The impact of statistical dependence on multiple life insurance programs.

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Abstract

Consider an insurance programs involving two dependent lives. We study the impact of several types of dependence relations on the present value of annuities, benefits and premiums. As an example for dependence we discuss a model where the couple is subject to a common cause of death.

Key words: joint life, last survivor, positive orthant dependent, concordance ordering, stochastic ordering, multi variate hazard rate, bi-exponential distribution.

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1 Introduction

An important issue in life insurance is pricing policies written on more than one individual. Consider, for example, the following policies issued to two lives:

1. First survivor policy: Here, the couple pays premiums until earlier death when benefits are paid to the survivor.

2. Last survivor policy: Premiums are paid until later death, when the heirs collect the benefits.

Such models are discussed in several texts. When the two lives are independent, formulas for the discounted premiums inflow and discounted sum assured were derived, compare Bowers et. al (1997) Chapter 8, and Neill (1977) Chapter 7.

Recently, experimental and theoretical studies have demonstrated dependence between lifetimes of paired lives such as husband and wife. It is of great interest to find the impact of the dependence on actuarial quantities such as annuities, expected present value (EPV) of benefits, and on pricing of insurance contracts.

Norberg (1989) considers the joint lifetime distribution of husband and wife. Assuming that the force of mortality of the husband and wife increases after death of the spouse, he showed that the lifetimes of the couple are positively correlated, while if the hazard rate of the husband and wife decreases after the spouse death then their life times are negatively correlated. He showed that the annuities for the first survivor is greater in the positively correlated case than in the independent case, while the EPV of the sum assured is smaller than in the independent case. Reverse results hold for the last survivor case.

Experimental results presented in Denuit et. al. (2001), show that mortality rate of widows is higher than the mortality rate in the entire population. Thus Norberg's assumptions hold. In the same paper the authors found upper and lower bounds for the EPV of annuities and insurance for paired life, based on comonotinicity. They studied also some parametric models for the cumulative distribution of paired lives and derived estimators for the parameters. They also derived estimators for the parameters in a Markovian model.

Dhaene and Goovaerts (1996) studied the impact of dependence on a portfolio containing several risks. They considered portfolios containing m couples, and $n - m$ singles. The risks of each couple are dependent, but the different couples and the $n - m$ single risks are independent. They showed that as the dependence among the members of a couple increases the sum of the risks in the portfolio increases in stop loss (increasing convex) ordering. In another paper (1997), they compared portfolios containing n risks, assuming the marginal distributions of the risks in the different portfolios are identical. They showed that the sum of risks in a portfolio with comonotone risks is the biggest in the sense of stop loss ordering .

As an example for dependence between the lifetimes of a couple consider the following model: The death of each life can be of two types: 1. natural death. 2. non-natural death caused, say, by an accident. In a given accident several lives might die at the same time. Assume that the times to natural deaths are independent and moreover they are independent of the times until non-natural death. The possibility of common death imposes dependence among lifetimes of insurdes. This model enables one to demonstrate the behavior of certain actuarial quantities such as annuities, EPV of sum assured, and premiums as dependence among lives increases.

The paper is organized as follows: In Section 2 we give definitions of dependence. In section 3 we introduce concepts of stochastic dependence. In Section 4 we discuss the impact of dependence on present value of annuity and insurance. In Section 5 we give an example of dependence due to common cause. Finally, in section 6 we give a numerical example.

2 Definitions and Notations

Throughout the paper, R denotes the real line. We use small bold letters to denote a row vector, i.e. $\mathbf{x} = (x_1, \cdots, x_n)$, and by capital bold letters we denote random vectors.

2.1 Notions of Dependence between Random Variables.

Definition 1 Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors in R^n. $\mathbf{X}$ is smaller than $\mathbf{Y}$ in stochastic order, written $\mathbf{X} \underset{st}{\leq} \mathbf{Y}$, if $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ for all increasing functions f, provided that the expectations exist

Remark 1 . 1. Note that $\mathbf{X} \underset{st}{\leq} \mathbf{Y}$ implies that $g(\mathbf{X}) \underset{st}{\leq} g(\mathbf{Y})$ for any increasing function $g : R^n \to R$.

2. If X, Y are random variables then $X \underset{st}{\leq} Y$ if and only if $P(X \leq t) \geq P(Y \leq t)$, or, equivalently, $P(X > t) \leq P(Y > t)$.

We give now the definition of supermodular function and supermodular ordering. A comprehensive reference for supermodular functions is Topiks (1978). For further discussion on supermodular ordering of random vectors see Marshall and Olkin (1979), Bäuerle (1997) and Shaked and Shanthikumar (1997).

Let $S \subseteq R$. For $\mathbf{x}, \mathbf{y} \in S^n$, let $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$, similarly $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, \cdots, x_n \vee y_n)$, where, $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$.

Definition 2 A function $\psi : S^n \to R$ is said to be supermodular if for all $\mathbf{x} \in S^n$,

$$\psi(\mathbf{x} + \varepsilon \mathbf{e}_i + \delta \mathbf{e}_j) + \psi(\mathbf{x}) \geq \psi(\mathbf{x} + \varepsilon \mathbf{e_i}) + \psi(\mathbf{x} + \boldsymbol{\delta} \mathbf{e_j}), \tag{1}$$

where $\mathbf{e}_i$ is a vector in which the $i'th$ components is 1, and all the others are 0.Or equivalently

$$\psi(\mathbf{x} \wedge \mathbf{y}) + \psi(\mathbf{x} \vee \mathbf{y}) \geq \psi(\mathbf{x}) + \psi(\mathbf{y}) \tag{2}$$

Definition 3 Let **X** and **Y** be two $n-$dimensional random vectors. **X** is larger than **Y** in supermodular ordering, **X** $\underset{sm}{\geq}$ **Y**, if for all supermodular functions ψ, $E\psi(\mathbf{X}) \geq E\psi(\mathbf{Y})$.

The following Lemma is stated and proved in Bäuerle (1997):

Lemma 1 If **X** $\underset{sm}{\geq}$ **Y**, then:

1. X_i and Y_i are identically distributed, $i = 1, \cdots, n$.
2. $(X_{i_1}, \cdots, X_{i_k}) \underset{sm}{\geq} (Y_{i_1}, \cdots, Y_{i_k})$, for all $i_1 \leq \cdots \leq i_k$, $k \leq n$.
3. $Cov(f(X_i), g(X_j)) \geq Cov(f(Y_i), g(Y_j))$ for all non-negative increasing functions f, and g.

Definition 4 Let **X** $= (X_1, X_2)$ and **Y** $=(Y_1, Y_2)$ be bivariate random vectors with the same marginals. Then **X** is said to be smaller than **Y** in concordance ordering denoted by, **X** $\underset{c}{\leq}$ **Y** if

$$P(X_1 \leq s, X_2 \leq t) \leq P(Y_1 \leq s, Y_2 \leq t) \quad \text{for all } s \text{ and } t. \tag{3}$$

Thus the concordance ordering is equivalent to the supermodular ordering for $n = 2$. Tchen (1980) showed that if **X** $\underset{c}{\leq}$ **Y** then the distribution of **Y** can be derived from that of **X** by finite number of repairing which add mass ε to (a, b) and to (a', b') while subtracting mass ε from (a', b) and (a, b'), where, $a' > a$, and $b' > b$. We intend to apply this technique in the current research.

Definition 5 A random vector **X** $=(X_1, \cdots, X_n)$ is said to be (positively) associated if

$$Cov(f(\mathbf{X}), g(\mathbf{X})) \geq 0 \tag{4}$$

for all increasing functions $f, g : R^n \to R$.

Definition 6 A random vector $\mathbf{X} = (X_1, \cdots, X_n)$ is said to be negatively associated if

$$Cov(f(\mathbf{X}_K), g(\mathbf{X}_L)) \leq 0 \qquad (5)$$

for all disjoint subsets K and L of $\{1, \cdots, n\}$, for all increasing functions f and g.

