ON THE LOADING OF LARGEST CLAIMS REINSURANCE COVERS

Erhard Kremer
Fachbereich Mathematik
Universität Hamburg

Bundesstrasse 55, 20146 Hamburg, Germany

Tel. 040/428 38-4934 — Fax 040/428 38-5260 — E-Mail: kremer@math.uni-hamburg.de

The paper consists of two parts:

Part 1: Estimating the loading of the largest claims reinsurance covers.
Part 2: On the loading of the ECOMOR.

Abstract on part 1

The largest claims reinsurance treaties are reconsidered. Two approaches for estimating a certain main part of the loading are given. For the first approach certain bounds are derived, for the second the Monte-Carlo-Integration-method adapted. The second, not so practicable approach can be used for finding adequate mixing coefficients for the first, quite practicable approach.

Abstract on part 2

Unfortunately the in part one given, very practicable, first approach can not be used in case of the ECOMOR-cover, though this treaty is related to the largest claims covers of part 1. So this note gives further, more special results on the loading of the ECOMOR, in case of using again the standard-deviation or variance principle.

Estimating the loading of the largest claims reinsurance covers

1 Introduction

Since the beginning of the eighties the author published many new papers on the theory of the largest claims reinsurance treaty. Mentioned can be his articles on the total claims amount and the efficiency (see e.g. Kremer (1990a), (1990b), (1992)), furthermore his general work on the premium (see e.g. Kremer (1984), (1985) (1986), (1988), (1994), (1998), (2001), (2002)). Many comparably handy results were given on the net premium, whereas for a long time nothing handy was developed for the security loading. General exact formulas for the loading were already given in Kremer (1985), and in addition under different model assumptions in Kremer (2002). Unfortunately those formulas are, like given, much too unhandy for practical application. Further more adequate results are needed. Two such are presented in the present paper. First a very practical approach is given that uses two crude bounds on a part of the loading formula. The second approach consists in adapting adequately the method of Monte-Carlo-Integration. Though this second solution is quite unpractical, it is very helpful for finding out most adequate mixing coefficients for the first approach. All is illustrated in a typical example.

2 The Treaty

Consider a collective of risks of a first insurer and let N denote the random variable of the number of claims. The corresponding claims amounts are described by the random variables $X_1, X_2, X_3, \ldots$. Suppose all random variables are defined on one and the same probability space $(\Omega, \mathcal{A}, P)$. Denote by

$$X_{N:1} \geq X_{N:2} \geq \ldots \geq X_{N:N}$$

the claims ordered in nonincreasing size. Furthermore let $c_1, c_2, c_3, \ldots$ be real constants such that

$$R_p := \sum_{i=1}^{p} c_i \cdot X_{N:i}$$

with a fixed $p \in \mathbb{N}$ describes a certain claims amount to be paid by the reinsurer to the first insurer. Consequently, the family $(c_1, \ldots, c_p)$ defines a reinsurance treaty, called by the author **(generalized) largest claims**

cover with parameter p (in short: **GLC(p)**). For the more special choice

$$c_i = 1, \quad \forall i$$

one gets the **(classical) largest claims treaty** covering the p largest claims (see e.g. Ammeter (1964), Kremer (1982)).

A topic of great practical actuarial interest is the calculation of the (risk) premium of that $GLC(p)$. In risk theory various premium principles were defined and analyzed (see e.g. De Vylder et. al. (1984) or Kremer (1999)). For the (generalized) largest claims cover it is most adequate to take the so-called **standard deviation principle**, giving as risk premium

$$\Pi_p = m_p + \Lambda_p \cdot s_p$$

with the **net premium**

$$m_p = E(R_p)$$

and the standard deviation

$$s_p = [Var(R_p)]^{1/2} \ .$$

The loading factor $\Lambda_p > 0$ shall be given, so that m_p and s_p^2 remain to be of interest. Handy formulas and methods on calculating m_p are given e.g. in Kremer (1985), (1986), (1994), (1998), (2002), so that because of

$$Var(R_p) = E(R_p^2) - m_p^2$$

only the $E(R_p^2)$ remains to be of further interest. Results on $E(R_p^2)$ can also be found in Kremer (1985), (2001), (2002), but they turn out to be too unhandy or too crude.

3 Bounding Approach

Assume for the sequel

$$c_i \geq 0, \quad \forall i \ .$$

One of the two main results of this section is

<u>Theorem 1</u>

Define new coefficients c_{pi} according

$$\begin{aligned} c_{pi} &= \left(c_i + 2 \cdot \sum_{j=i+1}^{p} c_j\right) \cdot c_i, \quad \text{for } i \leq p-1 \\ c_{pp} &= c_p^2 \end{aligned}$$

and with them

$$\begin{aligned} S_p &= \sum_{i=1}^{p} c_{pi} \cdot X_{N:i}^2 \\ q_p &= E(S_p) \ . \end{aligned}$$

