

Portfolio optimization under solvency constraints: a dynamical approach

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Abstract

We develop portfolio optimization problems to a non-life insurance company for finding the minimum capital required, which simultaneously satisfy solvency and portfolio performance constraints. Motivated by standard insurance regulations, we consider solvency capital requirements based on three criteria: Ruin Probability, Conditional Value-at-Risk and Expected Policyholder Deficit ratio. We propose a novel semiparametric formulation for each problem and explore the advantages of implementing this methodology over other potential approaches. When liabilities follow a Lognormal distribution, we provide sufficient conditions for convexity for all our problems. Using different expected Return on Capital target levels, we construct efficient frontiers when portfolio assets are modelled with a special class of multivariate GARCH models. We found that the correlation between assets plays an important role in the behaviour of the optimal capital required and the portfolio structure. The stability and out-of-sample performance of our optimal solutions are empirically tested with respect to both, the solvency requirement and the portfolio performance, through a double rolling window estimation exercise. Our results indicate that a time-varying correlation model outperforms the constant and no-correlation counterparts.

Keywords: Portfolio optimization; Capital requirements; Solvency constraint; Multivariate GARCH; Double rolling window.

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1 Introduction

Insurance regulation has played an important role in securing policyholders and investors against various types of risk. One of its primary objectives is the establishment of an initial capital amount required to be held by insurance companies, in order to offer protection in the case of unexpected events. There has been an extensive literature on capital adequacy and its relationship to risk measures. For example, the Value-at-Risk (VaR), one of the most popular tools used in financial risk management, constitutes the basis for the Solvency II regulatory standards which applies to insurance companies in European Union (EU) (e.g. Sandström, 2006). In order to overcome some of the VaR pitfalls (e.g. VaR is not sub-additive), Artzner *et al.* (1999) introduced the notion of coherent risk measures. A discussion about their applications to capital requirements in insurance is provided in Artzner (1999). Amongst the coherent risk measures, an important special case is represented by the Expected Shortfall (ES), which plays a crucial role for the development of the Swiss Solvency Test (SST) (FOPI, 2004). The class of coherent measures has been further extended to convex measures by Föllmer and Schied (2002). For an overview of theoretical properties of various well-known risk measures used as solvency capital requirements, we refer to Dhaene *et al.* (2006). For a more recent survey on applications of risk measures in portfolio management we refer to Krokmal *et al.* (2011).

The standard approach used in connecting minimum capital standards to risk measures relies on the investment of solvency capital into a single “eligible” security, often taken as a risk-free asset. However, if the regulator allows the financial institution (e.g. insurance company in our case) to use a portfolio of such “eligible” assets, investing only into the risk-free asset may not be efficient. For example, Balbas (2008) showed that the investment of the capital requirement into a risk-free asset is not optimal in several important cases, and he provided an example based on a Conditional Value-at-Risk ($CVaR$) (risk measure introduced by Rockafellar and Uryasev, 2000) and Black-Scholes assumptions. Artzner *et al.* (2009) provided a brief discussion on the efficient use of capital and risk measures in the case of multiple traded assets, while Farkas *et al.* (2012) gave a comprehensive theoretical background on the same issue. However, none of the above studies provide empirical examples on how minimum capital and its optimal allocation are obtained. Moreover, despite their popularity, these optimization problems are typically treated separately in the actuarial literature. The use of both initial capital and portfolio weights as decision variables for optimization problems has only been recently proposed. For example, Mankai and Bruneau

(2012) introduced a joint optimization problem by maximizing the expected return on risk-adjusted capital subject to a $CVaR$ constraint, while Asimit *et al.* (2012) developed a minimum capital requirement problem based on a Ruin Probability (RP) constraint. However, both studies assumed a static setting and did not investigate the behaviour of the optimal solutions and portfolio performance over time.

In this paper we introduce three joint optimization problems for a non-life insurance company in a dynamic framework. Each problem is constructed by minimizing the initial capital subject to two types of constraints. The first category is represented by solvency requirements according to a particular insurance regulation, while the second constraint, which is the same for all problems, is given through a portfolio performance measure.⁵ Since shareholders usually require a gain on their investment, we use the expected Return on Capital (ROC) as our portfolio performance measure. Other choices for measuring performance are suggested in Cherny and Madan (2009), among others.

Motivated by Solvency II and SST directives, the first two solvency criteria are based on a target value for the RP and a negative $CVaR$ of the insurer's net loss, respectively, both computed over a predefined period of time (e.g. one year horizon). Since the RP constraint is equivalent to a negative VaR , the two criteria considered agree with the mathematical definition of a solvency requirement given by Djehiche and Hörfelt (2007) for a general risk measure. The third solvency constraint uses an upper bound for the Expected Policyholder Deficit (EPD), which was introduced by Busic (1994) as a new measure of insolvency risk. The EPD criteria has played an important role in establishing the US Risk-based Capital (RBC) regulatory system (e.g. see NAIC, 2009). Analyses and comparisons of the three capital standards have been previously considered in the literature. For example, Holzmüller (2009) and Cummins and Phillips (2009) provided detailed assessments of the RBC, SST and Solvency II directives. Barth (2000) compared the RP and EPD approaches and found that the latter increases the insolvency risk for larger insurers. Eling *et al.* (2009) investigated the RP , EPD and ES in a mean-variance setup using data from a German non-life insurance company.

Our objective is to provide a detailed analysis of the optimal capital required and its portfolio allocation for all

⁵A conceptually similar problem has been very recently proposed in the financial literature by Santos *et al.* (2012). They develop an optimization problem which minimizes the capital required subject to a Basel II criteria (i.e. a target number of VaR violations within a year) and a lower bound for the expected portfolio return. However, their problem is constructed as a single optimization problem (i.e. the only decision variable is the portfolio weight), since the capital requirement is given explicitly by the maximum between current-day VaR and the average one-day-ahead VaR over the last 60 business days.

three solvency criteria. In constructing the insurer net loss, we model only two sources of risk, namely the market (assets) and insurance (liabilities) risks. The dynamical structure is introduced by assuming that the portfolio's assets follow a Multivariate Generalized Autoregressive Conditional Heteroskedastic (MV-GARCH) model. There is a considerable number of MV-GARCH specifications proposed in the financial econometrics literature (e.g. VEC, BEKK, CCC, DCC etc.), and for recent surveys we refer to Bauwens *et al.* (2006) and Silvennoinen and Terasvirta (2009). In this study, I we focus on the class of Dynamic Conditional Correlation (DCC) models introduced by Engle (2002). There are at least three important reasons for this choice. Firstly, these models are not heavily parametrized and therefore, are appropriate for large scale estimation and risk management problems (e.g. see Engle and Sheppard, 2008). Secondly, their forecasting performance is not significantly outperformed by more richer competitors (e.g. see Laurent *et al.*, 2012). Thirdly, we wish to analyze the effect of a time-varying correlation matrix between portfolio's assets on the optimal solutions, by comparing it with constant and zero correlation cases. The insurance liability is modelled with a univariate random variable.

One of the major issues with implementing the proposed problems is related to their convexity. Since the expected *ROC* constraint is linear in both capital and weights, the focus remains on the convexity of the solvency constraints. The standard approach for dealing with *CVaR* optimization is based on a Monte-Carlo type of approximation, and this leads to a linear programming (LP) reformulation for the initial problem (e.g. see Rockafellar and Uryasev, 2000, 2002, and Krokmal *et al.*, 2002, among others). Tian *et al.* (2010) used the similar prescription for solving asset-liability mean-variance portfolio optimization problems under *CVaR* constraints. Alexander *et al.* (2006) pointed out that the LP reformulation becomes less efficient when the number of Monte-Carlo paths becomes large. For the *RP* problem, closed-form expression and/or convex reformulation are rarely available. There are two streams of literature dealing with probability (chance) constraints. The first direction consists of using Monte-Carlo type estimators based on indicator functions and performing further appropriate approximations (e.g. see Boyd and Vandenbergue, 2004, for convex approximations by eliminating the indicator function, Nemirovski and Shapiro, 2006, for Bernstein scheme convex approximation, Luedtke and Ahmed, 2008, for non-convex mixed-integer programming (MIP) reformulation, among others). The second direction formulates and solves the chance constraints as *VaR*-constrained optimization (e.g. see Larsen *et al.*, 2002, for algorithms based on iterative *CVaR* optimizations, Gaivoronski and Pflug, 2004, for scenario-based methods and Wozabal *et al.*, 2008, for a difference of convex functions reformulation).

In order to avoid the above convexity issues, we propose a semiparametric approach for reformulating the solvency constraints using the empirical distribution based on asset returns scenarios generated according to the MV-GARCH models and the given parametric specification of the liability distribution. For the *RP*-constrained optimization this methodology can be viewed as a generalization of the semiparametric algorithm proposed in Asimit *et al.* (2012). When liabilities are Lognormal distributed, we derive sufficient convexity conditions for all three solvency constraints. Our numerical examples are constructed based on 3-asset portfolios formed with one “risk-free” asset (US T-Bills) and two risky assets (NASDAQ and NYSE). The parameters are estimated from daily returns using the two-stage estimation methodology introduced by Engle and Sheppard (2001) for three covariance specification: DCC-GARCH, CCC-GARCH (constant correlation) and UNI-GARCH (no-correlation). The liability parameters are estimated based on monthly aggregate claim amounts from property insurance provided by a Romanian insurance company. Using different level of shareholders’ expected *ROC*, we construct efficient frontiers for a one-month horizon. All three solvency constraints indicate a similar type of behaviour in the sense that the DCC-GARCH is the most conservative model in terms of capital requirements. The time-varying correlation plays an important role in the portfolio structure. For example, the DCC-GARCH allocates nothing into the NYSE index, while the corresponding weights for the other two covariance models are significantly different from zero. Generally speaking, most of the initial wealth is allocated into the “risk-free” asset. We also notice that the 99% *CVaR* optimization is more conservative than the corresponding *VaR* problem at 99.5% significance level. Finally, we run a double rolling window estimation exercise (re-estimate asset and liability parameters over a given period) to compare the out-of-sample performance of our models. The results indicate the DCC specification outperforms the CCC and the no-correlation ones, in terms of both the solvency constraint and the portfolio performance. We further found that the optimal capital and portfolio weights are very stable when only asset returns parameters are re-estimated. However, this is no longer the case when we allow liability re-estimation at each step in the rolling window experiment.

The rest of the paper is organized as follows. In the next section, we introduce the optimization problems based on the solvency and expected *ROC* constraints and illustrate the semiparametric approach for solving them. Models for both assets and liabilities and discussions on the convexity of the proposed methods are provided in Section 3. An extensive empirical analysis is performed in Section 4. We conclude the paper in Section 5.