The following Theorem is stated and proved in (Müller and Stoyan (2001), Chapter 3.8):

Theorem 2 Let $\mathbf{X}$ and $\mathbf{Y}$ be bivariate random vectors with the same marginals. Then the following conditions are equivalent:

- $\mathbf{X} \underset{c}{\leq} \mathbf{Y}$.
- $P(X_1 > s, X_2 > t) \leq P(Y_1 > s, Y_2 > t)$.
- $E(f_1(X_1)f_2(X_2)) \leq E(f_1(Y_1)f_2(Y_2))$, for all increasing functions f_1 and f_2 .
- $Cov(f_1(X_1), f_2(X_2)) \leq Cov(f_1(Y_1), f_2(Y_2))$ for all increasing functions f_1 and f_2 .
- $\mathbf{X} \underset{sm}{\leq} \mathbf{Y}$.

Let $\mathbf{X}$ be bivariate random vector. Let $\mathbf{X}^{\perp}$ the independent version of $\mathbf{X}$, that is the components of $\mathbf{X}^{\perp}$ are independent with the same marginal distribution as $\mathbf{X}$.

The Proof of the following Proposition is straight forward from the definition of associated random vectors.

Proposition 3 1. If $\mathbf{X}$ is positive associated then $\mathbf{X} \underset{c}{\geq} \mathbf{X}^{\perp}$.

2. If $\mathbf{X}$ is positive associated then $\mathbf{X} \underset{c}{\leq} \mathbf{X}^{\perp}$.

Remark 2 The Proposition holds also for m-dimensional random vectors where the supermodular ordering replaces the concordance ordering. (See Xu (1999)).

2.2 Actuarial Notations and Definitions

In this subsection we consider actuarial quantities involving two lives, (x,y), say husband aged x and wife aged y. Let T_x and T_y be the remaining lifetimes of the husband and wife respectively. Let $\bar{F}_x(t) = 1 - F_x(t)$ and $\bar{F}_y(t) = 1 - F_y(t)$ be the survival functions of T_x and T_y. The joint life or the first survivor, is min(T_x,T_y) $= T_x \wedge T_y$. Its survival function at t is $P(T_x > t, T_y > t)$. The last survivor or the longest life is max(T_x,T_y) $= T_x \vee T_y$. Its distribution function at t is $P(T_x \leq t, T_y \leq t)$.

Let $v = e^{-\delta}$ denote the discount factor corresponding to interest rate $i = e^{\delta} - 1$, where δ is the force of interest. In this paper we derive some expressions for the present value of annuities and a unit sum assured. We will discuss only the continuous case, that is

1. Annuity is paid continuously at rate 1.
2. Death benefit, taken to be 1 is paid out at the moment of death.
3. Interest is compound continuously.

Extensions to the discrete case are obvious.

In Table 1 we give some actuarial notations.

Table 1. Actuarial notation-continuous case.

Payment scheme	Present value	Expected present value
Term Annuity	$\int_{t=0}^{T\wedge n} e^{-\delta t} dt$ $= \frac{1-v^{(T\wedge n)}}{\delta}$	$\bar{a}_{\bar{n}\vert} = \frac{1-Ev^{(T\wedge n)}}{\delta}$
Term insurance	$v^T I(T \leq n)$ $= v^{T\wedge n} - v^n I(T > n)$	$\bar{A}_{\bar{n}\vert} = 1 - \delta \bar{a}_{\bar{n}\vert}$ $-v^n P(T > n)$
Whole life annuity	$\frac{1-v^T}{\delta}$	$\bar{a} = \frac{1-Ev^{-\delta T}}{\delta}$
Whole life insurance	v^T	$\bar{A} = 1 - \delta \bar{a}$

Here, $I(A)$ denotes the indicator of the event A.

For detailed actuarial notations see Appendix A.

Consider a Term insurance issued to an insured aged x. The equation of value gives the rate of the net premium stream

$$\pi = \frac{\bar{A}_{\overset{1}{x}\overline{n}|}}{\bar{a}_{x\overline{n}|}} \tag{6}$$

3 Impact of dependence on joint life and last survivor life.

In this section, we study the impact of the dependence relations defined in Section 2.1 on the distribution of the present value of annuities and insurance involving two lives.

Consider a couple (husband and wife) aged x and y respectively. Let T_x, T_y be the residual life times of the husband and wife respectively. Assume that (T_x, T_y), has a joint distribution $F_{xy}(u, v)$ with marginal distributions F_x and F_y respectively. In this section we use the notation (x, y) for the joint life status and $\overline{xy}$ for the last survivor status. We compare the present values of annuities or of of life insurance benefits for two couples with residual life times as described above. The symbol $'$ is attached to variables of the second couple.

We apply two dependence relations between two vectors of life times. In Section 3.1 we study the impact of concordance ordering and in Section 3.2 we study the the multivariate hazard rate ordering.

3.1 Concordance ordering

Proposition 4 If $(T_x, T_y) \underset{c}{\leq} (T'_x, T'_y)$, then $(T_x \wedge T_y) \underset{st}{\leq} (T'_x \wedge T'_y)$.

Proof. Theorem 2 imply that

$$P((T_x \wedge T_y) > t) = P(T_X > t, T_Y > t) \leq (T'_x > t, T'_y > t) = P((T'_x \wedge T'_y) > t)$$

Thus the result follows from Remark 1.

Proposition 5 If $(T_x, T_y) \underset{c}{\leq} (T'_x, T'_y)$, then for $n \geq 1$:

1. $v^{T_x \wedge T_y \wedge n} - v^n I(T_x \wedge T_y > n) \underset{st}{\geq} v^{T'_x \wedge T'_y \wedge n} - v^n I(T'_x \wedge T'_y > n).$

Here we compare the present value (PV) of benefits of life insurance given at the moment of death of the first survivor.

2. $\frac{1-v^{(T_x \wedge T_y \wedge n)}}{\delta} \underset{st}{\leq} \frac{1-v^{(T'_x \wedge T'_y \wedge n)}}{\delta}.$

Here we compare the PV of two annuities having rate 1 paid until the first death or up to n.

3. (i) $\bar{A}'_{\overset{1}{x}y:\bar{n}|} \leq \bar{A}_{\overset{1}{x}y:\bar{n}|}$, (ii) $\bar{A}'_{xy} \leq \bar{A}_{xy}$, (iii) $a_{xy:\bar{n}|} \leq a'_{xy:\bar{n}|}$ and (iv) $\bar{a}_{xy} \leq \bar{a}'_{xy}$.

Proof 1-2. Since $v < 1$, v^z is decreasing in z. Thus the results of the proposition follows from Proposition 4 and 1 of Remark 1.

3. Just take expectation of both sides of 1-2 and apply Definition 1 and Remark 1.

Proposition 6 If $(T_x, T_y) \underset{c}{\leq} ((T'_x, T'_y)$ then, $(T_x \vee T_y) \underset{st}{\leq} (T'_x \vee T'_y)$.

Proof. The result follows directly from Theorem 2.

The next Proposition studies how dependence affects last survivor

Proposition 7 If $(T_x, T_y) \underset{c}{\leq} ((T'_x, T'_y)$ then,

1. $v^{(T_x \vee T_y) \wedge n} - v^n I(T_x \vee T_y > n) \underset{st}{\geq} v^{(T'_x \vee T'_y) \wedge n} - v^n I(T'_x \vee T'_y > n).$

   Here we compare the PV of unit sum assured given at the moment of death of the last survivor if it occurred before time n.

2. $\frac{1-v^{(T_x \vee T_y) \wedge n}}{\delta} \underset{st}{\leq} \frac{1-v^{(T'_x \vee T'_y) \wedge n}}{\delta}.$

   Here we compare the PV of annuity of rate 1 paid until the last death.