One has the upper bound

$$\boxed{E(R_p^2) \leq q_p} \tag{3.1}$$

Proof

One takes the splitting up

$$R_p^2 = \sum_{i=1}^{p} c_i^2 \cdot X_{N:i}^2 + 2 \cdot \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} c_i \cdot c_j \cdot X_{N:i} \cdot X_{N:j} \tag{3.2}$$

where obviously the double sum is bounded from above by

$$\sum_{i=1}^{p-1} c_i \cdot \Big(\sum_{j=i+1}^{p} c_j \Big) \cdot X_{N:i}^2$$

since for $j > i$ holds $X_{N:i} \geq X_{N:j}$. Altogether one has

$$R_p^2 \leq S_p$$

what implies at once the statement. □

The problem arises how to calculate q_p. On this

Theorem 2

Define the random variables

$$\begin{aligned} U_p &= \sum_{i=1}^{p} c_i \cdot X_{N:i}^2 \\ V_p &= \sum_{i=1}^{p} c_i^2 \cdot X_{N:i}^2 \end{aligned}$$

and their means

$$\begin{aligned} u_p &= E(U_p) \\ v_p &= E(V_p) \ . \end{aligned}$$

With these one has the recursion

$$\boxed{q_p = q_{p-1} + 2 \cdot c_p \cdot u_{p-1} + v_p - v_{p-1}} \tag{3.3}$$

for $p \geq 2$ starting with

$$q_1 = c_1^2 \cdot E(X_{N:1}^2) \ .$$

Proof

One simply gets by the definition of the c_{pi} for $i \leq p-1$

$$c_{pi} - c_{(p-1)i} = 2 \cdot c_i \cdot c_p$$

and consequently

$$S_p - S_{p-1} = 2 \cdot c_p \cdot U_{p-1} + c_p^2 \cdot X_{N:p}^2 \ .$$

Since

$$c_p^2 \cdot X_{N:p}^2 = V_p - V_{p-1} \ ,$$

one arrives at the statement by taking the expectations. □

Now the new problem of calculating u_p, v_p is there. On this see the following section. As second main result

Theorem 3

Suppose $X_1, X_2, X_3, \ldots$ are i.i.d. with distribution function F and density f and that they are independent of N. Assume that f is continuous and strictly positive on $\{x : 0 < F(x) < 1\}$. Then one has the lower bound

$$\boxed{E(R_p^2) \geq v_p + 2 \cdot t_p} \tag{3.4}$$

with

$$\begin{aligned} t_p &= \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} (m_i - m_{i-1})(m_j - m_{j-1}) \\ &= m_p \cdot m_{p-1} - \sum_{i=2}^{p-1} (m_i - m_{i-1}) \cdot m_i - m_1^2 \ . \end{aligned} \tag{3.5}$$

(with convention $m_0 := 0$).

Proof

By (3.2) one has

$$E(R_p^2) = v_p + 2 \cdot \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} c_i\, c_j \cdot E(X_{N:i} \cdot X_{N:j}) \tag{3.6}$$

In analogy to exercise 3.1.12 in David (1981) the inequality

$$Cov(X_{N:i}, X_{N:j}) \geq 0$$

holds, what is equivalent with

$$E(X_{N:i} \cdot X_{N:j}) \geq E(X_{N:i}) \cdot E(X_{N:j}) \ .$$

This one substitutes into (3.6) and uses

$$E(X_{N:i}) = \frac{m_i - m_{i-1}}{c_i}$$

for $E(X_{N:i})$ and $E(X_{N:j})$. So one gets (3.4) with the t_p according to (3.5). □

Also in the bound (3.4) the new problem of calculating (efficiently) v_p appears.

Note that (3.5) can be written more elegantly as a recursion

$$\boxed{t_p = t_{p-1} + (m_p - m_{p-1}) \cdot m_{p-1}} \tag{3.7}$$

with start

$$t_1 = 0 \ .$$

For estimating $E(R_p^2)$ crudely, one is willing to take an adequate mixture of the upper bound q_p and the lower bound

$$w_p = v_p + 2 \cdot t_p \ . \tag{3.8}$$

The simplest choice would be

$$\frac{1}{2} \cdot q_p + \frac{1}{2} \cdot w_p \ . \tag{3.9}$$

A discussion of this and a more adequate proposal is given for the classical largest claims reinsurance cover in the section 6.

4 Helpful Results

For calculating the q_p and w_p one needs handy results on m_p, u_p, v_p. According to Kremer (1985) one has for continuous distribution function F of the i.i.d. X_i, independent of the N, as general formulas for the m_p, u_p, v_p

$$\begin{aligned} m_p &= \sum_{i=1}^{p} \frac{c_i}{\Gamma(i)} \cdot \int_0^1 F^{-1}(u) \cdot (1-u)^{i-1} M^{(i)}(u)\, du \\ u_p &= \sum_{i=1}^{p} \frac{c_i}{\Gamma(i)} \cdot \int_0^1 [F^{-1}(u)]^2 \cdot (1-u)^{i-1} M^{(i)}(u)\, du \\ v_p &= \sum_{i=1}^{p} \frac{c_i^2}{\Gamma(i)} \cdot \int_0^1 [F^{-1}(u)]^2 \cdot (1-u)^{i-1} M^{(i)}(u)\, du \end{aligned}$$

where

$$\begin{aligned} F^{-1}(u) &= \inf\{x : F(x) \geq u\} \\ \Gamma(i) &= (i-1)! \end{aligned}$$

and with the $i-th$ derivative $M^{(i)}(u)$ of the **probability generating function** of N

$$M(u) = \sum_{n=0}^{\infty} P(N=n) \cdot u^n \ .$$

For the author it is quite sure that these general fomulas are too terrible for the practical actuary. He likes to have more handy results. Under additional model assumptions one gets such. So assume that N is Poisson-distributed with parameter $\lambda = E(N) > 0$, i.e.

$$P(N=n) = \frac{\lambda^n}{n!} \cdot \exp(-\lambda), \quad n = 0, 1, 2, 3, \ldots$$

and the X_i is Pareto-distributed with parameter $\alpha > 2$ and start value $a > 0$, i.e.