2 Preliminaries and solvency constrained optimization

We consider a discrete-time framework with the set of trading dates indexed by $\mathcal{T} = \{0, 1, \dots, T\}$. The market consists of a portfolio of n assets with the gross return⁶ process over the period $[t, t + 1]$ defined by $\mathbf{R}_{t+1} = (R_{1,t+1}, \dots, R_{n,t+1})^T$. We denote by \mathcal{F}_t , the historical information on the asset return evolution up to time t , so that $\mathcal{F}_t = \sigma(\mathbf{R}_1, \dots, \mathbf{R}_t)$. For convenience, we use the following notations for conditional probabilities, expectations and variances: $\Pr(\cdot|\mathcal{F}_t) = \Pr_t(\cdot)$, $E[\cdot|\mathcal{F}_t] = E_t[\cdot]$ and $Var[\cdot|\mathcal{F}_t] = Var_t[\cdot]$. Moreover, we use majuscules for random variables (except for cases when an upper script associated to a random variable may be interpreted as a realization of that random variable) and non-capital letters for deterministic quantities.

We introduce three optimization problems based on different solvency criteria for a non-life insurance company within a one-period setting, $[t, t + \tau]$, where τ is the solvency horizon satisfying $\tau \leq T - t$. First, we denote by p_t the aggregate premium available for investment at time t . In addition, we assume that shareholders provide a regulatory initial capital of size c_t . Without loss of generality, no other premiums are collected and no capital is issued or retired between t and $t + \tau$, and therefore, the total invested amount is $p_t + c_t$. Let $\mathbf{x}_t = (x_{1,t}, \dots, x_{n,t})^T$ be the portfolio weights whose components satisfy the standard budget constraint, $\sum_{i=1}^n x_{i,t} = 1$ and the no short sales constraint, $x_{i,t} \geq 0$, $i = 1, \dots, n$. Since our problem is designed as a single-period optimization, no rebalancing is allowed during the solvency period.

To fully describe the setup, we let the insurer's liability be modelled by a univariate random variable $Y_{t+\tau}$. This represents the aggregate claim amount over the solvency horizon which is assumed to be paid at time $t + \tau$. At this point, no particular assumptions regarding the conditional distributions of $\mathbf{R}_{t+\tau}$ and $Y_{t+\tau}$ are made, and no premium calculation principle is assigned for p_t . We define the insurer's net loss as the difference between the liability and portfolio value over the solvency horizon:

$$L_{t,t+\tau} = Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T \mathbf{x}_t.$$

Since both the capital requirement and portfolio allocations are decision variables in our joint optimization problems, we can view the net loss r.v. as a function of these quantities (i.e. $L_{t,t+\tau} := L(c_t, \mathbf{x}_t)$). For convenience, we assume there are no other sources of risk other than the ones modelled through Y and \mathbf{R} , and there are no transaction or other friction costs.

⁶The gross return process is defined here as the ratio between the terminal and initial asset prices, and thus is non-negative.

Each optimization problem proposed in the following subsections is characterized by minimizing the capital requirement c_t , subject to two key constraints. The first one consists of the solvency capital requirement imposed by the insurer's regulator, and is based on one of the following criteria: RP , $CVaR$ and EPD . Next, we define the gross ROC over the investment period:

$$ROC_{t,t+\tau} = -\frac{L_{t,t+\tau}}{c_t}. \quad (2.1)$$

Since shareholders typically require a rate of return on the provided capital, the second constraint is introduced as a portfolio performance measure based on a target level for shareholders' expected return. For each type of solvency, we provide a novel a semiparametric approach, which allows us to reformulate the constraints and further implement the optimization without costly computational effort.

2.1 Optimization with RP constraint

The use of ruin probability constraints is motivated by the Solvency II Regime, which applies to any EU based insurance company, and consist of identifying the capital required to maintain a target level for the ruin probability over a specified period of time. Thus, we define the RP -constrained problem as follows:

$$\begin{aligned} & \min_{c_t, \mathbf{x}_t} c_t \\ \text{s.t. } & E_t[\mathbb{1}_{\{L_{t,t+\tau} > 0\}}] \leq 1 - \alpha, \\ & E_t[ROC_{t,t+\tau}] \geq ROC^\alpha, \\ & \mathbf{1}^T \mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0. \end{aligned} \quad (2.2)$$

Here, α represents the specified solvency level, $\mathbb{1}_{\{\cdot\}}$ is the indicator function and ROC^α is the lower bound for the shareholders' expected return on capital, which also depends on α . We notice that the solvency chance constraint in (2.2) can be reformulated as a Value-at-Risk constraint, where the VaR of a loss random variable Z at a confidence level α is defined by

$$VaR^\alpha(Z) := \inf\{z \in \mathfrak{R} : \Pr(Z \leq z) \geq \alpha\}.$$

Indeed, it immediately follows that:

$$E_t[\mathbb{1}_{\{L_{t,t+\tau} > 0\}}] \leq 1 - \alpha \Leftrightarrow \Pr_t(L_{t,t+\tau} > 0) \leq 1 - \alpha \Leftrightarrow VaR_t^\alpha(L_{t,t+\tau}) \leq 0,$$

where VaR_t^α is the value at risk conditional on the historical asset return evolution up to time t . The main difficulty in dealing with this type of problems is the convexity of the chance constraint. Closed-form expressions for the

ruin probability only exist in very few special cases. For example, if we assume that $L_{t,t+\tau}$ has a multivariate Gaussian distribution, then (2.2) can be rewritten as a Second Order Cone optimization, which can be efficiently implemented with appropriate solvers. Asimit *et al.* (2012) found a closed form expression for such a problem in the absence of the short-sales and *ROC* constraints. However, when $Y_{t+\tau}$ and $R_{t+\tau}$ do not belong to the same family of distributions, we may not be able to even identify the distribution of $L_{t,t+\tau}$.

A standard approach in the chance constrained programming literature is to use a fully nonparametric method for approximating the conditional expectation in (2.2). This can be done by using Monte-Carlo simulations for both assets and liabilities. The solvency condition can be thus reformulated as:

$$\frac{1}{m} \sum_{j=1}^m \mathbb{1}_{\{Y_{t+\tau}(j) - (p_t + c_t) \mathbf{R}_{t+\tau}^T(j) \mathbf{x}_t > 0\}} \leq 1 - \alpha. \quad (2.3)$$

Here, m is the number of Monte-Carlo simulations and $Y_{t+\tau}(j)$ and $\mathbf{R}_{t+\tau}(j)$ represent the j^{th} generated path for liabilities and assets conditional on \mathcal{F}_t . Due to the presence of the indicator function, the optimization problem is still non-convex. As it was already mentioned in the introduction, several approaches such as convex approximations or non-convex Mixed Integer Programming (MIP) representations have been recently proposed in the literature to handle the non-parametric constraint. In general, their implementation becomes less efficient when m is large, which is generally required for a better accuracy of the Monte-Carlo estimator. Another alternative is to construct an equivalent chance constrained to (2.3) by finding an appropriate confidence level which requires a reasonable small value for the number of Monte-Carlo paths; however, this depends on the data used and requires a calibration procedure. In order to avoid such issues, we use a conditional version of the semiparametric approach proposed in Asimit *et al.* (2012). This methodology is based on a pre-specified parametric conditional liability distribution and scenario-based asset returns. Using the notation, $E[\cdot | \mathcal{F}_t \cup \{\mathbf{R}_{t+\tau} = \mathbf{R}_{t+\tau}(j)\}] = E_t^{(j)}[\cdot]$, and using the double expectation rule, we reformulate the initial problem:

$$\begin{aligned} & \min_{c_t, \mathbf{x}_t} c_t \\ \text{s.t.} \quad & \frac{1}{m} \sum_{j=1}^m E_t^{(j)} [\mathbb{1}_{\{Y_{t+\tau} - (p_t + c_t) \mathbf{R}_{t+\tau}^T(j) \mathbf{x}_t > 0\}}] \leq 1 - \alpha, \\ & E_t[ROC_{t,t+\tau}] \geq ROC_\alpha, \\ & \mathbf{1}^T \mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0. \end{aligned} \quad (2.4)$$

The expectation in the solvency constraint is taken with respect to the r.v. Y . A sufficient condition for the convexity of (2.4) is that $E_t^{(j)} [\mathbb{1}_{\{Y_{t+\tau} - (p_t + c_t) \mathbf{R}_{t+\tau}^T(j) \mathbf{x}_t > 0\}}]$ is convex in (c_t, \mathbf{x}_t) , for any $j = 1, \dots, m$. This is

equivalent to having a conditionally convex survival function for the liability $Y_{t+\tau}$. Most of the survival functions used for modelling claim data possess this property (some not on their entire domain) and all our empirical results in Section 4 will be based on such a distribution.

2.2 Optimization with $CVaR$ constraint

The $CVaR$ was introduced by Rockafellar and Uryasev (2000) as an alternative coherent risk measure to VaR , which quantifies the loss severity in the case of default. For general random variables, the $CVaR$ is defined as a weighted average of the corresponding VaR and conditional expected losses which strictly exceed VaR . When losses have a continuous distribution function, $CVaR$ coincides with ES (e.g. see Acerbi and Tasche, 2002, and Hürliman, 2003), which constitute the basis for quantifying the target capital according to the Swiss Solvency Test (EIOPA, 2011), that applies to all Swiss based insurance companies.

Following a similar approach as in the RP -constrained case, we define the following optimization problem with a $CVaR$ solvency constraint:

$$\begin{aligned}
& \min_{c_t, \mathbf{x}_t} c_t \\
\text{s.t. } & CVaR_t^\beta(L_{t,t+\tau}) \leq 0, \\
& E_t[ROC_{t,t+\tau}] \geq ROC^\beta, \\
& \mathbf{1}^T \mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0.
\end{aligned} \tag{2.5}$$

Here, β is the confidence level for $CVaR$ and ROC^β is the associated lower bound for our performance measure. $CVaR$ is a more conservative measure of risk than VaR given the same confidence level. In the empirical analysis from Section 4, we shall relate the confidence levels for each of the risk measures by, $\beta = 1 - 2(1 - \alpha)$, such that VaR^α is the median of the worst $1 - \beta$ events. This is also satisfied by the values used in the Solvency II and SST directives ($\alpha = 99.5\%$ and $\beta = 99\%$).