3. (i) $\bar{A}'_{\overset{1}{\overline{xy}}:\bar{n}|} \geq \bar{A}_{\overset{1}{\overline{xy}}:\bar{n}|}$, (ii) $\bar{A}'_{\overline{xy}} \geq \bar{A}_{\overline{xy}}$, (iii) $\bar{a}_{\overline{xy}:\bar{n}|} \leq \bar{a}'_{\overline{xy}:\bar{n}|}$ and (iv) $\bar{a}_{\overline{xy}|} \leq \bar{a}'_{\overline{xy}|}$.

Proof Since $v < 1$, v^z is decreasing in z, thus the results follow from proposition 6 and 1 of Remark 1.

Consider a couple (x, y). In some annuity schemes, upon the death of x, y starts collecting continuous term annuity at rate 1. This annuity is paid until the earlier between the death of y or the term n. The EPV of this annuity, ${}_x|\bar{a}_{y:n}$ is

$$
{}_x|\bar{a}_{y:n} = \bar{a}_{y:n} - \bar{a}_{xy:n} = E\left[\frac{v^{(T_x \wedge T_y \wedge n)} - v^{(T_y \wedge n)}}{\delta}\right] \tag{7}
$$

Proposition 8 If $(T_x, T_y) \underset{c}{\leq} (T'_x, T'_y)$ then

$$
{}_x|\bar{a}_{y:n} \geq {}_x|\bar{a}'_{y:n} \tag{8}
$$

Proof. Note that by Lemma 1 $(T_x, T_y), (T'_x, T'_y)$ have the same marginals. Proposition 4 implies that $T'_x \wedge T'_y \underset{st}{\geq} T_x \wedge T_y$. Thus the result follows from Definition 1.

In another type of contracts a continuous term annuity, at rate 1 is paid until earlier death. Then, the annuity is reduced to θ if y dies before x, otherwise there are no payments. Assuming the term is n. The EPV of this annuity is:

$$
\begin{aligned}
{}_y|\bar{a}_{\overset{\theta}{xy}:\overline{n}|} &= E\left[\frac{1 - v^{(T_x \wedge n)}}{\delta} - (1-\theta)\frac{v^{(T_x \wedge T_y \wedge n)} - v^{(T_x \wedge n)}}{\delta}\right] \\
&= \bar{a}_{x:\overline{n}|} - (1-\theta)(\bar{a}_{x:\overline{n}|} - \bar{a}_{xy:\overline{n}|})
\end{aligned} \tag{9}
$$

Proposition 9 If $(T_x, T_y) \underset{c}{\leq} (T'_x, T'_y)$ then for $n \geq 1$

$$
{}_y|\bar{a}'_{\overset{\theta}{xy}:\overline{n}|} \geq {}_y|\bar{a}_{\overset{\theta}{xy}:\overline{n}|} \tag{10}
$$

Proof. Similar to the proof of Proposition 8.

Consider a continuous term annuity, at rate 1 paid whenever both members are alive. A fraction θ is paid to the survivor after earlier death before the end of the term. Let $\varphi_n(x,y)$ be the PV of the annuity for term n, and let $\bar{a}_{\underset{xy:n}{\theta}} = E(\varphi_n(x,y))$, $n \geq 1$. Note that

$$\varphi_n(x,y) = \frac{1 - v^{(T_x \wedge T_y \wedge n)}}{\delta} + \theta \frac{v^{(T_x \wedge T_y \wedge n)} - v^{(n \wedge (T_x \vee T_y))}}{\delta}. \tag{11}$$

Proposition 10 $\varphi_n(x,y)$ is supermodular for $0 \leq \theta \leq 1/2$, and $\varphi_n(x,y)$ is submodular for $1/2 \leq \theta \leq 1$, $n = 1, 2 \cdots$.

Proof. We prove the proposition for the case $0 \leq \theta \leq 1/2$, and $n = 1, 2, \cdots$. The proof for the case $1/2 \leq \theta \leq 1$ is similar. Let $0 < x_1 < x_2$, $0 < y_1 < y_2$, $0 < v < 1$. For $0 \leq \theta \leq 1/2$, $(1/2 \leq \theta \leq 1)$. According to Definition 3 we have to prove that

$$\begin{aligned} & (1-\theta)(v^{x_1 \wedge y_1 \wedge n} + v^{x_2 \wedge y_2 \wedge n}) + \theta(v^{n \wedge (x_1 \vee y_1)} + v^{n \wedge (x_2 \vee y_2)}) \\ \leq \; & (1-\theta)(v^{x_1 \wedge y_2 \wedge n} + v^{x_2 \wedge y_1 \wedge n}) + \theta(v^{n \wedge (x_1 \vee y_2)} + v^{n \wedge (x_2 \vee y_1)}) \end{aligned} \tag{12}$$

case 1. $x_2 < y_1$. In this case the two sides of (12) are equal to

$$(1-\theta)(v^{x_1 \wedge n} + v^{x_2 \wedge n}) + \theta(v^{y_1 \wedge n)} + v^{y_2 \wedge n)})$$

case 2. $y_2 < x_1$. In this case the two sides of (12) are equal to

$$(1-\theta)(v^{y_1 \wedge n} + v^{y_2 \wedge n}) + \theta(v^{x_1 \wedge n} + v^{x_2 \wedge n}).$$

case 3. $x_1 < y_1, x_2 < y_2, x_2 > y_1,$. The left hand side of (12) is

$$(1-\theta)(v^{x_1 \wedge n} + v^{x_2 \wedge n}) + \theta(v^{y_1 \wedge n} + v^{y_2 \wedge n}).$$

The right hand side of (12) is

$$(1-\theta)(v^{x_1} + v^{y_1}) + \theta(v^{y_2} + v^{x_2}),$$

thus difference between the right hand side and the left hand side of (12) is

$$(1-2\theta)(v^{x_2 \wedge n} - v^{y_1 \wedge n}) \leq 0,$$

since $0 \le \theta < 1/2$, $x_2 > y_1$, and $v < 1$.

case 4. $x_1 < y_1, x_2 > y_2$. The left hand side of (12) is

$$(1-\theta)(v^{x_1 \wedge n} + v^{y_2 \wedge n}) + \theta(v^{y_1 \wedge n} + v^{x_2 \wedge n}).$$

The right hand side of (12) is

$$(1-\theta)(v^{x_1} + v^{y_1}) + \theta(v^{y_2} + v^{x_2})$$

Thus difference between the right hand side and the left hand side of ((12)is $(1-2\theta)(v^{y_2 \wedge n} - v^{y_1 \wedge n}) \le 0$ since $0 \le \theta < 1/2$, $y_2 > y_1$, and $v < 1$.

case 5. $x_1 > y_1, x_2 < y_2$. The left hand side of (12) is

$$(1-\theta)(v^{y_1 \wedge n} + v^{x_2 \wedge n}) + \theta(v^{x_1 \wedge n} + v^{y_2 \wedge n})$$

the right hand side of (12) is

$$(1-\theta)(v^{x_1} + v^{y_1}) + \theta(v^{y_2} + v^{x_2}),$$

thus difference between the right hand side and the left hand side of (12)is $(1-2\theta)(v^{x_2 \wedge n} - v^{x_1 \wedge n}) \le 0$ since $0 \le \theta < 1/2$, $x_2 > x_1$, and $v < 1$.

case 6. $x_1 > y_1, x_2 > y_2, x_1 < y_2$. The left hand side of (12) is $(1-\theta)(v^{y_1 \wedge n} + v^{y_2 \wedge n}) + \theta(v^{x_1 \wedge n} + v^{x_2 \wedge n})$ the right hand side of (12) is $(1-\theta)(v^{x_1 \wedge n} + v^{y_1 \wedge n}) + \theta(v^{y_2 \wedge n} + v^{x_2 \wedge n})$. Thus the difference between the right hand side and the left hand side of (12)is $(1-2\theta)(v^{y_2 \wedge n} - v^{x_1 \wedge n})$. The letter term is negative since $0 \le \theta < 1/2$, $y_2 > x_1$, and $v < 1$.