$$F(x) = 1 - \left(\frac{x}{a}\right)^{-\alpha}, \ x \geq a \ .$$

Then the above formulas simplify to

$$m_p = \lambda^{1/\alpha} \cdot \sum_{i=1}^{p} \frac{c_i}{\Gamma(i)} \cdot \Gamma_\lambda(i - 1/\alpha) \cdot a \tag{4.1}$$

$$u_p = \lambda^{2/\alpha} \cdot \sum_{i=1}^{p} \frac{c_i}{\Gamma(i)} \cdot \Gamma_\lambda(i - 2/\alpha) \cdot a^2 \tag{4.2}$$

$$v_p = \lambda^{2/\alpha} \cdot \sum_{i=1}^{p} \frac{c_i^2}{\Gamma(i)} \cdot \Gamma_\lambda(i - 2/\alpha) \cdot a^2 \tag{4.3}$$

with the incomplete Gamma-function

$$\Gamma_\lambda(s) = \int_0^\lambda \exp(-t) \cdot t^{s-1}\, dt \ .$$

For larger λ one can replace Γ_λ by $\Gamma = \Gamma_\infty$. But often also these formulas will appear to be too unhandy or inadequate. In these situations the author proposes to use his recursive method in Kremer (1994) for approximate computation of the m_p, u_p, v_p. So for m_p one has the approximate recursion

$$\begin{aligned} m_p \ &\doteq \ m_{p-1} \cdot \left[1 + K_p \cdot \left(1 - \left(\frac{k}{p-1}\right)\right)\right] \\ &\quad -m_{p-2} \cdot K_p \cdot \left(1 - \left(\frac{k}{p-1}\right)\right), \quad p \geq 3 \end{aligned} \tag{4.4}$$

with $K_p = c_p/c_{p-1}$ and an adequate small number $k > 0$. The counterpart for u_p is

$$\begin{aligned} u_p \ &\doteq \ u_{p-1} \cdot \left[1 + K_p \cdot \left(1 - \left(\frac{2k}{p-1}\right)\right)\right] \\ &\quad -u_{p-2} \cdot K_p \cdot \left(1 - \left(\frac{2k}{p-1}\right)\right), \quad p \geq 3 \end{aligned} \tag{4.5}$$

and for v_p

$$\begin{aligned} v_p \ &\doteq \ v_{p-1} \cdot \left[1 + K_p^2 \cdot \left(1 - \left(\frac{2k}{p-1}\right)\right)\right] \\ &\quad -v_{p-2} \cdot K_p^2 \cdot \left(1 - \left(\frac{2k}{p-1}\right)\right), \quad p \geq 3 \end{aligned} \tag{4.6}$$

Remember that behind these recursions there stands the general assumption that the claims number N satisfies with given parameters a, b the recursion

$$P(N = n) = P(N = n - 1) \cdot (a + b/n)$$

for $n = 1, 2, 3, \ldots$ (compare Kremer (1986), (1994)). Note also that these recursions are only adequate for „not too large“ p.

5 Monte-Carlo-Approach

For the sequel assume that the $X_i, i = 1, 2, 3, \ldots$ are i.i.d. with continuous distribution function F and that they are independent of N. Again a result in Kremer (1985) gives that

$$E(R_p^2) = v_p + \rho_p \tag{5.1}$$

where v_p is that given already in sections 3–4 and

$$\rho_p = 2 \cdot \sum_{j=2}^{p} \sum_{i=1}^{j-1} \frac{c_j}{\Gamma(j-i)} \cdot \frac{c_i}{\Gamma(i)} \cdot S_{ij}$$

where

$$S_{ij} = \int_0^1 \int_0^v F^{-1}(v) \cdot F^{-1}(u) \cdot (v-u)^{j-i+1} \cdot (1-v)^{i-1} \cdot M^{(j)}(u) du\, dv \ ,$$

in what

$$F^{-1}(u) := \inf\{x : F(x) \geq u) \ ,$$

is the so-called **Pseudo-Inverse** of F, and $M^{(j)}$ is the j-th derivative of

$$M(t) = \sum_{n=0}^{\infty} P(N = n) \cdot t^n \ ,$$

the probability generating function of the distribution of N. Practicable results on computing v_p where already given in section 4. So the crucial part in (5.1) is the ρ_p. A simple lower bound on ρ_p was already fiven in Theorem 3. Since that bound is expected to be quite crude, the question arises whether one perhaps could calculate nearly exactly the ρ_p with methods of numerical mathematics. It is nearlying to apply numerical integration methods to the S_{ij} in ρ_p. Most elegant appears to be the method of **Monte-Carlo-Integration** (see e.g. Robert & Casella (2000)).

So suppose one has $2K$ standard-pseudo-random numbers

$$U_1, V_1, U_2, V_2, \ldots, U_K, V_K \ .$$

The simulated value for S_{ij} is then

$$\hat{S}_{ij} = \frac{1}{K} \cdot \sum_{k=1}^{K} f_k(i,j) \tag{5.2}$$

with

$$f_k(i,j) = 0 \ , \quad \text{if} \quad U_k \geq V_k$$

and

$$f_k(i,j) = F^{-1}(V_k) \cdot F^{-1}(U_k) \cdot (V_k - U_k)^{j-i-1} \cdot (1-V_k)^{i-1} \cdot M^{(j)}(U_k) \ , \quad \text{if} \quad U_k < V_k \ .$$

According to classical probability theory the $\hat{S}_{ij}$ converges almost surely against the exact S_{ij}, when K goes to infinity. Consequently, K has to be chosen considerably large. The $\hat{S}_{ij}$ one puts into the formula for ρ_p, resulting in an estimated $\hat{\rho}_p$.