There are various ways of formulating $CVaR$ in the literature. The most appropriate representation for our context is the one provided by Pflug (2000) and Rockafellar and Uryasev (2000), who define $CVaR$ as the solution of an optimization problem. For a general loss r.v. Z we have:

$$CVaR^\beta(Z) = \inf_{s \in \mathbb{R}} \left\{ s + \frac{1}{1 - \beta} E[(Z - s)_+] \right\},$$

where $(Z - s)_+ = \max(Z - s, 0)$. Using the above definition, the optimization (2.5) becomes:

$$\begin{aligned}
& \min_{s, c_t, \mathbf{x}_t} c_t \\
\text{s.t. } & s + \frac{1}{1-\beta} E_t[(L_{t,t+\tau} - s)_+] \leq 0, \\
& E_t[ROC_{t,t+\tau}] \geq ROC^\beta, \\
& \mathbf{1}^T \mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0.
\end{aligned} \tag{2.6}$$

There are different potential strategies for reformulating the solvency constraint. The traditional method used in the literature is based on approximating the above conditional expectation with a Monte-Carlo type estimator and transforming (2.6) into a Linear Programming (LP) problem. Indeed, under a fully non-parametric prescription, the *CVaR* constraint can be rewritten as:

$$s + \frac{1}{m(1-\beta)} \sum_{j=1}^m (Y_{t+\tau}(j) - (p_t + c_t) \mathbf{R}_{t+\tau}^T(j) \mathbf{x}_t - s)_+ \leq 0,$$

which can be further reformulated as a system of linear inequalities by introducing m additional decision variables (e.g., see Rockafellar and Uryasev, 2000). Despite the attractiveness of having the LP representation, the implementation of (2.6) with standard solvers becomes less efficient when the number of Monte-Carlo paths is large, since the dimension of the problem increases with m . Therefore, alternative convex approximations for the conditional expectation should be investigated to accommodate such scenarios. For example, Alexander *et al.* (2006) use a continuously differentiable piecewise quadratic approximation. As in the *RP*-constrained optimization case, we propose here a semiparametric approach which reformulates (2.6) as:

$$\begin{aligned}
& \min_{s, c_t, \mathbf{x}_t} c_t \\
\text{s.t. } & s + \frac{1}{m(1-\beta)} \sum_{j=1}^m E_t^{(j)} [(Y_{t+\tau} - (p_t + c_t) \mathbf{R}_{t+\tau}^T(j) \mathbf{x}_t - s)_+] \leq 0, \\
& E_t[ROC_{t,t+\tau}] \geq ROC^\beta, \\
& \mathbf{1}^T \mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0.
\end{aligned} \tag{2.7}$$

A sufficient condition which ensures the convexity of (2.7) is that $E_t^{(j)} [(Y_{t+\tau} - (p_t + c_t) \mathbf{R}_{t+\tau}^T(j) \mathbf{x}_t - s)_+]$ is a convex function in s, c_t and \mathbf{x}_t . This issue is discussed in Section 3, once the liability r.v. is fully specified.

2.3 Optimization with *EPD* constraint

The *EPD* concept was introduced by Busic (1994) as an alternative method to the ruin probability for measuring insolvency risk, and constitutes a useful tool in establishing the US RBC system. *EPD* is defined as the expected

loss in the event of insolvency, and thus, it is similar to the *ES* concept (for a detailed discussion, see Cummins and Phillips, 2009). Translating this definition into our setting, we write:

$$EPD(L_{t,t+\tau}) = E_t[(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T \mathbf{x}_t)_+]. \quad (2.8)$$

The solvency constraint based on this measure can be constructed by imposing a maximum allowance level for *EPD*. However, since an a priori choice of such threshold is not straightforward and it depends on the insurer expected liability, we introduce a solvency criteria based on a target level of a deficit ratio. Consequently, the *EPD* constraint is defined as:

$$\frac{EPD(L_{t,t+\tau})}{E_t[Y_{t+\tau}]} \leq f.$$

Here, f is the maximum level for the *EPD* ratio with $0 \leq f < 1$. Since (2.8) contains a similar expectation term as in the *CVaR* definition, the whole discussion on dealing with *CVaR*-constrained problems applies here as well. For consistency, we only give the semiparametric representation for our *EPD*-constrained optimization problem:

$$\begin{aligned} & \min_{c_t, \mathbf{x}_t} c_t \\ \text{s.t. } & \frac{1}{m} \sum_{j=1}^m E_t^{(j)} [(Y_{t+\tau} - (p_t + c_t)\mathbf{R}_{t+\tau}^T(j)\mathbf{x}_t)_+] \leq f E_t[Y_{t+\tau}], \\ & E_t[ROC_{t,t+\tau}] \geq ROC^f, \\ & \mathbf{1}^T \mathbf{x}_t = 1, \quad \mathbf{x}_t \geq \mathbf{0}, \quad c_t \geq 0. \end{aligned} \quad (2.9)$$

The convexity of the solvency constraint in (2.9) will be discussed in the same manner as in the *CVaR* case in Section 3.

3 Modelling assets and liabilities

MV-GARCH processes are probably the most popular tools for modelling the variances and covariances of different assets in discrete time. Depending of the conditional covariance matrix structure, a large number of MV-GARCH models have been proposed in the literature. We consider here the class of DCC-GARCH models of Engle (2002), for modelling the multivariate dynamic of the log-return process. Due to their relative simple estimation procedure, the DCC framework is also convenient for large scale risk management problems.

We assume that the vector of asset log-returns are observed at a higher frequency than solvency is observed.

In particular, we sample returns on a daily basis:

$$\log \mathbf{R}_{t+1} = \mathbf{m}_{t+1} + \boldsymbol{\varepsilon}_{t+1}, \quad \boldsymbol{\varepsilon}_{t+1} | \mathcal{F}_t \sim \text{MVN}(\mathbf{0}, H_{t+1}). \quad (3.1)$$

Here, \mathbf{m}_{t+1} is the n -dimensional \mathcal{F}_t -measurable conditional mean log-return vector and $\boldsymbol{\varepsilon}_{t+1} = (\varepsilon_{1,t}, \dots, \varepsilon_{n,t})^T$ has a conditionally multivariate Gaussian distribution with mean $\mathbf{0}$ and covariance matrix H_{t+1} .

One of the main features of the DCC structure is that it allows for separate dynamics for the individual conditional variances and the time-varying conditional correlation matrix. In the following, we briefly illustrate Engle's (2002) formulation:

$$H_{t+1} = D_{t+1}^{1/2} \Sigma_{t+1} D_{t+1}^{1/2}, \quad (3.2)$$

$$D_{t+1} = \text{diag}(h_{1,t+1}, \dots, h_{n,t+1}), \quad (3.3)$$

$$\Sigma_{t+1} = \text{diag}(q_{11,t+1}^{-1/2}, \dots, q_{nn,t+1}^{-1/2}) Q_{t+1} \text{diag}(q_{11,t+1}^{-1/2}, \dots, q_{nn,t+1}^{-1/2}), \quad (3.4)$$

$$Q_{t+1} = (1 - \theta_1 - \theta_2) \bar{Q} + \theta_1 \mathbf{u}_t \mathbf{u}_t^T + \theta_2 Q_t. \quad (3.5)$$

Here, D_{t+1} is the $n \times n$ diagonal matrix formed with the univariate conditional variances which are assumed to follow a standard GARCH(1,1) process as below:

$$h_{i,t} = \omega_i + \alpha_i \varepsilon_{i,t-1}^2 + \beta_i h_{i,t-1}, \quad i = 1, \dots, n \quad (3.6)$$

The time-varying conditional correlation matrix of \mathbf{R}_{t+1} is denoted by Σ_{t+1} and its elements $\rho_{ij,t+1}$ are of the form $\rho_{ij,t+1} = q_{ij,t+1} q_{ii,t+1}^{-1/2} q_{jj,t+1}^{-1/2}$, for any $1 \leq i, j \leq n$; $q_{ij,t+1}$ are the elements of Q_{t+1} and are assumed to follow another GARCH(1,1) dynamic given in (3.5). The process \mathbf{u}_t represents the $n \times 1$ vector of devolatilized, but correlated innovations (i.e. $u_{i,t} = h_{i,t}^{-1/2} \varepsilon_{i,t}$) and \bar{Q} is the unconditional covariance matrix of \mathbf{u}_t . We assume that all univariate GARCH parameters in (3.6), ω_i, α_i and β_i , and the DCC parameters in (3.5), θ_1, θ_2 , satisfy the conditions required for covariance stationarity, and positive definiteness of H_{t+1} , for any t .

In order to investigate the effect of the time-varying conditional correlations between the portfolio's assets, we shall also look at two other models, which can be viewed as particular cases of the DCC-GARCH. The first one is the Conditional Constant Correlation (CCC) model of Bollerslev (1990) that can be obtained by replacing the time-varying correlation matrix by a symmetric positive definite matrix with constant elements (i.e. $\Sigma_t = \Sigma$). The second alternative analyzed assumes the assets are uncorrelated, and this is immediately obtained by letting $\Sigma_t = I_n$ in (3.5), where I_n is the $n \times n$ identity matrix.

Historical data for modelling claim amounts is commonly fitted using one-component parametric distributions such as, Pareto, Lognormal, Gamma, Weibull etc., or more recently using composite distributions (see for example, Scollnik and Sun, 2012, and the references therein for Pareto composite models). Since the objective of the paper is not to investigate goodness-of-fit of different alternatives, we restrict our attention only to a one parametric family. In particular, we consider that claims are modelled with a Lognormal distribution. Since our semiparametric method requires the computations of various conditional expectations given historical information on the asset evolutions, we further assume in our numerical examples that $Y_{t+\tau}$ is independent of the enlarged filtration $\mathcal{F}_t \cup \sigma(\mathbf{R}_{t+\tau})$, for any time t and a given solvency horizon τ . Although this allows us for a more convenient implementation, the optimization problems can be solved under more general dependence structures between assets and liabilities, as long as the resulting constraints are convex. Therefore, we let:

$$Y_{t+\tau} \sim \text{LGN}(\mu_{t+\tau}, \sigma_{t+\tau}). \quad (3.7)$$

The model parameters are assumed to be time-dependent as they will be re-estimated using a double rolling-window exercise. In the remainder of this section, we discuss the convexity of the solvency constraints under the lognormality assumption from (3.7).

First, we let $\mathbf{z}_t = (p_t + c_t)\mathbf{x}_t$ in all three optimization problems (2.4), (2.7) and (2.9). With this notational change, the new decision variables are c_t and \mathbf{z}_t , and the budget constraint becomes $\mathbf{1}^T \mathbf{z}_t = p_t + c_t$ with $\mathbf{z}_t \geq \mathbf{0}$. Under the above assumption, the solvency constraint for the *EPD* problem can be rewritten as:

$$\frac{1}{m} \sum_{j=1}^m E[(Y_{t+\tau} - \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t)_+] \leq fE[Y_{t+\tau}].$$

A sufficient condition for convexity is that $E[(Y_{t+\tau} - \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t)_+]$ is convex for any $j = 1, \dots, m$. We notice that the quantity under the expectation represents the payoff of a European Call option written on $Y_{t+\tau}$. Under the lognormality assumption of $Y_{t+\tau}$, we can write the above expectation as the present value at maturity of a Black-Scholes Call price, $e^{rT}BS(S, K, T, \sigma, r)$, with the following parameter matching:

$$S = 1, \quad K = \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t, \quad T = 1, \quad \sigma = \sigma_{t+\tau}, \quad r = \mu_{t+\tau} + \frac{\sigma_{t+\tau}^2}{2}.$$

Thus, the solvency constraint can be reformulated as:

$$\sum_{j=1}^m \left[\exp\left(\mu_{t+\tau} + \frac{\sigma_{t+\tau}^2}{2}\right) \Phi\left(\frac{-\log(\mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t) + \mu_{t+\tau} + \frac{\sigma_{t+\tau}^2}{2}}{\sigma_{t+\tau}}\right) - \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t \Phi\left(\frac{-\log(\mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t) + \mu_{t+\tau}}{\sigma_{t+\tau}}\right) \right] \leq b. \quad (3.8)$$

Here, $b = fm \exp\left(\mu_{t+\tau} + \frac{\sigma_{t+\tau}^2}{2}\right)$ and $\Phi(\cdot)$ is the cumulative distribution function of a standard Gaussian random variable. The convexity of (3.8) follows now from the convexity property of the European Call price with respect to the strike price, which is itself an affine function of \mathbf{z}_t .

