Credibility evaluation for heterogenous populations

ASTIN topic: Risk evaluation

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Abstract

Simple linear credibility formulae can be obtained when the structure distribution of the risk parameter is conjugate and where claims belong to the Exponential Dispersion Model. The paper focuses on the case when a portfolio is heterogenous and the structure distribution is given by a mixture of conjugate distributions. The resulting credibility formula is derived and its properties studied.

Key words: Exponential Dispersion Model, credibility, heterogenous population.

1 Introduction

Let θ be a risk parameter characterizing a member of a risk collective, and given θ , let $f(x|\theta)$, the distribution of his claim X, be a member of a family of distributions $\{f(x|\theta)\ \theta \in \Theta \subset R^1\}$. Further more, let $\pi(\theta)$ be the prior distribution of θ, the so called *structure distribution.* The estimation of the fair premium $\mu(\theta) = E(X|\theta)$, given n years individual experience $x_{1,}x_{2,}...x_n$ and the collective fair premium $m = \int_\Theta \mu(\theta) d\pi(\theta)$, is traditionally done by means of a credibility formula of the type $(1-z)m + z\bar{x}$.

It is assumed that the claim distribution is a member of the *Exponential Dispersion Model* (EDM) given by

$$f(x|\theta,\lambda) = q_\lambda(x)\exp\{\lambda[\theta x - k(\theta)]\}, \tag{1}$$

The EDM was considered in Tweedie (1984), Nelder and Wedderburn (1972), Nelder and Verrall (1997), and Jorgensen (1986, 1987, 1992, 1997). Current interest in the EDM is due to Jorgensen who outlines EDM as one of the main classes of dispersion models, which includes most standard distribution families. In Landsman and Makov (1998, 1999a,b, 2001), the insurance (credibility) aspects of the Exponential Dispersion Models were discussed and the intrinsic relationship between linear Bayes estimators and stochastic approximation was established. See also Jorgensen and Paes de Souza (1994).

The EDM has certain analogies with location and scale models, where location is expressed by the population mean

$$E_{\theta,\lambda}X = \int x f(x|\theta,\lambda)dx = k'(\theta) = \mu, \tag{2}$$

and the role of the scale parameter is played by λ, $\sigma^2 = 1/\lambda$. It follows from (1) that the population variance is given by

$$V_{\theta,\lambda}(X) = k''(\theta)/\lambda = V_f(\mu)/\lambda, \tag{3}$$

where $V_f(\mu)$ is called the *variance function.*

Let the conjugate prior distribution of θ be given by

$$\pi(\theta|n_0,x_0) = e^{n_0[x_0\theta - k(\theta)] - h(n_0,x_0)}, \tag{4}$$

where

$$h(n_0,x_0) = \ln \int \exp\{n_0[x_0\theta - k(\theta)]\}\, d\theta,$$

and the hyperparameters, n_0, x_0 are properly chosen (Diaconis and Ylvisaker, 1979). Consequently, given λ, the Bayesian credibility formula takes the form

$$E(X_{n+1}|x_1,...,x_n) = E(\mu(\theta)|x_1,...,x_n,\lambda) = \frac{n_0}{n_0+n\lambda}m + \frac{n\lambda}{n_0+n\lambda}\bar{x}. \quad (5)$$

The simple, and often regarded as desirable, structure of this credibility formula is due to the choice of a conjugate prior distribution (4). For analogous results in the context of the exponential family, see Jewell(1975), Schmidt (1980), Goel (1982), Herzog (1990) and Gerber (1995).

While the use of conjugate prior distributions is clearly attractive from a mathematical point of view, it cannot always be justified. In this paper we extend the result of Landsman and Makov (1998, 1999a) by considering a mixture of conjugate prior distributions

$$g(\theta|\alpha) = \alpha\pi(\theta|n_{01},x_{01}) + (1-\alpha)\pi(\theta|n_{02},x_{02}), \quad (6)$$

where $0 < \alpha < 1$.