Corollary 11 Assume that $(T_x, T_y) \underset{c}{\le} (T'_x, T'_y)$. Then $\bar{a}_{\overset{\theta}{xy:\overline{n}|}} \le \bar{a}'_{\overset{\theta}{xy:\overline{n}|}}$ for $0 \le \theta \le 1/2$, and $\bar{a}_{\overset{\theta}{xy:\overline{n}|}} \ge \bar{a}'_{\overset{\theta}{xy:\overline{n}|}}$ for $1/2 \le \theta \le 1$. If $\theta = 1/2$ then $\bar{a}_{\overset{\theta}{xy:\overline{n}|}} = \bar{a}'_{\overset{\theta}{xy:\overline{n}|}}$.

In the next Remark we show that the EPV of annuities and sum assured when life times of the couple are associated can be bounded by the respected quantities for independent life times.

Let $(T_x^{\perp}, T_y^{\perp})$ be the indpendent version of (T_x, T_y). That is $(T_x^{\perp}, T_y^{\perp})$ has the same marginal distributions as (T_x, T_y) but are independent.

Remark 3 Proposition 3 part 1, yields that if (T'_x, T'_y) is positively associated then $(T'_x, T'_y) \underset{c}{\geq} (T^{\perp}_x, T^{\perp}_y)$. Thus, Propositions 4-10, and Corollary 11, hold when substituting $(T^{\perp}_x, T^{\perp}_y)$ with (T_x, T_y).

Similarly, if (T_x, T_y) is negatively associated then $(T_x, T_y) \underset{c}{\leq} (T^{\perp}_x, T^{\perp}_y)$. It implies that Propositions 4-10, and Corollary 11, hold when substituting, $(T^{\perp}_x, T^{\perp}_y)$ with (T'_x, T'_y).

For the following definition see Tchen (1980) or Chapter 3 in Müller and Stoyan (2002).

Definition 7 Consider the class of all bi-variate distributions with fixed marginals, say F_X and F_Y. The Fréchet lower bound is the joint distribution defined by

$$W(x, y) = \max\{F_X(x) + F_Y(y) - 1, 0\} \tag{13}$$

The Fréchet upper bound is the joint distribution defined by

$$M(x, y) = \min\{F_X(x), F_Y(y)\} \tag{14}$$

Let (X, Y) be a bi-variate random vector with joint distribution function having marginals F_X and F_Y. Let $(\hat{X}, \hat{Y})$ be bi-variate random vector with joint distrbution $W(x, y)$ and $(\check{X}, \check{Y})$ be a bi-variate random vector with joint distribution $M(x, y)$. Tchen (1980) showed that

$$(\hat{X}, \hat{Y}) \underset{c}{\leq} (X, Y) \underset{c}{\leq} (\check{X}, \check{Y}) \tag{15}$$

In the sequael we denote by $\wedge$ and $\vee$ the quantities relating to $(\hat{T}_x, \hat{T}_y)$ and to $(\check{T}_x, \check{T}_y)$ respectively.

Proposition 12 Assume that (T_x, T_y) has an arbitrary joint distribution with marginals F_X, and F_Y. Then for $n \geq 1$:

$$\begin{aligned} &v^{\check{T}_x \wedge \check{T}_y \wedge n} - v^n I(\check{T}_x \wedge \check{T}_y > n) \underset{st}{\leq} v^{T_x \wedge T_y \wedge n} - v^n I(T_x \wedge T_y > n) \\ &\underset{st}{\leq} v^{\hat{T}_x \wedge \hat{T}_y \wedge n} - v^n I(\hat{T}_x \wedge \hat{T}_y > n) \end{aligned}$$

Proof. Note that the functions $w(s,t) = I(T_x > t, T_y > s)$ and $m(s,t) = I(T_x \leq t, T_y \leq s)$ are supermodular. By Theorem 2, in the bivariate case, concordance ordering is equivalent to supermodular ordering. Thus,

$$\hat{T}_x \wedge \hat{T}_y \underset{st}{\leq} T_x \wedge T_y \underset{st}{\leq} \check{T}_x \wedge \check{T}_y \tag{16}$$

$$\check{T}_x \vee \check{T}_y \underset{st}{\leq} T_x \vee T_y \underset{st}{\leq} \hat{T}_x \vee \hat{T}_y \tag{17}$$

The result follows from (15), Proposition 4 and 1 of Remark 1.

Corollary 13 1. $\check{A}_{\overset{1}{xy}:\overline{n}|} \leq \bar{A}_{\overset{1}{xy}:\overline{n}|} \leq \hat{A}_{\overset{1}{xy}:\overline{n}|}$

2. $\frac{1-e^{-\delta(\hat{T}_x \wedge \hat{T}_y \wedge n)}}{\delta} \underset{st}{\leq} \frac{1-e^{-\delta(T_x \wedge T_y \wedge n)}}{\delta} \underset{st}{\leq} \frac{1-e^{-\delta(\check{T}_x \wedge \check{T}_y \wedge n)}}{\delta}$

3. $\hat{a}_{xy:\overline{n}|} \leq \bar{a}_{xy:\overline{n}|} \leq \check{a}_{xy:\overline{n}|}$.

4. $v^{(\hat{T}_x \vee \hat{T}_y) \wedge n} - v^n I(\hat{T}_x \vee \hat{T}_y > n) \underset{st}{\leq} v^{(T_x \vee T_y) \wedge n} - v^n I(T_x \vee T_y > n)$ $\underset{st}{\leq} v^{(\check{T}_x \vee \check{T}_y) \wedge n} - v^n I(\check{T}_x \vee \check{T}_y > n)$

5. $\hat{A}_{\overset{1}{\overline{xy}}:\overline{n}|} \leq \bar{A}_{\overset{1}{xy}:\overline{n}|} \leq \check{A}_{\overset{1}{\overline{xy}}:\overline{n}|}$

6. $\frac{1-e^{-\delta((\check{T}_x \vee \check{T}_y) \wedge n)}}{\delta} \underset{st}{\leq} \frac{1-e^{-\delta((T_x \vee T_y) \wedge n)}}{\delta} \underset{st}{\leq} \frac{1-e^{-\delta((\hat{T}_x \vee \hat{T}_y) \wedge n)}}{\delta}$

7. $\check{a}_{\overline{xy}:\overline{n}|} \leq \bar{a}_{\overline{xy}:\overline{n}|} \leq \hat{a}_{\overline{xy}:\overline{n}|}$.

8. ${}_x|\overset{\vee}{\bar{a}}_{y:\overline{n}} \leq_x |\bar{a}_{y:\overline{n}|} \leq {}_x|\overset{\wedge}{\bar{a}}_{y:\overline{n}}$, where ${}_x|\bar{a}_{y:\overline{n}}$ is given in (8).

9. ${}_y|\hat{a}_{\overset{\theta}{xy}:\overline{n}|} \leq_y |\bar{a}_{\overset{\theta}{xy}:\overline{n}|} \leq {}_y|\check{a}_{\overset{\theta}{xy}:\overline{n}|_y}$, where, ${}_y|\bar{a}_{\overset{\theta}{xy}:\overline{n}|}$ is given in (9)

10. For $0 \leq \theta \leq 1/2$, $\hat{a}_{\overset{\theta}{xy}:\overline{n}|} \leq \bar{a}_{\overset{\theta}{xy}:\overline{n}|} \leq \check{a}_{\overset{\theta}{xy}:\overline{n}|}$ For $1/2 \leq \theta \leq 1$, $\check{a}_{\overset{\theta}{xy}:\overline{n}|} \leq \bar{a}_{\overset{\theta}{xy}:\overline{n}|} \leq \hat{a}_{\overset{\theta}{xy}:\overline{n}|}$, where, $\bar{a}_{\overset{\theta}{xy}:\overline{n}|}$ is given in (11)

Proof. 2,4,6, follow from from Proposition 12, and Remark 1. 1,3,5,and 7-10, follow by taking expectations of the related quantities and applying Definition 1.