Using a similar prescription, we can show the convexity of the *CVaR* optimization problem (2.7) based on the Black-Scholes formula. The only difference consists of having a different strike price $K = \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t + s$, which is a linear function of the decision variables s and \mathbf{z}_t .

We now turn our attention to the *RP* problem (2.4). The solvency constraint is equivalent to:

$$\frac{1}{m} \sum_{j=1}^m \Phi\left(\frac{-\log(\mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t) + \mu_{t+\tau}}{\sigma_{t+\tau}}\right) \leq 1 - \alpha. \quad (3.9)$$

Since the standard Gaussian c.d.f. is convex only on its negative domain, a sufficient condition for the convexity of (3.9) is the following:

$$\min_{1 \leq j \leq m} \mathbf{R}_{t+\tau}^T(j)\mathbf{z}_t \geq \exp \mu_{t+\tau}. \quad (3.10)$$

Thus, according to condition (3.10), convexity of (2.4) is satisfied when the terminal value of the total assets investment in the worst case scenario is greater than the median of the liability distribution. Although this requirement cannot be verified analytically as in the previous two cases, our numerical simulations from Section 4 indicate that (3.10) is never violated.

4 Empirical Analysis

In this section, we investigate the empirical performance of three MV-GARCH models for all optimization problem considered. We provide two main numerical experiments. Firstly, for a specified solvency target, we construct efficient frontiers for (2.4), (2.7) and (2.9) by varying the expected *ROC*. Secondly, the out-of-sample performance analysis is carried out through a detailed double rolling window estimation exercise.

4.1 Data Description

We consider a 3-asset portfolio formed with two types of financial instruments: one "risk-free" asset represented by three month US T-Bills and two risky assets consisting of the NASDAQ and NYSE Composite indices. These indices have been previously used for multivariate option pricing under various MV-GARCH models by Rombouts

and Stentoft (2011). Both index series were observed on a daily basis from January 3, 2005 to July 29, 2011 for a total of $l = 1,656$ observations. Descriptive statistics for NASDAQ and NYSE log-returns are provided in Table 4.1.1. We notice that there are not significant differences between the two series in terms of the first two

Table 4.1.1: Descriptive statistics for NASDAQ and NYSE log-returns from January 3, 2005 - July 29, 2011 for a total of 1,656 observations.

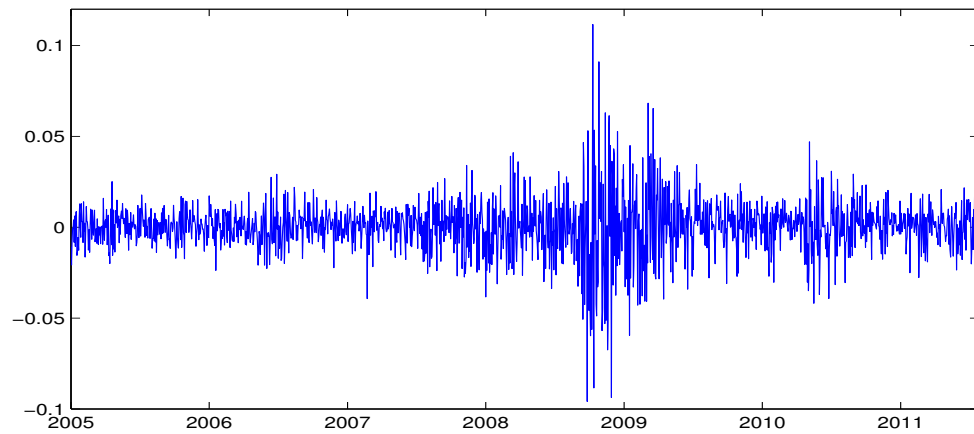
Index	Min	Max	Mean	Std	Skewness	Kurtosis
NASDAQ	-0.0959	0.1116	0.0001	0.0149	-0.1670	10.2725
NYSE	-0.1023	0.1153	0.0001	0.0150	-0.3480	12.7329

moments. However, the NYSE log-returns have a more pronounced negative skewness (i.e. a higher chance of extremely negative outcomes) and a greater kurtosis than the NASDAQ counterpart. The data set is divided into two samples: Sample A consists of $l_A = 1,259$ daily observations for a 5-year period from January 3, 2005 through December 31, 2009, and it is used for the in-sample estimation and analysis of the efficient frontiers. Sample B , which covers the period from January 1, 2010 through July 29, 2011 with $l_B = 397$ daily points, is used for testing the out-of-sample performance in the rolling window exercise. The evolution of both indices over the entire period is depicted in Figure 4.1.1. The most significant part of the financial crisis period is included into Sample A . Figure 4.1.1 also indicates a strong correlation between the two risky assets, with similar volatility clustering shapes.

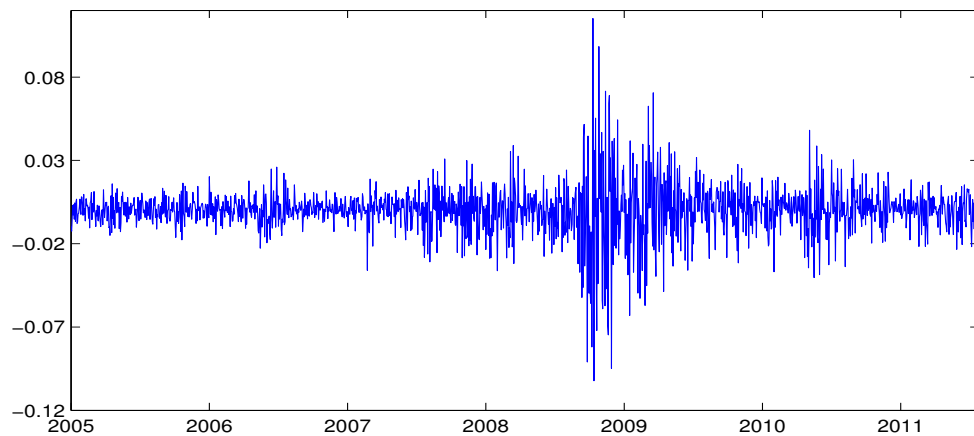
For liabilities, we use a data set on property insurance claim amounts provided by a Romanian insurance company for the same period used in the assets case. However, the main difference is that the sampling frequency is different from the one used for assets. There are 79 observations representing aggregate monthly claim amounts, which are divided into two samples according to a similar prescription (i.e. Sample A' consist of $l_{A'} = 60$ monthly observations and Sample B' has $l_{B'} = 19$ data points which are used for the out-of-sample comparison). The main characteristics of the entire sample are illustrated in Table 4.1.2.

4.2 Estimation results

We first estimate the parameters for the asset returns. There are various ways which one can specify the conditional mean vector in the MV-GARCH log-return equation (3.1). For example, Rombouts and Stentoft (2011) use a



(a) NASDAQ



(b) NYSE

Figure 4.1.1: Time series of log-returns for NASDAQ and NYSE over the period January 3, 2005 - July 29, 2011 for a total of 1656 observations.

Table 4.1.2: Descriptive statistics for monthly claim amounts from January 3, 2005 - July 29, 2011 for a total of 79 observations (figures are in thousands €).

Min	Max	Mean	StDev	Skewness	Kurtosis
8.2465	2049.2119	603.2802	375.1311	1.2434	5.5068

multivariate risk premium specification for \mathbf{m}_t when pricing options under a DCC-GARCH model, while Hlouskova *et al.* (2009) consider an autoregressive structure for deriving multistep predictions with applications in risk management. Since our objective is to analyze the conditional correlation effect to the optimal capital and its allocation, we perform our estimation ignoring the mean effect.

The estimation procedure follows the two-stage Maximum Likelihood Estimator (MLE) algorithm proposed by Engle and Sheppard (2001). In the first stage, the univariate GARCH parameters are estimated by replacing the conditional correlation matrix of \mathbf{R}_k , Σ_k , with the identity matrix in the log-likelihood function below:

$$\log L = -\frac{1}{2} \sum_{k=1}^l (\log(|H_k|) + \boldsymbol{\varepsilon}'_k H_k^{-1} \boldsymbol{\varepsilon}_k),$$

where l represents the number of observations in the dataset. Given the parameters estimated in the first stage, the DCC and CCC parameters are estimated based on the correct log-likelihood specification with Σ_k and Σ , respectively. Thus, at the second stage only θ_1 and θ_2 for DCC, and ρ for CCC are estimated. The results are reported in Table 4.2.1. According to all three selection criteria, the DCC specification is preferred to the CCC

Table 4.2.1: Parameter estimates (with corresponding asymptotic variances reported the brackets) using log-returns for NASDAQ and NYSE during January 3, 2005 - July 29, 2011 for a total of 1656 observations. AIC and BIC are the Akaike and Bayesian Information Criteria.

Estimation Stage	Model parameters				Selection criteria		
1	Index	ω_i	α_i	β_i			
	NASDAQ	2.0E-06 (5.06E-13)	0.0736 (1.35E-04)	0.9146 (1.40E-04)			
	NYSE	1.4E-06 (3.99E-13)	0.0856 (1.46E-04)	0.9061 (1.41E-04)			
	Covariance model						
2	DCC	θ_1	θ_2		$\log L$	AIC	BIC
		0.0432 (5.58E-05)	0.9409 (9.28E-05)		11,630	-23,245	-23,201
	CCC	ρ			$\log L$	AIC	BIC
		0.9061 (1.64E-05)			11,574	-23,134	-23,096

one. The parameter estimates for the DCC-GARCH are in the same range with the values obtained in other previous studies. Each univariate series is characterized by a high degree of persistence (i.e., $\alpha_1 + \beta_1 = 0.988$ for NASDAQ, and $\alpha_2 + \beta_2 = 0.992$ for NYSE). Similar persistence can be observed in the conditional correlation dynamic, since $\theta_1 + \theta_2 = 0.984$. The value of $\rho = 0.91$ in the CCC case suggests a high degree of positive

correlation over the considered period. The implied univariate GARCH conditional variances and conditional correlation between the two assets are illustrated in Figure 4.2.1. We notice the conditional correlation implied by the DCC-GARCH structure has a minimum value of around 0.7 (reached at the beginning of 2007), while its maximum is approximately 0.95 (reached at the beginning of 2009).