6 Example

Take just the (classical) largest claims treaty, covering the p largest claims. Assume basically the conditions underlying the formulas (4.1) - (4.3). Take the parameter $\lambda > 0$ of the Poisson distribution to be larger, more concretely such that the Γ_λ in (4.1) - (4.3) (with $c_i = 1, \forall i$) can be replaced by $\Gamma_\infty = \Gamma$. Then the formulas (4.1) - (4.3) reduce (with $c_i = 1 \ \forall i$) to

$$m_p = \lambda^{1/\alpha} \cdot \left(\frac{\alpha}{\alpha - 1}\right) \cdot \frac{\Gamma(p + 1 - 1/\alpha)}{\Gamma(p)} \cdot a \qquad (6.1)$$

$$u_p = v_p = \lambda^{2/\alpha} \cdot \left(\frac{\alpha}{\alpha - 2}\right) \cdot \frac{\Gamma(p + 1 - 2/\alpha)}{\Gamma(p)} \cdot a^2 \qquad (6.2)$$

(compare already Ammeter (1964)). Consequently one has for the startvalues in (3.3), (3.7), (4.4), (4.5) ($\equiv$ (4.6))

$$m_1 = \lambda^{1/\alpha} \cdot \frac{\alpha}{\alpha - 1} \cdot \Gamma(2 - 1/\alpha) \cdot a \qquad (6.3)$$

$$m_2 = \lambda^{1/\alpha} \cdot \frac{\alpha}{\alpha - 1} \cdot \Gamma(3 - 1/\alpha) \cdot a \qquad (6.4)$$

$$u_1 = \lambda^{2/\alpha} \cdot \frac{\alpha}{\alpha - 2} \cdot \Gamma(2 - 2/\alpha) \cdot a^2 \qquad (6.5)$$

$$v_1 = q_1 = u_1 \qquad (6.6)$$

$$u_2 = \lambda^{2/\alpha} \cdot \frac{\alpha}{\alpha - 2} \cdot \Gamma(3 - 2/\alpha) \cdot a^2 \tag{6.7}$$

$$v_2 = u_2 \ . \tag{6.8}$$

Finally in (4.4), (4.5) ((4.5)$\equiv$ (4.6)) one has simply to insert

$$K_p = K_p^2 = 1 \ .$$

Note that for $k = 1/\alpha$ the recursions (4.4), (4.5) give the same results like the *rhs* of (6.1) and (6.2) (see on this Kremer (1994)).

With the calculated $m_p, u_p = v_p$ one can apply the bounds (3.1), (3.8), where q_p has to be computed with recursion (3.3) (with $c_p = 1$ and $v_p = u_p$) and t_p with recursion (3.7). With (3.9) one gets a first crude estimate of $E(R_p^2)$, denoted by $r_p^{(1)}$. With that a first estimate of s_p is just

$$s_p^{(1)} = (r_p^{(1)} - m_p^2)^{1/2} \ .$$

Now on the method of section 5. Under the given assumptions the S_{ij} can be rearranged to

$$S_{ij} = a^2 \cdot \lambda^j \cdot \int_0^1 \int_t^1 t^{i-1-1/\alpha} \cdot s^{-1/\alpha} \cdot (s-t)^{j-i-1} \cdot \exp\left(-\lambda \cdot s\right) \, ds \, dt \ .$$

For giving this, one has to take the substitutions

$$s = 1 - u, \quad t = 1 - v$$

and to remember that

$$\begin{aligned} F^{-1}(u) &= a \cdot (1-u)^{-1/\alpha} \\ M^{(j)}(u) &= \lambda^j \cdot \exp\left(\lambda \cdot (u-1)\right) \end{aligned}$$

As modified, simulated value for S_{ij} one takes now (5.2) with

$$f_k(i,j) = 0, \quad \text{if } S_k \leq T_k$$

and

$$f_k(i,j) = a^2 \cdot \lambda^j \cdot T_k^{i-1-1/\alpha} \cdot S_k^{-1/\alpha} \cdot (S_k - T_k)^{j-i-1} \cdot \exp\left(-\lambda \cdot S_k\right), \quad \text{if } S_k > T_k.$$

Here

$$S_1, T_1, S_2, T_2, \ldots, S_K, T_K$$

are again $2K$ standard-pseudo-random numbers. For very large K the

$$r_p = v_p + \hat{\rho}_p$$

($\hat{\rho}_p$ is the ρ_p with S_{ij} replaced by $\hat{S}_{ij}$) can be expected to be nearly exactly the exact value of $E(R_p^2)$.

In practice one is mostly more interested in **rates** than in the absolute values. This means that one divides $u_p, v_p, q_p, w_p, r_p^{(1)}, r_p$ by μ^2 with

$$\mu = E(N) \cdot E(X_i)$$

and $s_p^{(1)}, m_p$ by μ. Note that under the given Poisson-Pareto-model one has just

$$\mu = \lambda \cdot \left(\frac{\alpha}{\alpha - 1} \right) \cdot a \ .$$

The so calculated rates corresponding to $m_p, u_p, v_p, q_p, w_p, r_p^{(1)}, s_p^{(1)}, r_p$ shall be denoted by

$$\tilde{m}_p, \tilde{u}_p, \tilde{v}_p, \tilde{q}_p, \tilde{w}_p, \tilde{r}_p^{(1)}, \tilde{s}_p^{(1)}, \tilde{r}_p \ .$$

The author made numerical calculations for the choices

i) $\lambda = 100, \ a = 1, \ \alpha = 2.5$

First the computations of $\tilde{q}_p, \tilde{w}_p$ were made with the (nearly) exact results (6.1), (6.2), resulting in table 1. Then for a second time the computations were done with the recursions (4.4), (4.5) (remember $v_p = u_p$) with $K_p = 1$ and $k = 0.3$, giving as result table 2. Note that with $k = 1/\alpha = 0.4$ one would have got the same results like with (6.1), (6.2). Finally the simulation fo $\tilde{r}_p$ was carried through with $K = 2\,5000\,000$. The resulting values are given in table 3. Note, all numbers are in percent of μ^2 or μ.

ii) $\lambda = 100, \ a = 1, \ \alpha = 3.0$

The computations of $\tilde{q}_p, \tilde{m}_p$ were done with (4.4), (4.5) with $K_p = 1$ and $k = 1/3$. The results are given in table 4, again in percent of μ^2 or μ. Finally the simulation of $\tilde{r}_p$ was carried through again with $N = 2\,500\,000$, resulting in table 5.

