597	9821	6290	4441	3281	2472	1862	1369	928
0.10	18137	8975	5788	4094	3024	2275	1709	1250	841
0.11	15957	8137	5278	3739	2760	2072	1551	1128	751
0.12	13953	7292	4754	3370	2484	1860	1386	1001	658
0.13	12039	6422	4202	2978	2191	1634	1210	866	560
0.14	10122	5490	3602	2549	1868	1385	1017	718	454
0.15	8033	4415	2897	2042	1486	1091	790	546	333
0.16	4951	2739	1783	1240	886	634	443	290	160

TABLE 15

Solutions with respect to the volatility coefficient σ (b.p.) when r=0.2

i \ η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00		32549	15477	10256	7412	5559	4217	3158	2223
0.01		29128	14737	9850	7142	5364	4071	3047	2142
0.02		26507	14035	9454	6876	5171	3926	2936	2060
0.03		24365	13366	9069	6614	4980	3781	2826	1979
0.04		22543	12727	8692	6356	4790	3637	2716	1898
0.05		20948	12114	8322	6100	4601	3493	2606	1817
0.06		19525	11523	7960	5847	4414	3350	2496	1736
0.07		18234	10952	7603	5596	4226	3207	2386	1655
0.08		17049	10398	7251	5346	4040	3063	2276	1574
0.09		15948	9858	6903	5098	3853	2920	2166	1492
0.10	56492	14917	9330	6559	4849	3665	2775	2055	1410
0.11	36556	13941	8812	6216	4601	3477	2630	1943	1327
0.12	30434	13011	8302	5873	4352	3288	2483	1830	1244
0.13	26410	12116	7796	5531	4100	3096	2334	1715	1160
0.14	23314	11250	7293	5186	3846	2901	2183	1598	1074
0.15	20738	10403	6789	4836	3587	2702	2029	1479	987
0.16	18488	9566	6279	4480	3321	2498	1870	1357	898
0.17	16447	8729	5758	4113	3046	2287	1705	1230	805
0.18	14532	7878	5219	3729	2758	2064	1532	1098	709
0.19	12671	6992	4647	3319	2449	1826	1347	956	607
0.20	10772	6034	4018	2865	2106	1561	1142	800	496
0.21	8656	4906	3267	2320	1694	1244	897	616	368
0.22	5306	3026	1999	1400	1003	718	500	325	174

Once again, we choose the extreme scenarios considered in Tables 11 and 15 in order to catch some numerical feelings about our findings. When r=3% (a scenario similar to the Italian one at the present time) and the participation level is rather high, then a fair pricing is attainable only with the choice of a reference portfolio characterized by a very low volatility. For instance, if η is between 70% and 90%, a fair pricing would require a volatility coefficient between 6.28% and 3.33% for i=0, and respectively between 1.03% and 0.37% for i=3% (see Table 11). When instead r=20%, there are no solutions in σ if η=10% and $i < 10\%$. In this case, moreover, a technical rate of 3% and a participation level between 70% and 90% lead to a fair volatility coefficient between 37.81% and 19.79% (see Table 15).

5. Concluding remarks

In this paper we have analysed a life insurance endowment policy, paid either by a single premium at issuance or by a sequence of periodical premiums, in which both the benefit and the periodical premiums are annually adjusted according to the performance of a special investment portfolio. The premium calculation technique and the adjustment mechanism are defined in such a way that a minimum interest rate is guaranteed to the policyholder and, moreover, a special bonus is annually credited to the mathematical reserve of the policy. These features introduce in the contract some embedded options, of European style, that can be priced in a contingent-claims framework once an independence assumption allows us to keep apart the financial risk from the mortality one. Under the Black and Sholes model for the evolution of the reference portfolio and exploiting the martingale approach, we derive a very simple closed-form relation that characterizes "fair" contracts, i.e., contracts priced consistently with the usual assumptions on financial markets and, in particular, with no-arbitrage. This relation links together the contractual parameters (i.e., the minimum interest rate guaranteed and a "participation" coefficient) with the market interest rate and the riskiness of the reference portfolio.

Undoubtedly a quality of our valuation model is its simplicity, although it includes almost all the features of Italian participating policies. However, taking into account that life insurance policies are usually long-term contracts and bearing in mind the experience on the evolution of the market interest rates in the last two decades, it must be admitted that a framework with deterministic interest rates, such as the Black and Scholes one, is not suitable to represent the real world. Therefore a natural extension of the present paper is certainly the inclusion of stochastic interest rates. Moreover, since endowment policies are usually equipped with a surrender option, obviously of American style, the valuation of such option constitutes another topic of future research.

References

Benjamin, J. and A.S. Soliman (1993): *Mortality on the Move*, Actuarial Education Service, Oxford.

Black, F. and M. Scholes (1973): "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy* 81, 637-654.

Boyle, P.P. and E.S. Schwartz (1977): "Equilibrium Prices of Guarantees under Equity-Linked Contracts", *Journal of Risk and Insurance* 44, 639-660.

Brennan, M.J. and E.S. Schwartz (1976): "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee", *Journal of Financial Economics* 3, 195-213.

Brennan, M.J. and E.S. Schwartz (1979a): "Alternative Investment Strategies for the Issuers of Equity Linked Life Insurance Policies with an Asset Value Guarantee", *Journal of Business* 52, 63-93.

Brennan, M.J. and E.S. Schwartz (1979b): *Pricing and Investment Strategies for Equity-Linked Life Insurance*, The S.S. Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, Philadelphia.

Briys, E. and F. de Varenne (1997): "On the Risk of Life Insurance Liabilities: Debunking Some Common Pitfalls", *Journal of Risk and Insurance* 64, 673-694.

Grosen, A. and P.L. Jørgensen (1999): "Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies", Working Paper n. 32, Centre for Analytical Finance, Aarhus, Denmark.

Harrison, M.J. and D. Kreps (1979), "Martingales and Arbitrage in Multiperiod Securities Markets", *Journal of Economic Theory* 20, 381-408.

Harrison, M.J. and S. Pliska (1981), "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and Their Applications* 11, 215-260.

Harrison, M.J. and S. Pliska (1983), "A Stochastic Calculus Model of Continuous Trading: Complete Markets", *Stochastic Processes and Their Applications* 15, 313-316.

Macdonald, A.S., A.J.G. Cairns, P.L. Gwilt and K.A. Miller (1998): "An International Comparison of Recent Trends in Population Mortality", *British Actuarial Journal* 4, 3-141.

Miltersen, K.R. and S.-A. Persson (1999): "Guaranteed Investment Contracts: Distributed and Undistributed Excess Returns", Working Paper.

Pentikäinen, T. (1968): "Linking Life and Private Pension Insurance to Price Index", *Transactions of the 18th International Congress of Actuaries*, vol. 2, 847-859.

Vasicek, O. (1977), "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics* 5, 177-188.