Next, we use the MLE to fit a Lognormal distribution on the monthly claim amounts for the same period. The results are reported in Table 4.2.2. The Kolmogorov-Smirnov test indicates that a Lognormal distribution cannot be rejected at 5% significance level.

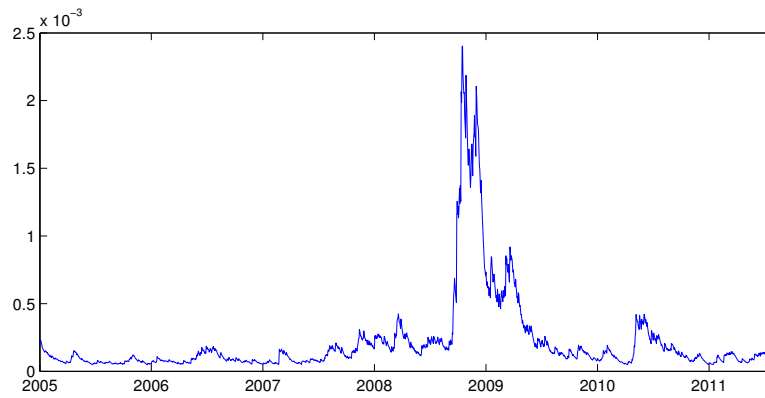
Table 4.2.2: Parameter estimates (with corresponding asymptotic variances reported the brackets) for Lognormal distribution using monthly claim amounts for property insurance during January 3, 2005 - July 29, 2011 for a total of 79 observations. KStest stands for the Kolmogorov-Smirnov test and its p -value is reported in the brackets.

$\hat{\mu}$	$\hat{\sigma}$	Log L	KStest
6.160460	0.829457	584.00	0.1297
(0.0087)	(0.0044)		(0.1279)

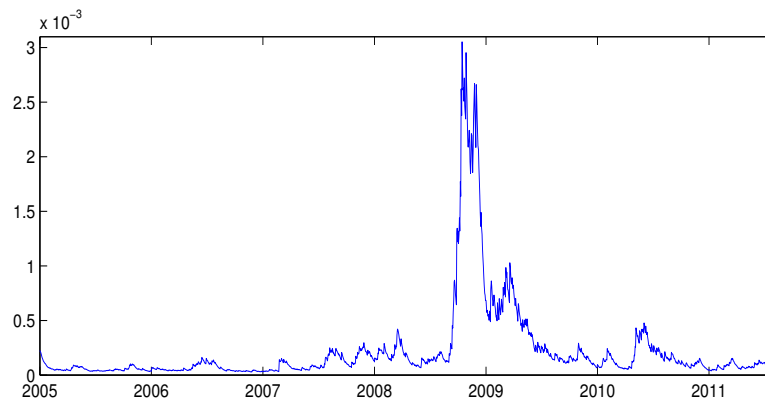
4.3 Implementation of solvency constrained optimization

All three optimization problems (2.4), (2.7) and (2.9), combined with the convex reformulations for the solvency constraints from Section 3, are implemented using Matlab's non-linear optimization routine `fmincon` based on interior-point algorithms. The solvency targets are fixed as follows: $\alpha = 99.5\%$ (the standard value imposed by Solvency II) for the *RP*-constrained problem, $\beta = 99\%$ (the standard value imposed by SST) for the *CVaR*-constrained problem and $f = 0.25\%$ (arbitrarily chosen) for the *EPD*-constrained problem. Since losses are sampled on a monthly basis, we let the solvency horizon $\tau = 21$ days. Given all the information up to time t , each optimization is implemented according to the following algorithm:

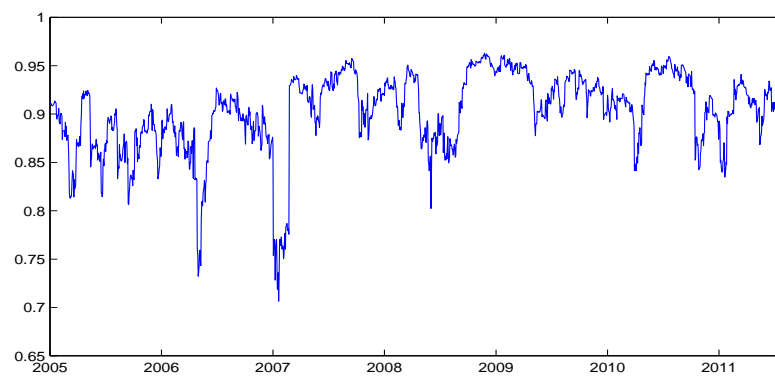
1. Estimate the asset and liability parameters according to the methodology described in Section 4.2.
2. Compute the insurance premiums using the expected premium principle, so $p_t = (1 + \eta)E[Y_{t+\tau}]$, where η is the relative security loading factor fixed at 0.1.



(a) Conditional variance for NASDAQ



(b) Conditional variance for NYSE



(c) Conditional correlation

Figure 4.2.1: Conditional variances and correlations for the DCC-GARCH models based on the MLE estimates over the period January 3, 2005 - July 29, 2011 for a total of 1656 observations.

3. Generate $m = 10,000$ Monte-Carlo paths, $\mathbf{R}_{t+\tau}(j)$, $j = 1, \dots, m$, for the asset returns, according to the corresponding covariance structure from equations (3.1)-(3.5). For the T-Bill rate, we use the three month rate corresponding to the period $[t, t + \tau]$.
4. Solve each optimization problem and find the optimal capital required c_t^* , and the optimal portfolio allocation $(x_{i,t}^*, i = 1, \dots, 3)$.

When liabilities are Pareto distributed, Asimit *et al.* (2012) showed that the differences between the semiparametric approximation implemented with Matlab and the SOC representation under Mosek are insignificant when $m = 10,000$. Moreover, in the Gaussian case, this solution converges to the theoretical one.⁷ Our analysis is based on three covariance models: DCC-GARCH, CCC-GARCH and UNI-GARCH (which is the no-correlation model and is obtained from the CCC specification by letting $\rho = 0$).

4.3.1 Efficient Frontier Analysis

Efficient frontiers are constructed by running the above algorithm for different expected return on capital targets. The minimum levels for the expected *ROC* are obtained by solving the unconstrained version of each optimization (i.e. we discard the performance measure constraint). All parameters are estimated from Sample A data.

The behaviours of (c_t^*) and $(x_{i,t}^*, i = 1, \dots, 3)$ are analyzed for all three covariance specifications. First, we plot the efficient frontiers in Figure 4.3.1. We notice that all efficient frontiers are smooth for all three optimizations. Moreover, the same pattern can be observed for each of the covariance model considered. On the one hand, the DCC-GARCH, which captures the best the correlation dynamic, is the most conservative model in the sense that it requires the highest minimum optimal capital for the same level of expected *ROC*. On the other hand, c_t^* has the smallest values for UNI-GARCH, as this model totally ignores the strong positive correlation between the two risky assets. The correlation dynamic seems to have a strong impact on the structure of the optimal portfolio. This is depicted in Figure 4.3.2. The optimal allocation into NASDAQ increases with the expected *ROC* level for all three models and for all problems considered, while the optimal allocation in T-Bills decreases in a similar fashion. Indeed, when no expected *ROC* is imposed, the optimal allocations are around 20% in NASDAQ, and 70% (UNI)

⁷In an unreported numerical exercise, we tested the accuracy for the Monte-Carlo approximation when $m = 10,000$ for the *CVaR* and *EPD* problems; our results suggested that the standard errors of the optimal solutions are in the same range as the ones obtained in the *RP* case.

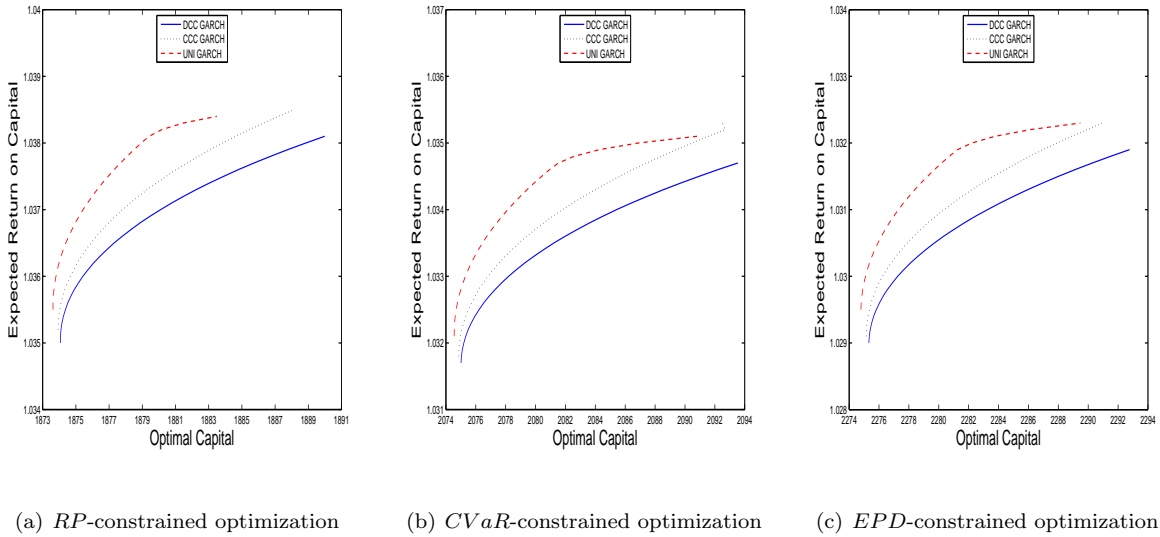


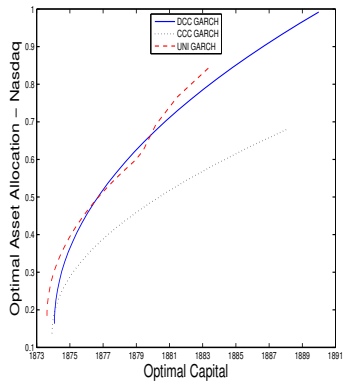
Figure 4.3.1: Efficient frontiers for DCC, CCC and UNI-GARCH models under the *RP*, *CVaR* and *EPF*-constrained problems. Solvency constraints are approximated based on 10,000 Monte-Carlo paths and scenarios are generated based on Sample A estimates.

and 80% (CCC and DCC) in T-Bills. When the shareholders' expected *ROC* approaches its maximum feasible value, the optimal portfolios are constructed based almost solely on the NASDAQ index. Interestingly, the optimal allocation in the riskiest asset (NYSE) is almost zero for the DCC-GARCH, while for the other two dynamics it first increases until a maximum is reached, and after that decreases approximately to zero as well.