Table 1 case i)

	p=1	2	3	4	5	6	7	8
$\tilde{m}_p$	5.64	9.02	11.73	14.07	16.18	18.12	19.94	21.65
$\tilde{u}_p = \tilde{v}_p$	0.66	0.79	0.87	0.93	0.97	1.01	1.04	1.07
$\tilde{q}_p$	0.66	2.11	3.76	5.56	7.46	9.44	11.50	13.62
$\tilde{w}_p$	0.66	1.17	1.74	2.35	2.99	3.65	4.34	5.06
$\tilde{r}_p^{(1)}$	0.66	1.64	2.75	3.95	5.22	6.54	7.92	9.34
$\tilde{s}_p^{(1)}$	5.85	9.09	11.72	14.05	16.13	18.06	19.85	21.56

Table 2 case i)

	p=1	2	3	4	5	6	7	8
$\tilde{m}_p$	5.64	9.02	11.90	14.48	16.88	19.13	21.26	23.31
$\tilde{u}_p = \tilde{v}_p$	0.66	0.79	0.88	0.96	1.02	1.07	1.12	1.17
$\tilde{q}_p$	0.66	2.16	3.78	5.62	7.59	9.68	11.88	14.18
$\tilde{w}_p$	0.66	1.17	1.78	2.47	3.23	4.04	4.90	5.83
$\tilde{r}_p^{(1)}$	0.66	1.67	2.78	4.05	5.41	6.86	8.39	10.00
$\tilde{s}_p^{(1)}$	5.85	9.23	11.68	13.97	16.00	17.89	19.68	21.38

Table 3 case i)

	p=2	3	4	5	6	7	8
$\tilde{r}_p$	1.24	1.85	2.49	3.14	3.82	4.53	5.25

Table 4 case ii)

	p=1	2	3	4	5	6	7	8
$\tilde{m}_p$	4.19	6.98	9.31	11.38	13.28	15.05	17.73	19.31
$\tilde{u}_p = \tilde{v}_p$	0.26	0.34	0.40	0.44	0.48	0.51	0.41	0.57
$\tilde{q}_p$	0.26	0.86	1.60	2.44	3.36	4.35	5.41	6.51
$\tilde{w}_p$	0.26	0.57	0.96	1.39	1.86	2.36	2.89	3.45
$\tilde{r}_p^{(1)}$	0.26	0.72	1.28	1.91	2.61	3.35	4.15	4.98
$\tilde{s}_p^{(1)}$	2.91	4.77	6.40	7.86	9.20	10.45	11.64	12.76

Table 5 case ii)

	p=2	3	4	5	6	7	8
$\tilde{r}_p$	0.60	1.01	1.46	1.95	2.47	3.02	3.60

The numerical results of the tables show the following

1.) It makes no great difference working with (4.4), (4.5) (with start values (6.3)-(6.8)) than with (6.1), (6.2) (compare tables 1 and 2) when k is chosen adequately.

2.) For longer p values the upper bound $\tilde{q}_p$ and the lower bound $\tilde{w}_p$ differ considerably. Obviously the more, the more risky the claims size distribution is (compare tables 1 and 4).

3.) The crude rule of thumb (3.9) gives senseful results. But $\tilde{r}_p^{(1)}, r_p$ differ still considerably (compare tables 1, 3 and 4, 5).

According to 3.) one likes to see what happens for the choice

$$\frac{1}{3} \cdot q_p + \frac{2}{3} \cdot w_p \ , \tag{6.9}$$

as estimator for $E(R_p^2)$, denoted by $r_p^{(2)}$. From tables 1 and 4 the author computed the relevant rates $\tilde{r}_p^{(2)} = r_p^{(2)}/\mu^2$, given in tables 6 and 7 (all in percent of μ^2). Also given there is $\tilde{s}_p^{(2)}$, the rate corresponding to

$$s_p^{(2)} = (r_p^{(2)} - m_p^2)^{1/2} \ .$$

Table 6 case i)

	p=1	2	3	4	5	6	7	8
$\tilde{r}_p^{(2)}$	0.66	1.48	2.41	3.42	4.47	5.58	6.72	7.90
$\tilde{s}_p^{(2)}$	5.85	8.18	10.10	12.00	13.63	15.16	16.59	17.95

Table 7 case ii)

	p=1	2	3	4	5	6	7	8
$\tilde{r}_p^{(2)}$	0.26	0.67	1.17	1.73	2.36	3.02	3.73	4.47
$\tilde{s}_p^{(2)}$	2.85	4.26	5.51	6.66	7.72	8.72	9.67	10.57

Comparison with tables 3 and 5 shows that choice (6.9) gives still too high values. One likes to try now e.g.

$$\boxed{\tfrac{1}{5} \cdot q_p + \tfrac{4}{5} \cdot w_p \ ,} \tag{6.10}$$

as estimator for $E(R_p^2)$ (in the bounding approach), denoted by $r_p^{(3)}$. Again the author computed the values $\tilde{r}_p^{(3)} = r_p^{(3)}/\mu^2$, given in tables 8 and 9 (all in percent of μ^2). Also given is $\tilde{s}_p^{(3)}$, the rate corresponding to

$$s_p^{(3)} = (r_p^{(3)} - m_p^2)^{1/2} \ .$$

Table 8 case i)

	p=1	2	3	4	5	6	7	8
$\tilde{r}_p^{(3)}$	0.66	1.34	2.14	2.99	3.88	4.80	5.77	6.77
$\tilde{s}_p^{(3)}$	5.85	7.85	8.76	10.06	11.25	12.35	13.40	14.44

Table 9 case ii)

	p=1	2	3	4	5	6	7	8
$\tilde{r}_p^{(3)}$	0.26	0.63	1.09	1.60	2.16	2.76	3.39	4.06
$\tilde{s}_p^{(3)}$	2.91	3.80	4.68	5.50	6.22	7.03	7.74	8.42

Comparison with tables 3 and 5 shows that choice (6.10) as estimate of $E(R_p^2)$ is quite acceptable.