Remark 4 1. If (X,Y) is positively associated then Proposition 3 part 1 implies that $(X,Y) \underset{c}{\geq} (X^{\perp}, Y^{\perp})$. In this case $(X^{\perp}, Y^{\perp})$ is an improved lower bound for (X, Y), in the sense of concordance order, over $(\check{X}, \check{Y})$, defined by (13).

2. If (T_x, T_y) is positive associated, then 1 implies that, $T_x \wedge T_y \underset{st}{\geq} T_x^{\perp} \wedge T_y^{\perp}$, and $T_x \vee T_y \underset{st}{\leq} T_x^{\perp} \vee T_y^{\perp}$. In this case 1-10 of Corollary 12 hold with $(T_x^{\perp} \wedge T_y^{\perp})$ substituting $(\hat{T}_x, \hat{T}_y)$. Thus, in this case tighter bounds can be derived .

3.2 Multivariate hazard rate ordering

Another stochastic ordering which indicates a different kind of dependence among random variables is the multivariate hazard rate ordering. The multivariate hazard rate ordering was introduced in Chapter 4 of Shaked and Shantikumar (1987). The definition here follows Chapter 3 in Müller and Stoyan (2002). Consider a non-negative random vector $\mathbf{X} = (X_1, \cdots, X_n)$ having joint density. Let $\mathbf{I}$ be a subset of $\{1, \cdots, n\}$ and assume that $k \in \mathbf{I}^c$, i.e. $k \notin \mathbf{I}$. Define

$$r_{\mathbf{X},k|I}(t|\mathbf{x}_\mathbf{I}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} P(t < X_k < t + \varepsilon | \mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_{\mathbf{I}^c} > t\mathbf{1}), \tag{18}$$

where $\mathbf{1} = (1, \cdots, 1)$ is a vector of 1's.

The interpretation of $r_{\mathbf{X},k|\mathbf{I}}(t|\mathbf{X}_\mathbf{I})$ in actuarial sciences is the force of mortality of life k given that individuals in $\mathbf{I}$ have already died and the individuals in $\mathbf{I}^c$ are still alive. When $n = 2$, we describe mortality rate of a couple. For $n = 2$, $r_{(T_x,T_y),x|\mathbf{y}}(t|y)$ is the force of mortality of the husband given that the wife died, similarly $r_{(T_x,T_y),y|\mathbf{x}}(t|x)$ is the force of mortality of the wife given that the husband has died.

Consider now the definition of multivariate hazard rate ordering:

Definition 8 The random vector $\mathbf{Y}$ is said to be larger than $\mathbf{X}$ in multivariate hazard rate order, written $\mathbf{X} \underset{hr}{\leq} \mathbf{Y}$, if $\mathbf{X}$ and $\mathbf{Y}$ having hazard rates $r_\mathbf{X}$ and $r_\mathbf{Y}$ satisfy

$$r_{\mathbf{X},k|\mathbf{J}}(t|\mathbf{x}_\mathbf{J}) \geq r_{\mathbf{Y},k|\mathbf{I}}(t|\mathbf{y}_\mathbf{I}) \tag{19}$$

for all positive t whenever $\mathbf{J} \supseteq \mathbf{I}$, $0 \leq \mathbf{x_I} \leq \mathbf{y_J} \leq t\mathbf{1}$ and $\mathbf{0} \leq \mathbf{x_I} \leq \mathbf{y_J} \leq t\mathbf{1}$ and $\mathbf{0} \leq \mathbf{x_J} \leq t\mathbf{1}$.

In the context of multiple life insurance $(T_x, T_y) \underset{hr}{\leq} (T'_x, T'_y)$, means:

1.The force of mortality of $T_x, (T_y)$ at t given $T_y > t$ $(T_x > t)$ is bigger than the rate of mortality of T'_x (T'_y) at t given $T'_y(T'_x) > t$.

2. The force of mortality of T_x (T_y) at t given that $T_y(T_x) = s < t$ is bigger than the rate of mortality of of T'_x (T'_y) at t given $T'_y(T'_x) > t$, or given that $T'_y(T'_x) = s' > s$, $s' < t$.

Shaked and Shanthikumar (1987) proved the following Theorem:

Theorem 14 If $\mathbf{X}$ and $\mathbf{Y}$ are two random vectors such that (19) holds then

$$\mathbf{X} \underset{st}{\leq} \mathbf{Y} \tag{20}$$

Remark 1 yields the following corollary:

Corollary 15 If $(T_x, T_y) \underset{hr}{\leq} (T'_x, T'_y)$ then the following relations hold:

1. Relation for the PV of unit sum assured given at the moment of earlier death.

$$v^{T_x \wedge T_y \wedge n} - v^n I(T_x \wedge T_y > n) \underset{st}{\geq} v^{T'_x \wedge T'_y \wedge n} - v^n I(T'_x \wedge T'_y > n)v$$

2. Relation for the PV of annuity of rate 1 paid until the up to n or earlier death.

$$\frac{1 - e^{-\delta(T_x \wedge T_y \wedge n)}}{\delta} \underset{st}{\leq} \frac{1 - e^{-\delta(T'_x \wedge T'_y \wedge n)}}{\delta}$$

3. Relation for the PV of unit sum assured given at the moment of death of last survivor.

$$v^{(T_x \vee T_y) \wedge n} - v^n I(T_x \vee T_y > n) \underset{st}{\geq} v^{(T'_x \vee T'_y) \wedge n} - v^n I(T'_x \vee T'_y > n)$$

4. Relation for the PV of annuity paid until the death of last survivor.

$$\frac{1 - e^{-\delta(T_x \vee T_y)}}{\delta} \underset{st}{\leq} \frac{1 - e^{-\delta(T'_x \vee T'_y)}}{\delta}$$

5. Relation for the PV of annuity at rate 1 given to x as long as both x and y live and reducing to θ upon earlier death of y

$$\frac{1-v^{(T_x\wedge n)}}{\delta}-(1-\theta)\frac{v^{(T_x\wedge T_y\wedge n)}-v^{(T_x\wedge n)}}{\delta}\underset{st}{\leq}\frac{1-v^{(T'_x\wedge n)}}{\delta}-(1-\theta)\frac{v^{(T'_x\wedge T'_y\wedge n)}-v^{(T'_x\wedge n)}}{\delta}$$

6. Relation for the PV of annuity at rate 1 given to a couple as long as both are alive, and reducing to θ upon earlier death.

$$\frac{1-e^{-\delta(T_x\wedge T_y)}}{\delta}+\theta\frac{e^{-\delta(T_x\wedge T_y)}-e^{-\delta(T_x\vee T_y)}}{\delta}\underset{st}{\leq}\frac{1-e^{-\delta(T'_x\wedge T'_y)}}{\delta}+\theta\frac{e^{-\delta(T'_x\wedge T'_y)}-e^{-\delta(T'_x\vee T'_y)}}{\delta}$$

4 The impact of common cause on life insurance contracts

4.1 Bi-variate common cause model

4.1.1 Model description

As an example for concordance ordering consider a couple (x, y). In addition to the natural cause of death (which occur at times $\tilde{T}_x$ or $\tilde{T}_y$) both members of the couple might suffer a non-natural death say, due to accidents. Consider now the case where there are three factors to non-natural death: 1. An accident that kills x only. The time until such an accident is a random variable U_{10}, with distribution F_{10} and hazard rate $r_{10}(\cdot)$ 2. An accident that kills y only. The time until such accident is a random variable U_{20}, with distribution F_{20} and hazard rate $r_{20}(\cdot)$ 3. An accident that kills both husband and wife. The time until such an accident is a random variable U_{12}, with distribution F_{12} and hazard rate $r_{12}(\cdot)$. We assume that these three random variables are independent and independent of $(\tilde{T}_x, \tilde{T}_y)$. We call this model the bi-variate common cause model (BCC). Let U_1, (U_2) be the time until an accident kills the husband (wife) respectively. Clearly, U_1, and U_2 are dependent, due to the common cause. Note that $U_1(U_2)$, is distributed as $\min(U_{10}, U_{12})$ $(\min(U_{20}, U_{12}))$. The hazard rates $r_j(t)$, $j = 1, 2$ of the distribution of U_j are:

$$r_j(t) = r_{j0}(t) + r_{12}(t) \tag{21}$$

The joint distribution of (U_1,U_2) is :

$$P(U_1 > u_1, U_2 > u_2) = \bar{F}_{10}(u_1)\bar{F}_{20}(u_2)\bar{F}_{12}(u_1 \vee u_2), \tag{22}$$

where, $\bar{F}(u) = 1 - F(u)$, and $u_1 \vee u_2 = \max(u_1, u_2)$. Note that the joint corresponding to the BCC model is singular.