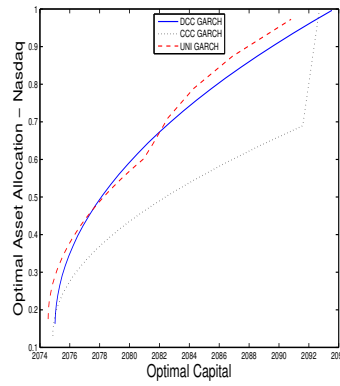
4.3.2 Out-of-Sample Performance

In this section, we carry out an out-of-sample analysis for the optimal portfolios based on *RP*, *CVaR* and *EPD* constraints. Our approach is similar to the standard rolling window methodology proposed in the portfolio optimization literature (e.g., see Santos *et al.*, 2012). However, since we have two main sources of risk, we further propose and analyze the effect of a double rolling window estimation on our optimal solutions.

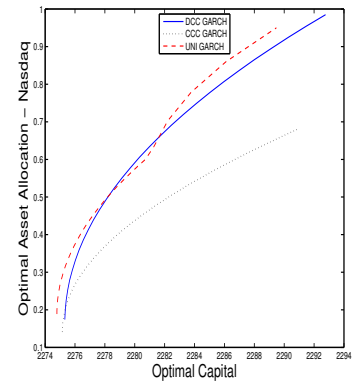
We set the length of the rolling window $l_A = 1,259$ for asset returns estimation and $l_{A'} = 60$ observations for liability estimation. First, we compute the optimal solutions (c_t^*, \mathbf{x}_t^*) for period $[t, t + \tau]$ using data from Samples *A* and *A'*. Next, we construct a new sample for assets by dropping the first $\tau = 21$ observations from Sample *A* and adding the same number of data points from Sample *B*. This corresponds to a monthly portfolio rebalancing.



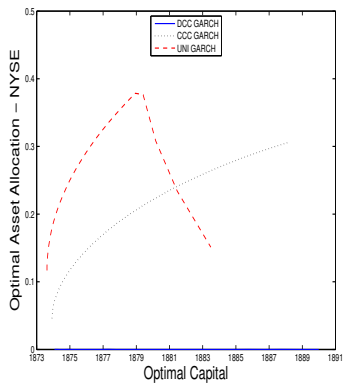
(a) *RP*-constrained optimization



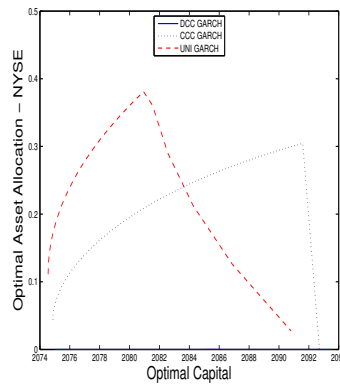
(b) *CVaR*-constrained optimization



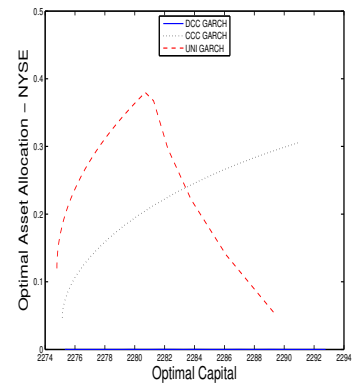
(c) *EPD*-constrained optimization



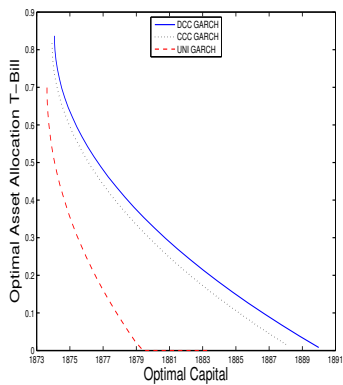
(d) *RP*-constrained optimization



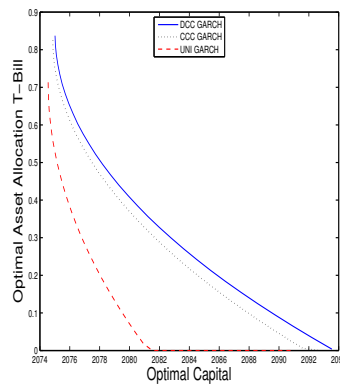
(e) *CVaR*-constrained optimization



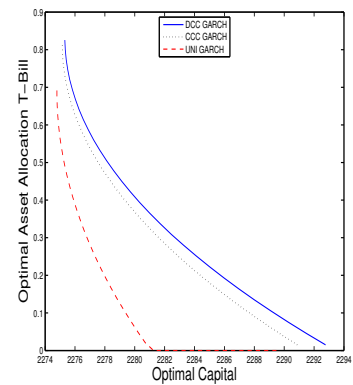
(f) *EPD*-constrained optimization



(g) *RP*-constrained optimization



(h) *CVaR*-constrained optimization



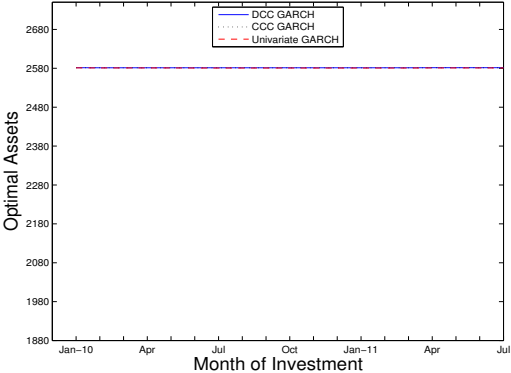
(i) *EPD*-constrained optimization

Figure 4.3.2: Optimal asset allocation for different levels of expected *ROC* for DCC, CCC and UNI-GARCH models under the *RP*, *CVaR* and *EPF*-constrained problems. Solvency constraints are approximated based on 10,000 Monte-Carlo paths and scenarios are generated based on Sample A estimates.

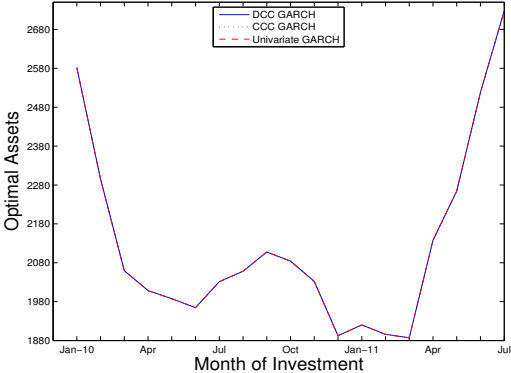
Similarly, we construct the new sample for liabilities by discarding the first observation from Sample A' and adding the first observation from Sample B' . With this new data set, we recompute the next period optimal solutions $(c_{t+\tau}^*, \mathbf{x}_{t+\tau}^*)$ based on Step 1 - 4. We repeat this sampling procedure and the corresponding optimization steps until the end of Sample B /Sample B' is reached. In other words, we have computed $(c_{t+(k-1)\tau}^*, \mathbf{x}_{t+(k-1)\tau}^*)$, with $k = 1, \dots, l_{B'}$, optimal solutions for each solvency constrained problem and MV-GARCH model. The effect of the liability estimates is investigated by comparing the optimal solutions obtained under the double rolling window experiment with the ones calculated from a single rolling window exercise (i.e. when only asset returns re-estimation is performed). In the latter case, all optimizations are solved based on liability parameters obtained from Sample A' . In order to avoid potential feasibility issues created by the expected return on capital constraint under liability re-estimation, all optimization problems are implemented without a lower bound for the expected ROC . The results illustrated in Figures 4.3.3-4.3.5.

Figure 4.3.3 plots the evolution of the total optimal assets invested $(p_t + c_t^*)$ into the optimal portfolio under the two exercises. A comparison only between optimal capital requirements would have been misleading, since insurance premiums change at each step under the double rolling window (they are linear functions of expected severity), and this may also cause changes in c_t^* . First, we notice that there are no significant differences among the covariance models. Under the single rolling window scenario, the optimal capital required remains unchanged for all three solvency constrained problems. This is no longer the case when we allow re-estimation for Y_t . The variation in total assets invested is quite large, ranging from 1,887 to 2,728 for RP , 1,986 to 2,949 for $CVaR$ and 1,868 to 3,222 for EPD ; thus, the EPD problem has the largest fluctuation, while RP is the smallest. Figures 4.3.4-4.3.5 suggest that the differences between the optimal portfolio allocations with and without liability re-estimation are much less pronounced. The portfolio structure also depends on the choice of the MV-GARCH model. For example, for the RP and $CVaR$ problems, the variation in optimal allocations for NASDAQ and NYSE are smaller for the UNI-GARCH when compared to the DCC and CCC counterparts. The double rolling window exercise indicates that the optimal investment is very sensitive to the liability parameters; however, this may also be attributed to a much smaller sample size used for liability estimation. Finally, we notice that most portion of amount investment is typically allocated to the $T - Bills$, the minimum value of approximately 70% corresponds to the UNI-GARCH for each problem.