As consequence the author fully can recommend the method of section 2 (combined with (4.4)-(4.6)), with choice (6.10) as final estimate of $E(R_p^2)$, when rating the (classical) largest claims reinsurance cover in motor-liability insurance.

7 Final Remarks

The numerical investigations of section 6 show that the method of section 3 can be used in practice, when calculating a security loading. When applying the method of section 5 one has to take a quite large K. Then, usually in case of that method the computing times are quite large, often too large for the taste of the practical actuary. The practical actuary surely is willing to prefer the recursive method of section 3 (with in addition the recursions (4.4)-(4.6)). Nevertheless, the method of section 5 can be used by him to find out the adequate mixing coefficient α in the final estimate

$$\alpha \cdot q_p + (1 - \alpha) \cdot w_p$$

of $E(R_p^2)$ in case of the recursive method.

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On the loading of the ECOMOR

1 Introduction

Recently the author investigated with bounds on the loading of the largest claims reinsurance covers (see Kremer (2001),(2003)). Unfortunately, the there given general bounds are not applicable to the ECOMOR-cover, though this treaty is related to the largest claims covers. Clearly the results in Kremer (1985), (2002) can be specialized also to the ECOMOR-cover. But those specializations are too unhandy for practical, actuarial applications. One likes to have further, more handy results. These are given in the following note.

2 The Set-Up

Consider a collective of risks of a first insurer and let N denote the random variable of the number of claims. The corresponding claims amounts are described by the (nonnegative) random variables $X_1, X_2, X_3, \ldots$. Suppose all random variables are defined on one and the same probability space $(\Omega, \mathcal{A}, P)$. Denote with

$$X_{N:1} \geq X_{N:2} \geq \ldots \geq X_{N:N}$$

the claims ordered in nonincreasing size. In this context the classical **largest claims reinsurance cover** (in short: $\mathbf{L(p)}$) is defined with a given $p \in \mathbb{N}$ through the reinsurer's claims amount

$$RL(p) := \sum_{i=1}^{p(N)} X_{N:i}$$

where

$$p\,(n) = \min(n, p)\ .$$

A related treaty is (for $p \geq 2$) the so-called **ECOMOR** (= d'exédent du coût moyen relativ) (in short: $\mathbf{E(p)}$).

Its reinsurer's claims amount is given as

$$RE(p) := \sum_{i=1}^{p(N)} (X_{N:i} - X_{N:p})$$

(with $X_{N:p} = 0$ for $p > N$). For both treaties one important topic is the determination of a risk adequate **premium**, defined, when choosing the so-called standard deviation principle (compare Kremer (1999)), as

$$\pi(T) = \nu(T) + \lambda \cdot \eta(T)$$

for $T = RL(p)$ or $T = RE(p)$ with the **net premium**

$$\nu(T) = E(T)$$

the (here given!) loading factor $\lambda > 0$, and the loading part

$$\eta(T) = [Var(T)]^{1/2} \ .$$

It is wellknown that

$$Var(T) = \rho(T) - (\nu(T))^2 \tag{2.1}$$

with

$$\rho(T) = E(T^2) \ .$$

So, when having a result on the net premium, one needs in addition results on $\rho(T)$. According to what was said in the introduction, the present paper gives additional (additional to Kremer (1985), (2002)), more simple results on $\rho(RE(p))$.

3 Exact Result

In Kremer (1990) it turned out that the concept of the so-called spacings can be helpful in deriving handy results also on the ECOMOR-treaty. The (normed) spacings $Z_{N1}, \ldots, Z_{NN}$ are defined as

$$Z_{Ni} = i \cdot (X_{N:i} - X_{N:(i+1)}) \qquad \text{for } i = 1, 2, \ldots, N-1$$

$$Z_{NN} = N \cdot X_{N:N}$$

With these one has

Theorem 1

Given $N = n$ let the spacings $Z_{n1}, \ldots, Z_{nn}$ be independent with the same mean μ and variance σ^2. Then one has

$$\rho(RE(p)) = [E(p\,(N)) - 1] \cdot \sigma^2 + Var(p\,(N)) \cdot \mu^2 + [\nu(RE(p))]^2$$

Proof:

One knows from probability theory that

$$Var(RE(p)) = E(Var(RE(p)|N)) + Var(E(RE(p)|N)) \ . \qquad (3.1)$$

Since

$$RE(p) = \sum_{i=1}^{p\,(N)-1} Z_{Ni}$$

one has

$$\begin{aligned} Var(RE(p)|N) &= \sum_{i=1}^{p\,(N)-1} Var(Z_{Ni}) \\ &= (p\,(N)-1)\cdot\sigma^2 \\ E(RE(p)|N) &= \sum_{i=1}^{p\,(N)-1} E(Z_{Ni}) \\ &= (p\,(N)-1)\cdot\mu \end{aligned}$$