This non-conjugate prior distribution can be justified on two grounds. On the one hand, (6) can be used to approximate a structure distribution in the case mathematical formulation does not lead to a simple credibility formula. On the other, (6) can genuinely portray the structure distribution of a heterogeneous population, where for a fraction α of the population the risk parameter θ has a structure distribution $\pi(\theta|n_{01},x_{01})$, and for the rest, $\pi(\theta|n_{02},x_{02})$. This interpretation will be adopted in this paper. Credibility evaluation using (6) is discussed in section 2. Methods of estimating unknown parameters are suggested in section 3 and numerical illustrations are given in section 4.

2 Credibility evaluation

Credibility assessment is based on the evaluation of the expectation of a future claim, given n years individual experience $x_1,x_2,...x_n$, $E[x_{n+1}|x_1,...x_n]$, which is known to be equal to the posterior mean of $\mu(\theta)$, $E[\mu(\theta)|x_1,...x_n]$.

The credibility formula is established in the following theorem:

Theorem 1 *Let the claim distribution be given by (1) and the structure distribution by (6). Then given n years individual experience, $x_1,x_2,...x_n$, the* *credibility formula is*

$$E\left[X_{n+1}|x_1,...,x_n\right]=$$
$$\left[\frac{n_{01}}{n_{01}+n\lambda}x_{01}\eta+\frac{n_{02}}{n_{02}+n\lambda}x_{02}(1-\eta)\right]+\left[\frac{n\lambda}{n_{01}+n\lambda}\eta+\frac{n\lambda}{n_{02}+n\lambda}(1-\eta)\right]\bar{x}, \tag{7}$$

where

$$\eta=\frac{\alpha f(x_1,...x_n|\lambda,\alpha=1)}{f(x_1,...x_n|\lambda,\alpha)}, \tag{8}$$

and

$$f(x_1,...,x_n|\lambda,\alpha)=$$
$$\sum_{i=1}^{2}\{\alpha^{2-i}(1-\alpha)^{i-1}\prod_j q_\lambda(x_j)\exp[h(n_{0i}+\lambda,\frac{n_{0i}x_{0i}+\lambda x_j}{n_{0i}+\lambda})-h(n_{0i},x_{0i})]\} \tag{9}$$

The credibility factor z_n is given by

$$z_n=\eta\frac{n\lambda}{n_{01}+n\lambda}+(1-\eta)\frac{n\lambda}{n_{02}+n\lambda}. \tag{10}$$

Some properties of this factor are discussed in the next theorem:

Theorem 2 *Let $n_{01}<n_{02}$ $(n_{01}>n_{02})$ then the credibility factor z_n is increasing (decreasing) in α*

Remark 1 *Landsman and Makov (1999b) indicated that the hyperparameters in (4) have the following meaning:*

$x_0=m$*, the fair premium*

$n_0=\frac{\lambda V(X)}{R_0}$*, where $V(X)=E_\lambda(X_j-\mu)^2$is the expectation of the variance of X_j, with respect to the structure distribution, and $R_0=E_\lambda(\mu-m)^2$ is the variance of this distribution.*

n_0 clearly has the interpretation of a ratio of the variability between claims and the variability between individuals. When $n_{01}<n_{02}$, a increase in α represents an increase in the share of the first group, in which this ratio is smaller, and hence larger weight is given to $\bar{x}$ by the credibility factor z_n.

Example: Suppose that the claim X is distributed gamma(α, β). Parameterizing the distribution as an EDM, we have $\lambda = \alpha$, $\theta = -\frac{\beta}{\alpha}$, $q_\lambda(x) \propto x^{\lambda-1}$ and $k(\theta) = -\ln(-\theta)$. Suppose further that the heterogeneity of the portfolio can be represented by

$$g(\theta) = \alpha \exp\{n_0 \left[x_{0i}\theta - k(\theta)\right] - h(n_{01}, x_{0i})\} + (1-\alpha)\exp\{n_{02}[x_{0i}\theta + \ln(-\theta)] - h(n_{02}, x_{0i})\}.$$