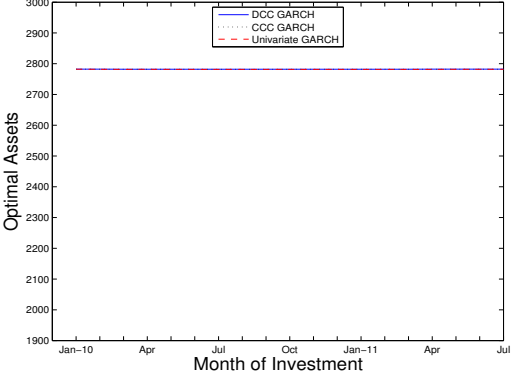
In the remainder of this section, we compute a variety of out-of-sample indicators to provide a comparison



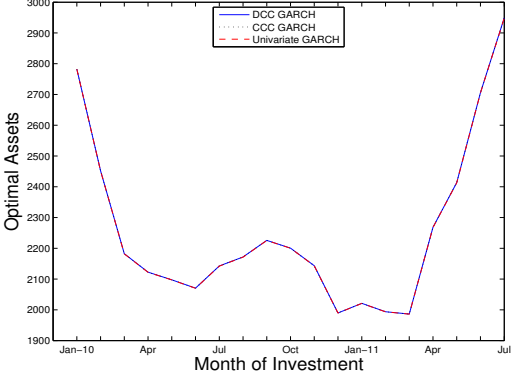
(a) *RP*-constrained and single rolling window



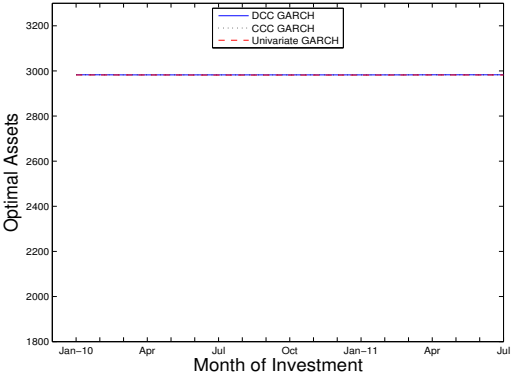
(b) *RP*-constrained and double rolling window



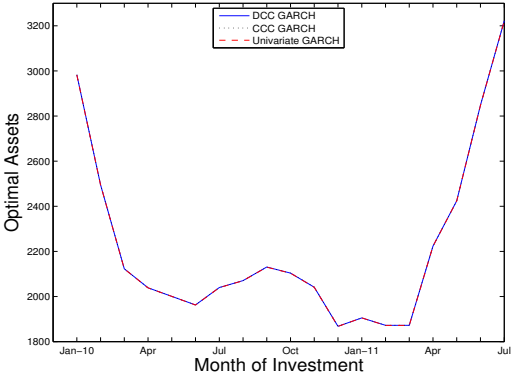
(c) *CVaR*-constrained and single rolling window



(d) *CVaR*-constrained and double rolling window

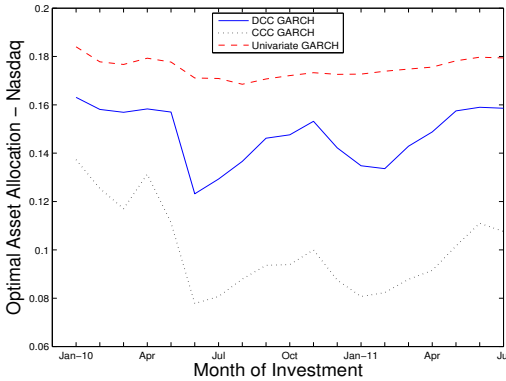


(e) *EPD*-constrained and single rolling window

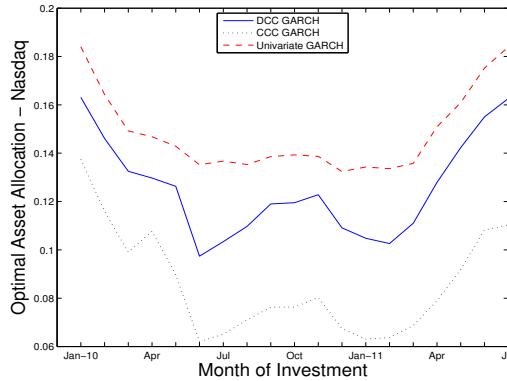


(f) *EPD*-constrained and double rolling window

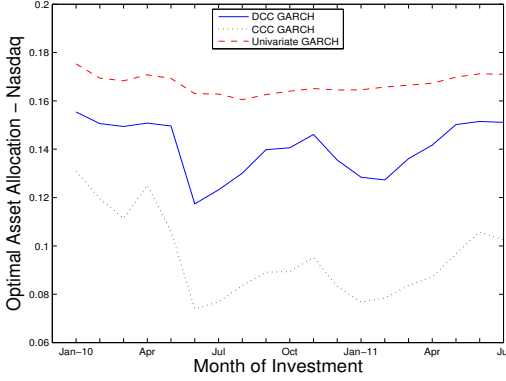
Figure 4.3.3: Comparison of optimal total assets invested, $p_t + c_t^*$, with single and double rolling window.



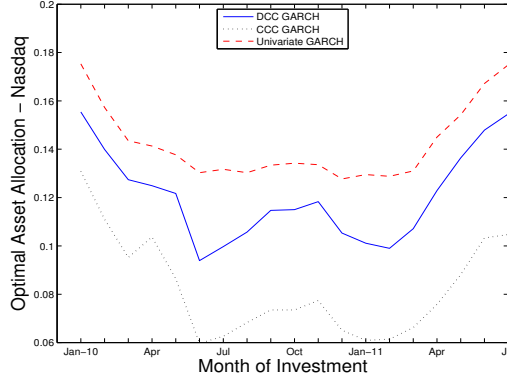
(a) *RP*-constrained and single rolling window



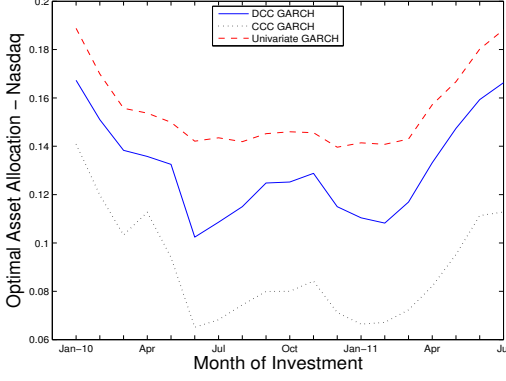
(b) *RP*-constrained and double rolling window



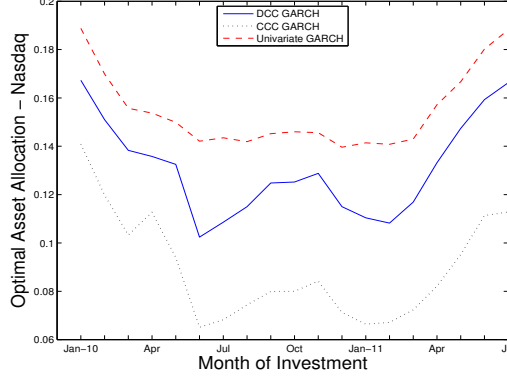
(c) *CVaR*-constrained and single rolling window



(d) *CVaR*-constrained and double rolling window

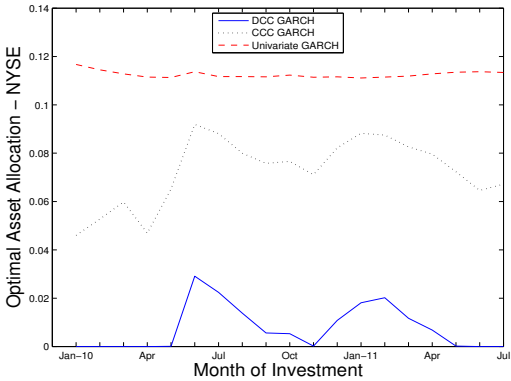


(e) *EPD*-constrained and single rolling window

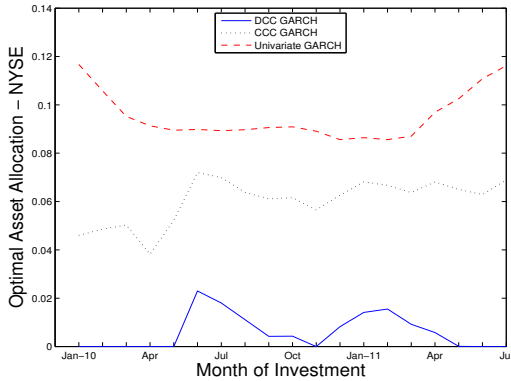


(f) *EPD*-constrained and double rolling window

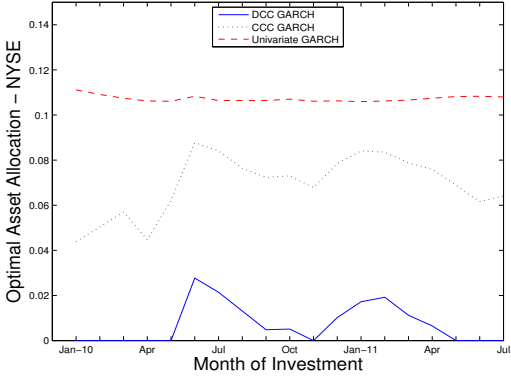
Figure 4.3.4: Comparison of optimal portfolio allocation in NASDAQ, with single and double rolling window.



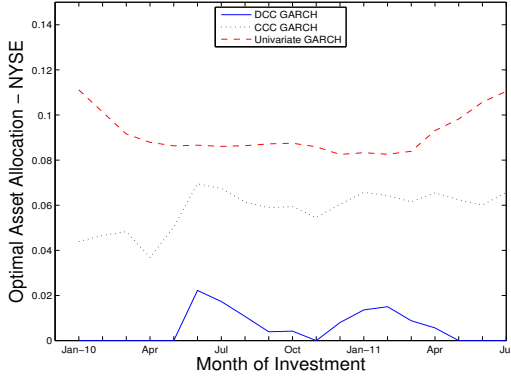
(a) *RP*-constrained and single rolling window



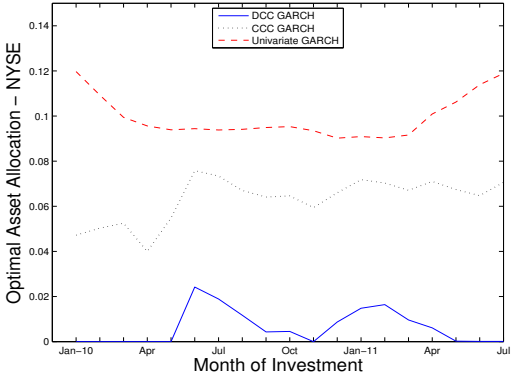
(b) *RP*-constrained and double rolling window



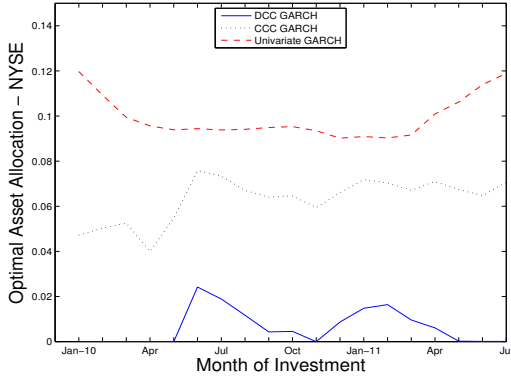
(c) *CVaR*-constrained and single rolling window



(d) *CVaR*-constrained and double rolling window



(e) *EPD*-constrained and single rolling window



(f) *EPD*-constrained and double rolling window

Figure 4.3.5: Comparison of optimal portfolio allocation in NYSE, with single and double rolling window.

between the covariance models relative to the solvency and portfolio performances. In order to measure the solvency requirement performance of the optimal solutions, we consider three metrics: the average assets invested, the average solvency value and the maximum solvency value. All averages are computed over the rolling window period, with and without re-estimation. Depending on the solvency criteria, the average solvency values are computed based on the following expressions:

$$\begin{aligned}\widehat{RP} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \Phi(d_{t+k\tau}), \\ \widehat{CVaR} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \left(\frac{E[Y_{t+k\tau}]}{1-\beta} \Phi(\sigma_{t+k\tau} - \Phi^{-1}(\beta)) - \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* \right), \\ \widehat{EPD} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} \left[E[Y_{t+k\tau}] \Phi(d_{t+k\tau} + \sigma_{t+k\tau}^2) - \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* \Phi(d_{t+k\tau}) \right].\end{aligned}$$

Here,

$$\begin{aligned}\mathbf{z}_{t+(k-1)\tau}^* &= (p_{t+(k-1)\tau} + c_{t+(k-1)\tau}^*) \mathbf{x}_{t+(k-1)\tau}^*, \\ d_{t+k\tau} &= \frac{-\log \mathbf{R}_{t+k\tau}^T \mathbf{z}_{t+(k-1)\tau}^* + \mu_{t+k\tau}}{\sigma_{t+k\tau}},\end{aligned}$$

where $(c_{t+(k-1)\tau}^*, \mathbf{x}_{t+(k-1)\tau}^*)$ and $\mathbf{R}_{t+k\tau}$ represents the optimal solution and the gross return vector, respectively, over the period $[t + (k-1)\tau, t + k\tau]$, for any $k = 1, \dots, l_{B'}$. The average assets invested are calculated by taking averages of all $p_{t+(k-1)\tau} + c_{t+(k-1)\tau}^*$ over the rolling period. The results are reported in the first panel of Tables 4.3.1 and 4.3.2.