From (3.1) one gets

$$Var(RE(p)) = (E(p\,(N))-1)\cdot\sigma^2 + Var(p\,(N))\cdot\mu^2$$

and consequently by (2.1) the statement. □

Remark 1

Note that

$$\begin{aligned} E(p\,(N)) &= \sum_{n=1}^{p-1} P(N=n)\cdot n + p\cdot P(N\geq p) \\ Var(p\,(N)) &= E[(p\,(N))^2] - [E(p\,(N))]^2 \end{aligned}$$

with

$$E((p\,(N))^2) = \sum_{n=1}^{p-1} P(N=n)\cdot n^2 + p^2\cdot P(N\geq p)$$

In case that $P(N < p)$ is neglectably small, then one has

$$\begin{aligned} E(p\,(N)) &\approx p \\ Var(p\,(N)) &\approx 0\ , \end{aligned}$$

giving simply

$$\boxed{Var(RE(p)) \approx (p-1) \cdot \sigma^2}\ .$$

$\square$

Remark 2

Furthermore note that the conditions of the Theorem hold in case that the $X_1, X_2, X_3, \ldots$ are i.i.d. with exponential-density

$$f(x) = \alpha \cdot \exp(-\alpha \cdot x), \quad x \geq 0$$

(parameter $\alpha < 0$). One can put

$$\mu = 1/\alpha\ , \qquad \sigma^2 = 1/\alpha^2\ .$$

For a proof, that the conditions are given, see David (1981), p. 20. $\square$

4 Bounding Results

As first trivial result one has

Theorem 2

It holds in general

$$\rho(RE(p)) \leq \rho(RL(p))\ .$$

Proof

One has

$$RL(p) = RE(p) + p\,(N) \cdot X_{N:p} \tag{4.1}$$

and consequently

$$(RL(p))^2 = (RE(p))^2 + \text{Rest}$$

with

$$\text{Rest} = 2 \cdot p\,(N) \cdot RE(p) \cdot X_{N:p} + (p(N))^2 \cdot X_{N:p}^2 \geq 0\ .$$

$\square$

Furthermore less trivial

Theorem 3

It is valid in general

$$\rho(RE(p)) \leq \rho(RL(p)) - r_1(p)$$

with

$$r_1(p) = p^2 \cdot (u_p - u_{p-1})$$

where

$$u_p = \nu(S_p)$$

with

$$S_p = \sum_{i=1}^{p} X_{N:i}^2 \ .$$

Proof

Obviously because of (4.1)

$$\begin{aligned}(RE(p))^2 =&(RL(p))^2 - 2 \cdot p\,(N) \cdot (RL(p) \cdot X_{N:p}) + \\ &+ (p\,(N))^2 \cdot X_{N:p}^2\end{aligned} \tag{4.2}$$

in what

$$RL(p) \geq p\,(N) \cdot X_{N:p} \ ,$$

implying

$$(RE(p))^2 \leq (RL(p))^2 - (p\,(N))^2 \cdot X_{N:p}^2 \ .$$

The result follows since

$$p\,(N)^2 \cdot X_{N:p}^2 = p^2 \cdot X_{N:p}^2 \tag{4.3}$$

and

$$X_{N:p}^2 \cdot 1_{\{N \geq p\}} = S_p - S_{p-1}$$

□

Finally

Theorem 4

Suppose $X_1, X_2, X_3, \ldots$ are i.i.d. with distribution function F, having density f. Assume that f is continuous and strictly positive on

$$\{x : 0 < F(x) < 1\} \ .$$

Finally suppose that N is independent of the $X_1, X_2, \ldots$. Then one has

$$\rho(RE(p)) \leq \rho(RL(p)) - r_2(p)$$

with

$$r_2(p) = 2 \cdot p \cdot m_p \cdot (m_p - m_{p-1}) - p^2 \cdot (u_p - u_{p-1})$$

where u_p is defined like in Theorem 3 and

$$m_p = \nu(RL(p)) \ .$$

Proof

One takes the expectation from equation (4.2), what gives

$$\rho(RE(p)) \leq \rho(RL(p)) - \text{Rest}$$

with

$$\text{Rest} = 2 \cdot E(p\,(N) \cdot [RL(p) \cdot X_{N:p}]) - E((p\,(N))^2 \cdot X_{N:p}^2) \ . \qquad (4.4)$$

In the term “Rest” one has

$$E(p\,(N) \cdot [RL(p) \cdot X_{N:p}]) = p \cdot \sum_{i=1}^{p} E(X_{N:i} \cdot X_{N:p}) \ . \qquad (4.5)$$

In analogy to the proof of Theorem 3 in Kremer (2003) it holds

$$E(X_{N:i} \cdot X_{N:p}) \geq E(X_{N:i}) \cdot E(X_{N:p})$$

what implies

$$\begin{aligned} \sum_{i=1}^{p} E(X_{N:i} \cdot X_{N:p}) &\geq E\Big(\sum_{i=1}^{p} X_{N:i}\Big) \cdot E(X_{N:p}) \\ &= m_p \cdot (m_p - m_{p-1}) \ . \end{aligned} \qquad (4.6)$$

Finally one has according to (4.3)

$$E(p\,(N)^2 \cdot X_{N:p}^2) = p^2 \cdot (u_p - u_{p-1}) \ . \qquad (4.7)$$

(4.4) - (4.7) imply

$$\text{Rest} \geq 2 \cdot p \cdot m_p(m_p - m_{p-1}) - p^2(u_p - u_{p-1}) \ ,$$

what proves the statement. □

Remark 3

As already mentioned, results on $\rho(RL(p))$ are given in Kremer (1985), (2001), (2002), (2003). □

Remark 4

The superiority of one bound to the other of the bounds of Theorems 3 and 4 could **not** be proved. □

Remark 5

On determining the m_p, u_p in Theorems 3, 4 the following:

