On Unknown Accumulations in Accident Insurance: An upper bound of the expected excess claim

Topic 2: Reinsurance (alternatively topic 1: pricing risk, rate making)

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Abstract

An upper bound of the expected excess claim is deduced depending on the following parameters: the expected number of victims per accident, the minimum number of victims, the maximum number of victims, the market share of the cedant, the expected indemnity per victim, the maximum indemnity per victim. If in addition the cedant's expected loss ratio is given then an upper bound of the reinsurance risk rate can be calculated.

Keywords

Binomial distribution, stop-loss order, dangerous distribution.

1. Introduction

Insurers are confronted with the problem of unknown accumulations mainly in air crashes and seek reinsurance protection against the risk of having a high number of their insured among the victims. Using some results of Bühlmann, Gagliardi, Gerber and Straub (1977) an upper bound of the expected excess claim and consequently also of the risk premium is deduced.

2. The number of customers among the number of victims

Consider an insurer with market share p. If for every plane passenger the probability of being a customer of the insurer is equal to p then the number K of customers among n passengers follows a binomial distribution with probabilities Prob(K = k) = $p_k(n)$:

$$(1)\quad p_k(n) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}.$$

If the plane crashes, the number K of customers among the victims follows the same binomial distribution. The expected number of customers in excess of a threshold s ≥ 0 is given by

$$(2)\quad E[(K-s)+] = \sum_{k>s}^{n} p_k(n) \cdot (k-s)$$

$$= E[K] - s - \sum_{k=0}^{[s]} p_k(n) \cdot k + s \cdot \sum_{k=0}^{[s]} p_k(n)$$

where [s] is the highest integer that is smaller or equal to s. Formally, (2) is a stop loss risk premium.

3. An upper bound of E[(K – s)+]

In this paragraph we use the concept of dangerous distribution in the terminology of Bühlmann et. al.: If for two risks X and Y with distribution functions $F_X(x)$ and $F_Y(x)$ a real number $c \geq 0$ exists such that

$$F_X(x) \leq F_Y(x) \quad \text{for } 0 \leq x < c$$
$$F_X(x) \geq F_Y(x) \quad \text{for } c \leq x$$

and moreover $E[X] \leq E[Y]$,

then X precedes Y in the stop-loss order and is called less dangerous than Y.

Normally, the number n of passengers and thus of potential victims in a crash is not known in advance. What is given in practice are estimates for the minimum number of passengers, u, and the maximum number of passengers, t, where $u \leq n \leq t$.

We show that the mixture of two binomial distributions with probabilities defined as

$$(3)\quad b_k(n) = \frac{n-u}{t-u} \cdot p_k(t) + \frac{t-n}{t-u} \cdot p_k(u)$$

is more dangerous than the distribution given by (1). If that is true then E[(K – s)+] for the distribution (3) is an upper bound of E[(K – s)+] for the distribution (1).

Proof:

$$\text{Let } F(x) = \sum_{k \leq x} p_k(n) \quad \text{and} \quad G(x) = \sum_{k \leq x} b_k(n)$$

where $p_k(n)$ and $b_k(n)$ are the probabilities defined by (1) and (3). For u = n and for t = n we have $p_k(n) = b_k(n)$ so that there is nothing to prove. We therefore assume u < n < t. Likewise we exclude the simple case p = 0 and assume p > 0.

Since $p_k(n)$ and $b_k(n)$ define the same expected value, $E[K] = n \cdot p$, G is a more dangerous distribution than F if the following three conditions hold:

- G(n) < F(n)
- G(0) > F(0)
- There is exactly one j such that F(k) < G(k) for k = 0, ..., j-1, and F(k) ≥ G(k) for k ≥ j.

To simplify the notation we write p_k and b_k instead of $p_k(n)$ and $b_k(n)$ where no confusion is possible.