For all models and for both rolling window exercises the average total investment is almost the same across each covariance model. We notice that the *CVaR* optimization at 99% requires a higher initial optimal capital than the corresponding *VaR* at 99.5% problem. The *EPD*-constrained optimization based on a ratio $f = 0.25\%$ is the most conservative method. The mean out-of-sample ruin probability is around 0.497 for both experiments. However, we observe scenarios under which the ruin solvency constraint is violated. Although not reported in the tables, the number of violations is the same across all models and typically corresponds to a negative monthly rate of return for both risky assets. More specifically, the maximum values for the ruin probabilities are attained when the asset monthly rate of returns are -11% and -12% , respectively. Potential improvements for reducing the number of constraint violations could be obtained using a more sophisticated conditional mean return (e.g. an autoregressive structure) and estimating the model parameters based on lower frequency data (e.g. weekly or monthly). The

Table 4.3.1: Solvency and portfolio out-of-sample performance under the single rolling window exercise.

Solvency Performance				Portfolio Performance			
	Avg. Assets invested	Avg. Solvency Value	Max. Solvency Value	Avg. AROC (%)	Std. AROC	Sharpe Ratio	Turnover
Problem 1.							
Ruin Constraint		Avg. Ruin Probability (%)	Max. Ruin Prob (%)				
Covariance Model							
DCC	2581.30	0.497	0.543	3.67	1.15	3.18	0.013
CCC	2580.77	0.498	0.550	3.66	1.25	2.92	0.018
UNI	2581.13	0.497	0.585	3.78	2.07	1.83	0.005
Problem 2.							
CVaR Constraint		Avg. CVaR	Max. CVaR				
Covariance Model							
DCC	2782.24	-3.665	46.718	3.32	1.07	3.11	0.013
CCC	2782.06	-3.236	53.702	3.31	1.16	2.85	0.018
UNI	2781.69	-5.220	87.822	3.42	1.92	1.79	0.005
Problem 3.							
EPD Constraint		Avg. EPD Ratio (%)	Max. EPD Ratio (%)				
Covariance Model							
DCC	2982.54	0.248	0.272	3.06	1.13	2.71	0.014
CCC	2982.33	0.249	0.275	3.05	1.22	2.49	0.019
UNI	2981.90	0.248	0.292	3.17	2.02	1.57	0.005

Table 4.3.2: Solvency and portfolio out-of-sample performance under the double rolling window exercise.

Solvency Performance				Portfolio Performance			
	Avg. Assets invested	Avg. Solvency Value	Max. Solvency Value	Avg. AROC (%)	Std. AROC	Sharpe Ratio	Turnover
Problem 1.							
Ruin Constraint		Avg. Ruin Probability (%)	Max. Ruin Prob (%)				
Covariance Model							
DCC	2129.17	0.497	0.546	5.08	1.46	3.46	0.019
CCC	2129.06	0.497	0.553	5.06	1.56	3.24	0.021
UNI	2128.81	0.496	0.589	5.17	2.25	2.29	0.020
Problem 2.							
CVaR Constraint		Avg. CVaR	Max. CVaR				
Covariance Model							
DCC	2258.85	-1.773	28.645	4.67	1.38	3.38	0.018
CCC	2258.75	-1.319	32.857	4.65	1.47	3.16	0.020
UNI	2258.48	-2.114	58.813	4.75	2.11	2.25	0.018
Problem 3.							
EPD Constraint		Avg. EPD Ratio (%)	Max. EPD Ratio (%)				
Covariance Model							
DCC	2222.25	0.248	0.273	4.92	1.67	2.95	0.019
CCC	2222.15	0.249	0.276	4.91	1.77	2.77	0.022
UNI	2221.86	0.248	0.294	5.03	2.46	2.05	0.019

latter reduces the number of simulation steps and thus improves the GARCH forecasting performance. The DCC-GARCH is the best in the sense that it gives the lowest maximum ruin probability of 0.543% (0.546% for liability re-estimation), as opposed to 0.585% (0.589% for liability re-estimation) observed in the no-correlation case. A similar pattern can be observed for the maximum levels of *CVaR* and *EPD* ratio. For example, the *CVaR* under the UNI-GARCH case is almost twice as large as the DCC-GARCH counterpart.

We now analyze the out-of-sample portfolio performance by computing averages, standard deviations and Sharpe ratios based on an adjusted rate of return on capital defined below:

$$AROC_{t,t+\tau} = \frac{(p_t + c_t^*) \mathbf{R}_{t+\tau}^T \mathbf{x}_t^* - E[Y_{t+\tau}]}{c_t} - 1.$$

The following quantities are calculated and reported in the second panel of Tables 4.3.1 and 4.3.2:

$$\begin{aligned} \hat{\mu}_{AROC} &= \frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} AROC_{t+(k-1)\tau, t+k\tau}, \\ \hat{\sigma}_{AROC} &= \sqrt{\frac{1}{l_{B'}} \sum_{k=1}^{l_{B'}} (AROC_{t+(k-1)\tau, t+k\tau} - \hat{\mu}_{AROC})^2}, \\ \widehat{SR}_{AROC} &= \frac{\hat{\mu}_{AROC}}{\hat{\sigma}_{AROC}}, \\ Turnover &= \frac{1}{l_{B'} - 1} \sum_{k=1}^{l_{B'} - 1} \sum_{i=1}^n |x_{i,t+k\tau}^* - x_{i,t+(k-1)\tau}^*|. \end{aligned}$$

DeMiguel and Nogales (2009) interpret the portfolio turnover as the average percentage of wealth traded in each period. From Table 4.3.1, we observe the no-correlation GARCH model outperforms the other two covariance specifications in terms of the average *AROC*. The DCC and CCC-GARCH models have slightly lower and approximately equal values for *AROC*. The risk-return trade-off is also visible from the fact that the average *AROC* is a decreasing function of capital invested. The DCC-GARCH provides the highest values of Sharpe Ratio in all situations. For example, for the *RP*-constrained optimization $\widehat{SR}_{AROC} = 3.18$, for the *CVaR* is 3.11 and for the *EPD* is 2.71. The smallest Sharpe Ratios are recorded for the no-correlation dynamic with values of 1.83, 1.79 and 1.57, respectively. Thus, we can conclude that the incorporation of a dynamic correlation increases the portfolio performance as measured by the Sharpe Ratio. However, the UNI-GARCH model provides the smallest portfolio turnover value. Similar patterns can be observed by examining the results from Table 4.3.2. Under the double rolling window exercise, the average values for the *AROC* increase across all models and optimization problems. This is a direct consequence of the fact that the optimal asset invested decreases due to changes in the liability

parameters. The Sharpe Ratio values indicate that DCC-GARCH is the best model for all three problems, the highest value being obtained for the *RP* constrained optimization (3.46) and the lowest for the *EPD*. Unlike in the single rolling exercise, the turnover ratios have now similar values for the DCC and no-correlation models. We further notice that the turnovers are higher when we re-estimate liability parameters.

5 Conclusions

In this paper we propose three problems to jointly solve for the optimal capital requirement and its optimal portfolio allocation. Each problem is constructed based on two types of constraints. The first set of constraints are dictated by standard solvency insurance requirements such as *VaR*, *CVaR* and *EPD* calculated for a specified horizon and for a given confidence level. The second constraint represents a performance measure constraint based on a lower bound for the shareholders' expected *ROC*. We provide a novel semiparametric approach for solving these problems based on a parametric distribution of the liability random variable and the empirical distribution for asset returns. In particular, we assume claim amounts follow a Lognormal distribution and portfolio's asset returns are generated according to a Dynamical Conditional Correlation multivariate GARCH model. We provide sufficient conditions such that each solvency constraint admits a convex representation; these are further implemented using the non-linear optimization Matlab solver based on interior-point algorithms. We examine optimal solutions for 3-asset portfolios (two indices and one risk-free asset) through two numerical experiments.

In the first numerical example, we construct efficient frontiers for the optimal capital based on different levels of expected *ROC*. The efficient frontiers have the same pattern for all constraints and covariance models considered. The correlation between the two entities plays an important role in the behaviour of the optimal capital required and the portfolio structure. For the same level of expected *ROC*, the minimum value of c_t^* is obtained for the no-correlation model, while DCC-GARCH is the most conservative model. Furthermore, the portfolio weights behaves differently; for example, DCC-GARCH allocates nothing to the NYSE index, while the optimal weights are non-zero for the other covariance specifications. When the shareholders increase their expected *ROC*, the optimal allocation in the T-Bills decreases from almost 70% (for UNI-GARCH) and 80% (for CCC and DCC-GARCH) to zero.

The out-of-sample performance of our portfolio is tested in a second detailed numerical example using a double rolling window estimation for both assets and liabilities. On the one hand, we found that the optimal required

capital is very stable when the liability parameters remain constant over the rolling horizon; however, the variation is substantial when the liability is re-estimated at each step. On the other hand, the differences between the optimal portfolio weights are not as pronounced for the single versus double rolling exercises. We compute two types of indicators for assessing the solvency and return on capital performances. Our results suggest that the DCC model outperforms the other candidates under both scenarios. More specifically, the DCC-GARCH has the smallest value of the maximum RP , $CVaR$ and EPD , and provides the highest out-of-sample Sharpe Ratio. Several extensions to our models can be further investigated by including more complex models for assets and liabilities, as well as by extending this work to allow for multiple business lines, friction costs and possibly a multiperiod setting.

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