Then the credibility factor is given in (7) where η for this case is

$$\eta = \frac{\alpha \exp\{h(n_{01} + \lambda, \frac{n_{01}x_{01} + \lambda x_j}{n_{01} + \lambda}) - h(n_{01}, x_{01})\}}{\sum_{i=1}^{2}\{\alpha^{2-i}(1-\alpha)^{i-1}\exp[h(n_{0i} + \lambda, \frac{n_{0i}x_{0i} + \lambda x_j}{n_{0i} + \lambda}) - h(n_{0i}, x_{0i})]\}}, \qquad (11)$$

where $h(a,b) = \ln \int \exp\left[a(b\theta + \ln(-\theta)\right] d\theta = \ln\left(\frac{\Gamma(a+1)}{b^{a+1}a^{a+1}}\right)$.

3 Estimating unknown parameters.

The following parameters need to be estimated. λ, the dispersion parameter of the claim distribution, and the hyperparameters, $\alpha, n_{0i}, x_{0i}, i = 1, 2$. The way these parameters are to be estimated, depends, to a large extent, on the nature of the heterogeneity of the portfolio. Suppose that a portfolio consists of two groups of insured, and each individual in the portfolio belongs to one of the two groups. We can distinguish between several cases:

Case a: The association of an individual with a group is known and is acceptable for rating. For example, individuals' age groups are known and are used for setting car insurance premium. In this case, the credibility formula is not to be based on (6) but on the structure distribution of the particular group an individual belongs to (like the structure distribution of 'young drivers' etc.). The credibility formula to be used is the classic formula corresponding to a particular group. This case clearly does not correspond to the heterogeneous portfolio this paper focuses on and, therefore, will not be discussed here.

Case b: .The association of an individual with a group is known and is unacceptable for rating. For example, individuals' gender is known and may not be used for premium setting. For example, though an insurance company can technically assess the structure distribution of male or female drivers, separate premiums are unacceptable. Instead, a 'mixed' structure distribution (6) is to be used, resulting in (7) as a credibility formula. This way, though records may show that male, or female, are better drivers, no differentiation is to be practiced and (7) allows the company to assess the expected loss (given individual experience) in the presence of known heterogeneity.

Case c: The association of an individual with a group is unknown and therefore cannot be used for rating. For example, some individuals cannot be classified, unequivocally, as 'risk avert' or 'risk prone'. In such a case (6) reflects the overall risk composition of the portfolio and credibility assessment is carried via (7).

We note that the assessment of the value of $n_{0i}, x_{0i}, i = 1, 2$ in **Case c** would only be possible if "training samples" are available, , i.e., past records or portfolios for which the association between individuals and groups is fully specified. In such a case, the estimation of these parameters would be similar to the estimation procedures for **Case b,** We shall, therefore, focus our attention on **Case b** only.

The estimates of λ, α, n_{0i}, x_{0i} $(l = 1, 2)$ for case b, can be obtained by maximizing the marginal distribution of the claims. We shall refer to these estimates as **Predictive distribution based estimates.** This is, in a way, the neo-Bayesian analog to maximum likelihood estimation[1]. The estimation procedure will, typically, require the use of mathematical software, as will be demonstrated below. The procedure, of course, is simpler if the maximization is done with respect to the approximate predictive distribution described in the next lemma.

Lemma 1 *The predictive distribution $f(x_1, ..., x_n|\lambda, \alpha)$ is approximated by*

$$\sum_{i=1}^{2}\{\alpha^{2-i}(1-\alpha)^{i-1}\prod_{j} q_\lambda(x_j)H \bullet T,$$

[1] A fully Bayesian approach would require the specification of a prior distribution for λ. the neo-Bayesian approach suggested here is simpler to implement.