1) G(n) < F(n) = 1 follows from n < t.
2) In order to prove G(0) > F(0) define

$$\alpha_0 = \frac{p_0}{b_0} = \frac{(1-p)^n}{\frac{n-u}{t-u}\cdot(1-p)^t + \frac{t-n}{t-u}\cdot(1-p)^u}.$$

If p = 0 then $\alpha_0 = 1$. For 0 < p < 1 we have $\alpha_0 <$ 1 since the derivative of α_0 with respect to p is equal to

$$\alpha_0{}' = -\left[\frac{n-u}{t-u}\cdot(1-p)^{t-n} + \frac{t-n}{t-u}\cdot(1-p)^{u-n}\right]^{-2}\cdot\frac{t-n}{t-u}\cdot(n-u)\cdot(1-p)^{u-n-1}\cdot\left[1-(1-p)^{t-u}\right]$$

$$< 0.$$

3) To prove assertion 3 define for k = 0, 1, ... , t

$$\alpha_k = \frac{p_k}{b_k}.$$

Using the notation $(n)_0 = 1$ and $(n)_k = n\cdot(n-1)\cdot\ldots\ldots\ldots\cdot(n-k+1)$ for k > 0 we can rewrite α_k in the form

$$\alpha_k = \frac{(n)_k\cdot(1-p)^n}{\frac{n-u}{t-u}\cdot(t)_k\cdot(1-p)^t + \frac{t-n}{t-u}\cdot(u)_k\cdot(1-p)^u}.$$

If k > n then $\alpha_k = 0$ because in this case $(n)_k = 0$. Hence

(4) $\alpha_{n+1} = \alpha_{n+2} = \ldots\ldots\ldots = \alpha_t = 0.$

Consider now the ratio

$$\frac{\alpha_{k+1}}{\alpha_k}$$

for k = 0, 1, ..., n - 1.

$$\frac{\alpha_{k+1}}{\alpha_k} = (n-k)\cdot\frac{\frac{n-u}{t-u}\cdot(t)_k\cdot(1-p)^t + \frac{t-n}{t-u}\cdot(u)_k\cdot(1-p)^u}{\frac{n-u}{t-u}\cdot(t)_k\cdot(t-k)\cdot(1-p)^t + \frac{t-n}{t-u}\cdot(u)_k\cdot(u-k)\cdot(1-p)^u}.$$

If k > u then $(u)_k = 0$ and therefore

$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{n-k}{t-k} < 1.$$

Hence

(5) $\alpha_{u+1} > \alpha_{u+2} > \ldots\ldots\ldots > \alpha_n$.

If $k \le u$ we have

$$\frac{\alpha_{k+1}}{\alpha_k} = (n-k) \cdot \frac{1}{w_k \cdot (t-k) + (1-w_k) \cdot (u-k)}$$

where in the denominator there is a weighted average of $(t – k)$ and $(u – k)$ with weights

$$w_k = \frac{\frac{n-u}{t-u} \cdot (t)_k \cdot (1-p)^t}{\frac{n-u}{t-u} \cdot (t)_k \cdot (1-p)^t + \frac{t-n}{t-u} \cdot (u)_k \cdot (1-p)^u}$$

$$= \frac{1}{1 + \frac{t-n}{n-u} \cdot \frac{(u)_k}{(t)_k} \cdot (1-p)^{u-t}}$$

and $1 - w_k$.

The weight w_k increases as k increases, ie $w_k < w_{k+1}$ because

$$\frac{(u)_{k+1}}{(t)_{k+1}} = \frac{(u)_k}{(t)_k} \cdot \frac{u-k}{t-k} < \frac{(u)_k}{(t)_k}.$$

Depending on p two cases have to be distinguished:

1st case: for all $k = 0, 1, \ldots, u$ the weighted average $w_k \cdot (t – k) + (1 – w_k) \cdot (u – k)$ is smaller than $(n – k)$. Consequently, using (4) and (5)

(6a) $\alpha_0 < \alpha_1 < \ldots < \alpha_{u+1} > \alpha_{u+2} > \ldots > \alpha_n > \alpha_{n+1} = \ldots\ldots\ldots = \alpha_t$.

2nd case: There exists k_0, $0 \le k_0 \le u$ so that
$w_k \cdot (t – k) + (1 – w_k) \cdot (u – k) < (n – k)$ for all $k < k_0$,
$w_k \cdot (t – k) + (1 – w_k) \cdot (u – k) \ge (n – k)$ for all $k = k_0$ and
$w_k \cdot (t – k) + (1 – w_k) \cdot (u – k) > (n – k)$ for all $k > k_0$. Consequently

(6b) $\alpha_0 < \ldots < \alpha_{k_0} \ge \alpha_{k_0+1} > \ldots > \alpha_{u+1} > \alpha_{u+2} > \ldots > \alpha_n > \alpha_{n+1} = \ldots\ldots\ldots = \alpha_t$.

Calling $k_0 = u + 1$ in (6a) we can summarise (6a) and (6b) in

(6) $\alpha_0 < .... < \alpha_{k_0} \geq \alpha_{k_0+1} > .... > \alpha_n > \alpha_{n+1} = ... = \alpha_t$.