Let $X_1, X_2, \ldots$ be i.i.d. and independent of N. Assume N to be Poisson-distributed with parameter $\lambda = E(N) > 0$

$$P(N = n) = \frac{\lambda^n}{n!} \cdot \exp(-\lambda), \quad n = 0, 1, 2, 3, \ldots$$

and X_i be Pareto-distributed with parameter $\alpha > 2$ and start value $a > 0$, i.e. for its distribution function F one has

$$F(x) = 1 - (x/a)^{-\alpha} \ , \quad x \geq 0 \ .$$

Then one has

$$m_p = \lambda^{1/\alpha} \cdot \sum_{i=1}^{p} \left(\frac{\Gamma_\lambda(i - 1/\alpha)}{\Gamma(i)} \right) \cdot a \tag{4.8}$$

$$u_p = \lambda^{2/\alpha} \cdot \sum_{i=1}^{p} \left(\frac{\Gamma_\lambda(i - 2/\alpha)}{\Gamma(i)} \right) \cdot a \tag{4.9}$$

with

$$\begin{aligned} \Gamma(i) &= (i-1)! \\ \Gamma_\lambda(s) &= \int_0^\lambda \exp(-t) \cdot t^{s-1} \, dt \ . \end{aligned}$$

For larger λ one can replace Γ_λ by $\Gamma = \Gamma_\infty$, what gives then

$$m_p \approx \lambda^{1/\alpha} \cdot \left(\frac{\alpha}{\alpha - 1} \right) \cdot \left(\frac{\Gamma(p + 1 - 1/\alpha)}{\Gamma(p)} \right) \cdot a \tag{4.10}$$

$$u_p \approx \lambda^{2/\alpha} \cdot \left(\frac{\alpha}{\alpha - 2} \right) \cdot \left(\frac{\Gamma(p + 1 - 2/\alpha)}{\Gamma(p)} \right) \cdot a \tag{4.11}$$

In case that the formulas (4.8) - (4.11) appear to be too unhandy or are inadequate, the author proposes to use the recursive method in Kremer (1994) for approximate computation of m_p, u_p. So one has

$$\begin{aligned} m_p &\approx m_{p-1} \cdot \left[2 - \frac{k}{p-1}\right] - m_{p-2} \cdot \left[1 - \frac{k}{p-1}\right] \\ u_p &\approx u_{p-1} \cdot \left[2 - \frac{2k}{p-1}\right] - u_{p-2} \cdot \left[1 - \frac{2k}{p-1}\right] \end{aligned}$$

for $p \geq 3$ and an adequate $k > 0$ (e.g. $k = 0.3$ in motor liability insurance).

The start values m_1, m_2, u_1, u_2 have to be computed from (4.8) - (4.11), or in general from

$$\begin{aligned} m_p &= \sum_{i=1}^{p} \frac{1}{\Gamma(i)} \cdot \int_0^1 F^{-1}(u) \cdot (1-u)^{i-1} \cdot M^{(i)}(u)\, du \\ u_p &= \sum_{i=1}^{p} \frac{1}{\Gamma(i)} \cdot \int_0^1 [F^{-1}(u)]^2 \cdot (1-u)^{i-1} \cdot M^{(i)}(u)\, du \end{aligned}$$

with

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}$$

and the i-th derivative $M^{(i)}$ of the probability generating function of N

$$M(u) = \sum_{n=0}^{\infty} P(N = n) \cdot u^n \ .$$

□

The paper shall be closed with a

Numerical Example

Consider the Poisson-Pareto-situation of Remark 5 with special

$$\lambda = 100,\ \alpha = 2.5,\ a = 1\,.$$

It is nearlying to consider instead of absolute just relative values. This means one divides $\rho(\cdot), r_1(p), r_2(p), u_p$ by τ^2 and m_p by τ with

$$\begin{aligned} \tau &= E(N) \cdot E(X_i) \\ &= \lambda \cdot \frac{\alpha}{\alpha - 1} \cdot a \ . \end{aligned}$$

The so calculated rates shall be denoted by $\tilde{\rho}(\cdot), \tilde{r}_1(p), \tilde{r}_2(p), \tilde{u}_p, \tilde{m}_p$. The computations of $\tilde{u}_p, \tilde{m}_p$ were done with (4.8) and (4.9). The $\tilde{\rho}_p := \tilde{\rho}(RL(p))$ was taken as $\tilde{r}_p^{(2)}$ from table 1 in Kremer (2003). $\tilde{\rho}_{p1}, \tilde{\rho}_{p2}$ let denote the upper bounds of Theorems 3, 4 for $\rho(RE(p))$. The results are given in the following table. They are all in percent of τ^2 or τ.

Table 1

	p=1	2	3	4	5	6	7	8
$\tilde{m}_p$	5.64	9.02	11.73	14.07	16.18	18.12	19.94	21.65
$\tilde{u}_p$	0.66	0.79	0.87	0.93	0.97	1.01	1.04	1.07
$\tilde{\rho}_p$	0.66	1.89	3.21	4.60	6.05	7.47	9.10	10.69
$\tilde{r}_1(p)$	—	0.52	0.72	0.95	1.00	1.44	1.47	1.92
$\tilde{r}_2(p)$	—	0.70	1.19	1.67	2.41	2.78	3.61	4.00
$\tilde{\rho}_{p1}$	—	1.37	2.49	3.65	5.05	6.03	7.63	8.77
$\tilde{\rho}_{p2}$	—	1.19	2.02	2.93	3.64	4.69	5.49	6.69

In this example the second bound is better (i.e. smaller) than the first bound. As already mentioned the author did not succeed in proving this under the general conditions of Theorem 4. Perhaps one of the readers likes to think about this and can give a proof or just a counterexample. □

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