Note that the time to the first accident is a random variable with hazard rate $\alpha(t)$, where:

$$\alpha(t) = r_{10}(t) + r_{20}(t) + r_{12}(t) \tag{23}$$

Consider another couple (x', y') with BCC distribution until accident having the following hazard rates $r'_{10}(t)$, $r'_{20}(t)$ and r'_{12}:

$$\begin{aligned} r'_{10}(t) &= r_{10}(t) + \varepsilon(t) && (24)\\ r'_{20}(t) &= r_{20}(t) + \varepsilon(t) && (25)\\ r'_{12}(t) &= r_{12}(t) - \varepsilon(t), && (26) \end{aligned}$$

where, $0 < \varepsilon(t) \leq r_{12}(t)$.

Proposition 16 Assumptions (24)-(26) imply that:

$$(U_1, U_2) \underset{c}{\geq} (U'_1, U'_2) \tag{27}$$

Proof. Note that $U_j = \min(U_{j0}, U_{12})$, and $U'_j = \min(U'_{j0}, U'_{12})$, thus for $j = 1, 2$

$$\begin{aligned} r_j(t) &= r_{j0}(t) + r_{12}(t)\\ &= r'_{j0}(t) + r'_{12}(t) = r'_j(t) && (28) \end{aligned}$$

Equation (28) implies that (U_1, U_2), (U'_1, U'_2) have the same marginals. By Theorem 2, to prove (27) one has to show that

$$P(U_1 > u_1, U_2 > u_2) \geq P(U'_1 > u_1, U'_2 > u_2). \tag{29}$$

Note that

$$
\begin{aligned}
& P(U_1 > u_1, U_2 > u_2) \\
= \; & P(U_{10} \wedge U_{12} > u_1, U_{20} \wedge U_{12} > u_2) \\
= \; & P(U_{10} > u_1, U_{20} > u_2, U_{12} > (u_1 \vee u_2) \\
= \; & \exp(-\int_{t=0}^{u_1} r_{10}(t)dt - \int_{t=0}^{u_2} r_{20}(t)dt - \int_{t=0}^{u_1 \vee u_2} r_{12}(t)dt) \qquad (30)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& P(U_1' > u_1, U_2' > u_2) \\
= \; & \exp(-\int_{t=0}^{u_1} (r_{10}(t) + \epsilon(t))dt - \int_{t=0}^{u_2} (r_{20}(t) + \epsilon(t))dt - \int_{t=0}^{u_1 \vee u_2} (r_{12}(t) - \epsilon(t))dt)
\end{aligned}
$$

Thus the result follows.

Note also that

$$
\alpha'(t) = r_{10}'(t) + r_{20}'(t) + r_{12}'(t) = \alpha(t) + \varepsilon(t) \qquad (31)
$$

Corollary 17 Assume that $(\tilde{T}_x, \tilde{T}_y), U_{10}, U_{20,},$ and U_{12} are independent. Similarly assume that $(\tilde{T}_x', \tilde{T}_y'), U_{10}', U_{20,}',$ and U_{12}' are independent. Assume that the random vectors $(\tilde{T}_x, \tilde{T}_y)$ and $(\tilde{T}_x', \tilde{T}_y')$ are identically distributed and $U_{10}, U_{20,},$ $U_{12},$ $U_{10}', U_{20,}',$ and U_{12}' have have hazard rates $r_{10}(t),$ $r_{20}(t),$ $r_{12}(t),$ $r_{10}'(t),$ $r_{20}'(t),$ $r_{12}'(t),$ respectively, such that (24)-(26) hold. Then

$$
(\tilde{T}_x \wedge U_1, \tilde{T}_y \wedge U_2) \underset{c}{\geq} (\tilde{T}_x' \wedge U_1', \tilde{T}_y' \wedge U_2') \qquad (32)
$$

Proof.

$$\begin{aligned}
& P(\tilde{T}_x \wedge U_1 > u, \tilde{T}_y \wedge U_2 > v) \\
= \;& P(\tilde{T}_x \wedge U_{10} \wedge U_{12} > u, \tilde{T}_y \wedge U_{20} \wedge U_{12} > v) \\
= \;& P(\tilde{T}_x > u, U_{10} > u, \tilde{T}_y > v, U_{20} > v, U_{12} > \max(u,v)) \\
= \;& P(\tilde{T}_x > u, \tilde{T}_y > v) \exp(-\int_{t=0}^{u_1} r_{10}(t)dt - \int_{t=0}^{u_2} r_{20}(t)dt - \int_{t=0}^{u_1 \vee u_2} r_{12}(t)dt) \\
\geq \;& P(\tilde{T}_x > u, \tilde{T}_y > v) \exp(-\int_{t=0}^{u_1} (r_{10}(t) + \epsilon(t))dt - \int_{t=0}^{u_2} (r_{20}(t) + \epsilon(t))dt - \int_{t=0}^{u_1 \vee u_2} (r_{12}(t) - \epsilon(t))dt) \\
= \;& P(\tilde{T}'_x \wedge U'_1 > u, \tilde{T}'_y \wedge U'_2 > v)
\end{aligned}$$

Thus, all the inequalities derived in Section 3 are valid for this model.

Remark 5

1. In the BCC model we can measure the impact of increasing dependence, on the actuarial quantities. Increasing dependence in this model is done by increasing $r_{12}(t)$ and decreasing both $r_{10}(t)$ and $r_{20}(t)$ such that $r_{10}(t) + r_{12}(t)$, and $r_{20}(t) + r_{12}(t)$ remain fixed.
2. Reducing dependence in the BCC model as described in (24)-(26) increases the rate of total number of accidents from $\alpha(t)$ to $\alpha'(t) = \alpha(t) + \epsilon(t)$. Note that the distribution of the number of accidents that hits either x or y do not change.

4.1.2 Bounds

Consider a BCC family of distributions where all joint distributions in the family have the samr margnal distribution with hazard rates $r_1(t)$ and $r_2(t)$. Let $(\breve{U}_1, \breve{U}_2)$ be a random vector with the following distribution:

$$P(\breve{U}_1 > u_1, \breve{U}_2 > u_2) = \exp(-\int_{t=0}^{u_1} \breve{r}_{10}(t)dt - \int_{t=0}^{u_2} \breve{r}_{20}(t)dt - \int_{t=0}^{u_1 \vee u_2} \breve{r}_{12}(t)dt), \quad (33)$$

where,

$$\begin{aligned}\check{r}_{10}(t) &= r_1(t) - r_1(t) \wedge r_2(t) \\ \check{r}_{20}(t) &= r_2(t) - r_1(t) \wedge r_2(t) \\ \check{r}_{12}(t) &= r_1(t) \wedge r_2(t)\end{aligned}$$

Remark 6 1. Note that marginal distributions of $(\breve{U}_1, \breve{U}_2)$ have hazard rates $r_1(t)$ and $r_2(t)$ respectively.

2. Note that equation (28) implies that $\check{r}_{12}(t) = r_1(t) \wedge r_2(t) \geq r_{12}(t)$, for all BCC distributions with the same marginals.