where,

$$H = \sqrt{\frac{(n_{0i})k''\left(\mu^{-1}\left[x_{0i}\right]\right)}{(n_{0i}+\lambda)k''\left(\mu^{-1}\left[\frac{n_{0i}x_{0i}+\lambda x_j}{n_{0i}+\lambda}\right]\right)}} \tag{12}$$

and

$$T = \frac{\exp\left\{(n_{0i}+\lambda)\left[\frac{n_{0i}x_{0i}+\lambda x_j}{n_{0i}+\lambda}\mu^{-1}\left[\frac{n_{0i}x_{0i}+\lambda x_j}{n_{0i}+\lambda}\right]-k\left(\mu^{-1}\left[\frac{n_{0i}x_{0i}+\lambda x_j}{n_{0i}+\lambda}\right]\right)\right]\right\}}{\exp\left\{(n_{0i})\left[n_{0i}\mu^{-1}\left[x_{0i}\right]-k\left(\mu^{-1}\left[x_{0i}\right]\right)\right]\right\}}.$$

Since we focus on **case b**, in which each an individual experience can be classified into one of the two groups and, therefore, can be used to estimate the parameters characterizing the underlying group, empirical bayes type estimators are available. We shall refer to this estimates as **Moment Bases Estimates**

Illustrating the methods discussed above, we simulated past claims of k=100 individuals, 50% males (M)and 50% females (F). Claims ($n_l = n = 10$) follow a gamma distribution $gamma(\alpha,\beta)$, such that $\lambda = \alpha$ and $\theta = -\frac{\beta}{\alpha}$.

θ has a structure distribution representing a mixture of gamma priors, where $gamma(11,22)$ represent the structure distribution for males and $gamma(11,11)$ for females.

Assumptions concerning heterogeneity:

A1: The population is regarded as homogeneous.

A2: The population is regarded as heterogeneous where gender is known.

Assumptions concerning the claim distribution:

D1: The claim distribution is unknown

D2: The claim distribution is assumed to belong to the EDF (gamma distribution in this case).

Naturally, we are interested in estimating parameters under assumption A2D2. However, for the sake of comparison, we also consider assumptions A1D1 and A1D2 . For assumptions A1D1 we used the classical Bühlmann credibility

In order to assess the standard error of the estimates under the various combination of assumptions, the generation of the data and the estimation of parameters were repeated 100 times. The mean and standard deviation (SD) of the estimates are provided in Table 1. Note that the correct value of the creditability factor under the specified assumptions is $z = 0.5$.

Table 1

Assumptions	Method	Mean($\hat{z}$)	SD($\hat{z}$)
A1D1	Bühlmann	0.80	0.076
A1D2	Moments	0.75	0.036
A1D2	Predictive	0.67	0.071
A2D2	Moments	0.69	0.059
A2D2	Predictive	0.48	0.172

The results shown in Table 1 are typical of many simulation studies carried on the model. The classical Bühlmann credibility formula is least accurate. It fails to capture the nature of the heterogeneity since it is based on the estimation of only two expressions $E_\theta\{V[x|\theta]\}$ and $V_\theta\{E[X|\theta]\}$. Better results are obtained when the loss model is assumed known, with the best results when heterogeneity is taken into account, and less so when heterogeneity is ignored. The predictive distribution based estimates are typically better then the moment based estimates, though the former have larger standard errors.

4 Discussion

Heterogeneity can take several forms. The one adopted here assumes that while all claims are generated from a common loss model, the structure distribution of the risk parameter depends on the association with one of the subgroups that make up the portfolio.

The paper has focused on loss distributions belonging to the EDM, with a risk parameter θ whose structure distribution reflects the heterogeneity through a mixture of conjugate prior distributions. The resulting credibility factor is derived, approximated and studied. Two estimation approaches are suggested. The moment based methods are easy to implement and are less accurate then the predictive distribution based methods, which are computationally more demanding.

Heterogeneity in a portfolio is probably more prevalent than assumed by most practitioners, who regard a portfolio as homogeneous. Consequently, premium setting results in either over estimation or under estimation of the future claims. This, respectively, is likely to reduce the company's competitiveness or to increase its financial risk.

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