Since G(0) > F(0) and G(n) < F(n) there is a j such that F(k) < G(k) for k = 0, 1, ..., j-1 and F(j) ≥ G(j). From p_j = F(j) – F(j – 1) > G(j) – G(j – 1) = b_j follows $\alpha_j > 1$.
The general case F(j) > G(j) and the special case F(j) = G(j) are analysed separately.

General case F(j) > G(j):
F(k) > G(k) for all k, j < k ≤ t – 1. Otherwise there would exist a first k > j for which F(k) ≤ G(k). From p_k = F(k) – F(k – 1) < G(k) – G(k – 1) = b_k would follow $\alpha_k < 1$. Because $\alpha_j > 1$, $\alpha_k < 1$, and the fact that α_i increase for i ≤ k_0 it follows that k > k_0. Hence $\alpha_i \leq \alpha_k$ for all i > k, therefore $p_i \leq b_i$ for all i > k F(n) ≤ G(n) which contradicts F(n) = 1 >G(n).

Special case F(j) = G(j):
As in the general case we can exclude that there is a k > j for which F(k) < G(k). However, it might be that F(j+1) = G(j+1). This means $p_{j+1} = b_{j+1}$, $\alpha_{j+1} = 1$, $\alpha_i \leq 1$ for all i > j+1 because α_i decrease for i > k_0, $p_i \leq b_i$ and F(n) ≤ G(n) which again contradicts the assumption F(n) = 1 > G(n).

4. The expected distribution of n
We now assume the number of passengers n is a random variable and call it N, u ≤ N ≤ t. Define $p_k(N)$ and $b_k(N)$ as in (1) and (3) for any value of N. Since the distribution F precedes G in the stop-loss order, according to Lemma 1 in Bühlmann et. al. also the expected distribution defined by the expected probabilities $E[p_k(N)]$ will precede the distribution defined by

$$(7)\quad E[b_k(N)] = \frac{E[N]-u}{t-u} \cdot p_k(t) + \frac{t-E[N]}{t-u} . p_k(u)$$

in the stop - loss order.
(7) can be used to calculate E[(K – s)+] in practice.

5. On the distribution of the indemnity
Let I be the indemnity of a victim with distribution function F, expected value m and a maximum value M, ie I ≤ M.
Define a random variable S with distribution function G given by

$$\mathrm{Prob}(S=0) = 1 - \frac{m}{M}$$

$$\mathrm{Prob}(S=M) = \frac{m}{M}.$$

Clearly, S is more dangerous than I. In Bühlmann et. al. it is shown that consequently the distribution

$$\sum_k p_k \cdot F^{*k} \text{ precedes } \sum_k p_k \cdot G^{*k} \text{ in the stop-loss order.}$$

This means that – provided the indemnities of victims are independent of each other – the expected total indemnity in excess of any threshold can be estimated by the following upper bound:

The market share p in (1) is replaced by

$$(8)\quad a = p \cdot \frac{m}{M}$$

and every indemnity is replaced by its maximum value, M.

6. A numerical example

Assumptions:
Number of passengers:
Expected number E[N] = 100
minimum number u = 40
maximum number t = 200
market share p = 5%
expected loss ratio of the ceding company lr = 50%
expected indemnity per victim m = 200 000
maximum indemnity per victim M = 1 000 000
maximum indemnity per accident t · M = 200 000 000
reinsurance deductible d = 2 500 000
reinsurance cover t · M – d = 197 500 000

Results:
expected number of insured victims per accident p· E[N] = 5
expected indemnity per accident p · E[N] · m = 1 000 000
modified market share according to (8) a = p · m / M = 1%
threshold s = d / M =2.5
[s] = 2

E[(K – s)+] is calculated using (1) and (2).
For the binomial distribution with parameters u and a E[(K – s)+] = 0.0044865
with parameters t and a E[(K – s)+] = 0.3769647
For the mixture according to (7) E[(K – s)+] = 0.1441658
expected reinsured indemnity per accident M · E[(K – s)+] = 144 165.8

reinsurance risk rate M · E[(K – s)+] / (p · E[N] · m) · lr = 7.2%.

7. Reference

Bühlmann H., Gagliardi B., Gerber H., Straub E. (1977): Some inequalities for stop-loss premiums. ASTIN bulletin, Vol. IX, 75-83.