Proposition 18 Assume that (U_1, U_2) has BCC distribution. Assume that hazard rate function of U_j is $r_j(t)$, $j = 1, 2$. The joint distribution of $(\breve{U}_1, \breve{U}_2)$ is as in (33). Let $(U_1^{\perp}, U_2^{\perp})$ be the independent version of (U_1, U_2). Then:

$$(U_1^{\perp}, U_2^{\perp}) \underset{c}{\leq} (U_1, U_2) \underset{c}{\leq} (\breve{U}_1, \breve{U}_2). \tag{34}$$

Proof. Note that

$$P(U_1 > u_1, U_2 > u_2) = \begin{cases} \exp(-\int\limits_{t=0}^{u_1} r_1(t)dt - \int\limits_{t=0}^{u_2} r_{20}(t)dt) & \text{if } u_1 \geq u_2 \\ \exp(-\int\limits_{t=0}^{u_2} r_2(t)dt - \int\limits_{t=0}^{u_1} r_{10}(t)dt) & \text{if } u_2 \geq u_1 \end{cases} \tag{35}$$

Thus, (35) and (38) imply that

$$\begin{aligned} P(U_1 &> u_1, U_2 > u_2) \geq \exp(-\int\limits_{t=0}^{u_1} r_1(t)dt - \int\limits_{t=0}^{u_2} r_2(t)dt) \\ &= P(U_1 > u_1)P(U_2 > u_2) \end{aligned} \tag{36}$$

Thus the left hand side of (34) holds.

To show right inequality, assume first that $u_1 \leq u_2$. Equations (21), (30) and Remark 6 imply that

$$\begin{aligned}
P(U_1 &> u_1, U_2 > u_2) = \exp(-\int\limits_{t=0}^{u_1}(r_1(t) - r_{12}(t))dt - \int\limits_{t=0}^{u_2} r_2(t)dt) \\
&\leq \exp(-\int\limits_{t=0}^{u_1}(r_1(t) - r_1(t) \wedge r_2(t))dt - \int\limits_{t=0}^{u_2} r_2(t)dt) \\
&= P(\breve{U}_1 > u_1, \breve{U}_2 > u_2). \qquad (37)
\end{aligned}$$

Similar inequality holds when $u_1 \geq u_2$. Thus inequality of (34) follows from the definition of concordance ordering.

Remark 7 Let $\breve{\alpha}(t)$, $\alpha^{\perp}(t)$ be the accident rates for the models with joint distributions corresponding to $(\breve{U}_1, \breve{U}_2)$ and $(U_1^{\perp}, U_2^{\perp})$ respectively. Then

1. $\breve{\alpha}(t) = r_1(t) \vee r_2(t)$.
2. $\alpha^{\perp} = r_1(t) + r_2(t)$.

Note that $\breve{\alpha}$ and $\alpha^{\perp}$ are the minimal and the maximal rates, of accidents in the BCC model with fixed marginals.

Corollary 19 1. $U_1^{\perp} \wedge U_2^{\perp} \underset{st}{\leq} U_1 \wedge U_2 \underset{st}{\leq} \breve{U}_1 \wedge \breve{U}_2$

2. $\breve{U}_1 \vee \breve{U}_2 \underset{st}{\leq} U_1 \vee U_2 \underset{st}{\leq} U_1^{\perp} \vee U_2^{\perp}$

In the next subsection we consider a well known special BCC model known as the Marshall and Olkin bivariate Exponential distribution.

4.2 Marshall and Olkin bivariate Exponential distribution.

In this section we assume that:

1. The time until an accident that kills x (y) only, U_{10}, (U_{20}), is exponentially distributed with parameter λ_{10} (λ_{20}). The time until an accident that

kills both husband and wife, U_{12}, is exponentially distributed with parameter λ_{12}. Recall that the density of an exponential random variable X with parameter θ is

$$f_X(x) = \theta e^{-\theta x} \quad x > 0$$

We assume that these three random variables are independent and independent of $(\tilde{T}_x, \tilde{T}_y)$. This model is known as Marshall and Olkin (MO) bivariate Exponential distribution. Let U_1, (U_2) be the time until an accident kills the husband (wife). Clearly, U_j, is exponentially distributed with parameter λ_j, where,

$$\begin{aligned} \lambda_1 &= \lambda_{10} + \lambda_{12} \\ \lambda_2 &= \lambda_{20} + \lambda_{12} \end{aligned} \tag{38}$$

The joint distribution of (U_1,U_2) is :

$$P(U_1 > u_1, U_2 > u_2) = \exp(-\lambda_{10}u_1 - \lambda_{20}u_2 - \lambda_{12}(u_1 \vee u_2) \tag{39}$$

Note that the time to the first accident is exponentially distributed with parameter α, where

$$\alpha = \lambda_{10} + \lambda_{20} + \lambda_{12} \tag{40}$$

Consider the family of Marshal-Olkin bi-exponential distributions with fixed marginals, say exponentials with parameters λ_1 and λ_2. Let $(\breve{U}_1, \breve{U}_2)$ be a random vector with the following MO distribution:

$$P(\breve{U}_1 > u_1, \breve{U}_2 > u_2) = \exp(-(\lambda_1 - \lambda_1 \wedge \lambda_2)u_1 - (\lambda_2 - \lambda_1 \wedge \lambda_2)u_2 - (\lambda_1 \wedge \lambda_2)(u_1 \vee u_2)) \tag{41}$$

Let $\bar{a}^{\alpha}_{xy:\bar{n}|/\delta}$ be the EPV of term annuity paid as long as both members of the couple are alive. Accidents occur according to bi-exponential model, with total rate α and the force of interest is δ. Then

$$\bar{a}^{\alpha}_{xy:\bar{n}|/\delta} = \int_{t=0}^{n} e^{-\delta t} \; {}_tp_{xy} \; e^{-\alpha t} dt = \bar{a}_{xy:\bar{n}|}/(\alpha + \delta) \tag{42}$$

where, $\bar{a}_{xy:\overline{n}|/\delta}$, and $\bar{a}_{xy:\overline{n}|/(\alpha+\delta)}$ are the EPV of the annuity when the force of interest is δ and $(\alpha+\delta)$ respectively. Clearly, $\bar{a}_{xy:\overline{n}|/(\alpha+\delta)}$ is decreasing in α. Thus the EPV of the annuity decreases as dependence increases. Similarly let $\bar{A}^{\alpha}_{\overset{1}{x}y:\overline{n}|/\delta}$ be the EPV of unit sum assured given upon the first death provided it is within the term n, when the force of mortality is δ .

$$\begin{aligned}\bar{A}^{\alpha}_{\overset{1}{xy}:\overline{n}|/\delta} &= 1-\delta\bar{a}^{\alpha}_{xy:\overline{n}|/\delta}-e^{-\delta n}P(T_x\wedge T_y\wedge U_{10}\wedge U_{20}\wedge U_{12}>n)\\ &= 1-\delta\bar{a}_{xy:\overline{n}|/(\delta+\alpha)}-e^{-(\delta+\alpha)n}P(T_x\wedge T_y>n) \end{aligned}\tag{43}$$

Clearly, $\bar{A}^{\alpha}_{\overset{1}{xy}:\overline{n}|/\delta}$ is increasing in α.

Similarly, we can find the EPV of annuity paid continuously for n years as long as at least one of the members of the couple is alive. Note that:

$$\bar{a}_{xy:\overline{n}|}+\bar{a}_{\overline{xy}:\overline{n}|}=\bar{a}_{x:\overline{n}|}+\bar{a}_{y:\overline{n}|}\tag{44}$$

Thus

$$\begin{aligned}\bar{a}^{\alpha}_{\overline{xy}:\overline{n}|/\delta} &= \bar{a}^{\alpha}_{x:\overline{n}|/\delta}+\bar{a}^{\alpha}_{y:\overline{n}|/\delta}-\bar{a}^{\alpha}_{xy:\overline{n}|/\delta}\\ &= \bar{a}_{x:\overline{n}|/(\lambda_1+\delta)}+\bar{a}_{y:\overline{n}|/(\lambda_2+\delta)}-\bar{a}_{xy:\overline{n}|/(\alpha+\delta)}\end{aligned}\tag{45}$$

Thus the EPV of the annuity is increasing in α.

The EPV of one unit sum insurance paid to the last survivor is

$$\begin{aligned}\bar{A}^{\alpha}_{\overset{1}{\overline{xy}}:\overline{n}|/\delta} &= 1-\delta\bar{a}^{\alpha}_{\overline{xy}:\overline{n}|/\delta}-e^{-\delta n}P((T_x\wedge U_{10}\wedge U_{12})\vee(T_y\wedge U_{20}\wedge U_{12})>n)\\ &= 1-\delta\bar{a}^{\alpha}_{\overline{xy}:\overline{n}|/\delta}-e^{-\delta n}P(U_{12}\wedge((T_x\wedge U_{10})\vee(T_y\wedge U_{20}))>n) \qquad (46)\\ &= 1-\delta\bar{a}^{\alpha}_{\overline{xy}:\overline{n}|/\delta}-e^{-\lambda_1 n}P(T_x>n)-e^{-\lambda_2 n}P(T_y>n)+e^{-(\alpha+\delta)n}P(T_x\wedge T_y>n)\end{aligned}$$

Thus $\bar{A}_{\overset{1}{\overline{xy}}:\overline{n}|}$ is decreasing in α, that is, increases as the dependence between T_x and T_y increases.

Similarly, define ${}_{x}|\bar{a}^{\alpha}_{y|/\delta}$ the EPV of annuity given to y after the death of x

$$\begin{aligned}{}_{x}|\bar{a}^{\alpha}_{y/\delta} &= E\left[\frac{e^{-\delta(T_x\wedge T_y\wedge U_{10}\wedge U_{20}\wedge U_{12})}-e^{-\delta(T_y v U_{20}\wedge U_{12})}}{\delta}\right]\\ &= \bar{a}_{y/(\lambda_2+\delta)}-\bar{a}_{xy/(\alpha+\delta)}\end{aligned}\tag{47}$$

Clearly for fixed λ_2, ${}_x|\bar{a}_{y/\delta}$ increases in α.

Let us consider annuity at rate 1 while both members of the couple are alive and at rate θ given to x if he survives after y.

$$\begin{aligned} {}_y|\bar{a}^{\alpha}_{\overset{\theta}{xy:\overline{n}|}}/\delta &= E\left[\frac{1-e^{-\delta(T_x\wedge U_{10}\wedge U_{12}\wedge n)}}{\delta} - (1-\theta)\frac{e^{-\delta(T_x\wedge T_y\wedge U_{10}\wedge U_{20}\wedge U_{12}\wedge n)} - e^{-\delta(T_x\wedge U_{10}\wedge U_{12}\wedge n)}}{\delta}\right] \\ &= \bar{a}_{x:\overline{n}|/(\lambda_1+\delta)} - (1-\theta)\big(\bar{a}_{x:\overline{n}|(\lambda_1+\delta)} - \bar{a}_{xy:\overline{n}|/(\alpha+\delta)}\big) \end{aligned} \tag{48}$$

Thus, also ${}_y|\bar{a}_{\overset{\theta}{xy:\overline{n}|/\delta}}$ is decreasing in α.

Similarly, (9) and 44 yield the following expression for the EPV of annuity at rate while both members of the couple are alive and at rate θ to the last survivor

$$\begin{aligned} &\bar{a}^{\alpha}_{\overset{\theta}{xy:\overline{n}|/\delta}} \\ &= \bar{a}_{xy:\overline{n}|/(\alpha+\delta)} + \theta\big(\bar{a}_{x:\overline{n}|/(\lambda_1+\delta)} + \bar{a}_{y:\overline{n}|/(\lambda_2+\delta)} - \bar{a}_{xy:\overline{n}|/(\alpha+\delta)} - \bar{a}_{xy:\overline{n}|/(\alpha+\delta)}\big) \\ &= (1-2\theta)\bar{a}_{xy:\overline{n}|/(\alpha+\delta)} + \theta\big(\bar{a}_{x:\overline{n}|/(\lambda_1+\delta)} + \bar{a}_{y:\overline{n}|/(\lambda_2+\delta)}\big) \end{aligned} \tag{49}$$

Thus for fixed λ_1 and λ_2, $\bar{a}^{\alpha}_{\overset{\theta}{xy:\overline{n}|/\delta}}$ is decreasing in α for $0 \leq \theta \leq 1/2$ and increasing in α for $1/2 \leq \theta \leq 1$.

5 An Example

In the following example we consider couple where the husband and the wife are at ages 35 and 30 respectively. The husband is subject to mortality Table a-(55m) and the wife to mortality table a-(55f), given in Formulae and Tables for Actuarial Examinations (1997). We assume that the times until natural death of the husband and wife are independent. We assume that the times to accident that might kill the husband and the wife follows M.O bi-exponential distribution, where the time until accident that kills the husband follows exponential distribution at rate 0.0002 and the time until accident that kills the husband follows exponential distribution at rate 0.00017. For different values of λ_{12}, the rate of accidents that kills both, we present in Table 2: 1. A whole life annuity paid until first death $\ddot{a}_{35,30}$. 2. The expected present value of the sum assured given at the end of the year of first death-$A_{30,35}$. 3. The premium paid in the case of joint life -$\Pi_{30,35}$. In Table 3 we calculate: 1. A whole life annuity paid until the death of the last survivor$\ddot{a}_{\overline{30,35}}$. 2 The expected present value of the sum assured given at the end of year of death of the last survival$A_{\overline{30,35}}$. 3. The annual premium paid in the case of last survival whole life insurance$\Pi_{\overline{30,35}}$

Table 2: annuity, EPV of sum assured and annual premium for joint life

λ_{12}	$\ddot{a}_{35,30}$	$A_{30,35}$	$\Pi_{30,35}$
0	21.77201928	0.365863516	0.016804299
0.00005	21.78857751	0.365381238	0.016769394
0.00007	21.79520644	0.365188162	0.016755435
0.00009	21.801838	0.364994992	0.016741478
0.00011	21.80847399	0.364801728	0.016727522
0.00013	21.81511261	0.36460837	0.016713568
0.00015	21.82175447	0.364414918	0.016699616
0.00017	21.82839956	0.364221372	0.016685665

Table 3: annuity, EPV of sum assured and annual premium for last survivor model

λ_{12}	$\ddot{a}_{\overline{30,35}}$	$A_{\overline{30,35}}$	$\Pi_{\overline{30,35}}$
0	26.79257784	0.219633655	0.008197556
0.00005	26.77601961	0.220115934	0.008220637
0.00007	26.76939068	0.220309009	0.008229885
0.00009	26.76275851	0.220502179	0.008239142
0.00011	26.75612312	0.220695443	0.008248409
0.00013	26.7494845	0.220888801	0.008257684
0.00015	26.74284265	0.221082253	0.008266969
0.00017	26.73619756	0.221275799	0.008276263

6 Appendix A

$\bar{A}_{\overset{1}{xy}:\overline{n}|}$ −EPV of unit sum insured payable at the moment at earlier death if it occurs before time n.

$\bar{A}_{xy}$ −EPV of unit sum insured payable at earlier death if it occurs before time n.

$A_{\overset{1}{\overline{xy}}:\overline{n}|}$ −EPV of unit sum insured payable at the end of year of death of the last survivor if this occurs before time n.

$A_{\overline{xy}}$ −EPV of unit sum insured payable at the end of year of death of the last survivor.

$\bar{A}_{\overline{xy}}$ −EPV of unit sum insured payable at the moment of death of the last survivor.

$\bar{a}_{xy:\bar{n}|}$−EPV of continuous stream of payments at rate 1, payable up to n periods or until the first death.

$\bar{a}_{xy}$−EPV of continuous stream of payments at rate 1, payable until the first death.

$\bar{a}_{\overline{xy}:\bar{n}|}$−EPV of continuous stream of payments at rate 1, payable up to n years or until the last death.

$\bar{a}_{\overline{xy}}$−EPV of continuous stream of payments at rate 1, payable until the last death.

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