Multirisks models in discrete time [1]

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Abstract

This paper proposes a multivariate model, which is a variant of the one introduced by Picard et al. (2003).
The evolution of this model is in discrete time where the premium incomes are arbitrary, and the successive claim amounts remain independent between periods but may be mutually dependent inside each period. Multirisks models can arise in various insurance contexts.
In this work we present some applications like the reinsurance to describe the surplus processes of the cedent and the reinsurers. The ceding and reinsurance companies are jointly liable for covering losses generated by the total mass of claims; hence their risks are mutually dependent.

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1 Introduction

In this paper, we examine a multirisks generalization of a discrete time risk model with non-homogeneous conditions (Castañer et al. (2011)). Thus, the insurance portfolio is now assumed to cover not just a single risk, but several risks that may be interdependent. This multivariate model is a variant of the one introduced by Picard et al. (2003); see also Denuit et al. (2007).

Multirisks models can arise in various insurance contexts. A typical situation is with certain damages or catastrophic events that often cause losses in several branches. Another application is in reinsurance to describe the surplus processes of the cedent and the reinsurers. The ceding and reinsurance companies are jointly liable for covering losses generated by the total mass of claims and hence, their risks are mutually dependent. Kaishev et al. (2008) investigated a bivariate model of this type under an excess of loss reinsurance treaty, and assuming Poisson claim arrivals, any premium income function and arbitrary joint distributions of the claims; see also the references in that paper and e.g. Avram et al. (2008) for a model with proportional reinsurance that may be reduced to a univariate case.

The multivariate model considered here has the same general structure as in Picard et al. (2003). It describes the evolution in discrete-time of r ($\in \mathbb{N}_0$) risk processes where the (deterministic) premium incomes are arbitrary, and the successive claim amounts remain independent between periods but may be mutually dependent inside each period. The vectors of the r claim amounts per period have again non-stationary joint distribution functions that are continuous with a positive probability for no claim.

The paper is organized as follows. In Section 2, the multirisk discrete model is introduced. In Section 3 we make an application of the model in the bivariate case when insurer and reinsurer make a limited stop-loss reinsurance treaty. Some numerical examples of the joint ruin probability of the cedent and the reinsurer are included and commented.

2 Model

In the notation used below, bold letters indicate r-dimensional vectors. So, $\mathbf{x} = (x_1, \dots, x_r)$, $\mathbf{0} = (0, \dots, 0)$, $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_r + y_r)$ and $\mathbf{x} \le \mathbf{y}$ means $(x_1 \le y_1, \dots, x_r \le y_r)$. The initial reserves are $\mathbf{u} = (u_1, \dots, u_r)$. During $(t, t+1]$, the fixed vector of premiums is $\mathbf{c}_t = (c_{1,t}, \dots, c_{r,t})$, and the random vector of claim amounts is $\mathbf{X}_t = (X_{1,t}, \dots, X_{r,t})$, with distribution function F_t. The cumulated premiums and claim amounts for the first t periods are denoted $\mathbf{c}(t)$ and $\mathbf{S}(t)$ (with distribution function $F_{\mathbf{S}(t)}$). Thus, the joint surplus at time t is given by $\mathbf{U}(0) = \mathbf{u}$ and

$$\mathbf{U}(t) = \mathbf{u} + \mathbf{c}(t) - \mathbf{S}(t), \quad t \in \mathbb{N}_0. \tag{2.1}$$

Various definitions of ruin are possible. One could consider that ruin occurs as soon as one of the r surplus process is insolvent, or 2 of them ... or all the r processes. Another possible definition could be that the sum of the r surpluses must become negative (this is basically a single risk case). For the sequel, we say that there is ruin at time T if at least one of the r processes then runs insolvent, i.e. $T = \min(T_1, \ldots, T_r)$ where T_i denotes the ruin time for risk i. This classical choice is reasonable in many applications.

Inside the model (2.1), consider the probability

$$\phi(t, \mathbf{x}) \equiv P[T > t \text{ and } \mathbf{U}(t) \geq \mathbf{x}], \quad t \in \mathbb{N} \text{ and } \mathbf{0} \leq \mathbf{x} \leq \mathbf{u} + \mathbf{c}(\mathrm{t}).$$

Associated with any $\mathbf{s} \in \mathbb{R}^{\mathrm{r},+}$, define the (integer) times $v_{\mathbf{s}} = 0$ if $\mathbf{s} \leq \mathbf{u}$, and

$$v_{\mathbf{s}} = \sup\{t \in \mathbb{N} : \mathrm{u_l} + \mathrm{c_l(t)} < \mathrm{s_l} \text{ for some risk } \mathrm{l} = 1, \ldots, \mathrm{r}\} \quad \text{if } \mathbf{s} \not\leq \mathbf{u}.$$

So, $v_{\mathbf{s}}$ is the last time where a claim amounts vector of $\mathbf{s}$ would lead to ruin for at least one risk, if ever; otherwise, it is equal to 0.

Proposition 2.1

$$\phi(t, \mathbf{x}) = F_{\mathbf{S}(t)}(\mathbf{0}) + \int_{\mathbf{w}=\mathbf{0}}^{\mathbf{u}+\mathbf{c}(t)-\mathbf{x}} b_{\mathbf{w}}\, F_{\mathbf{S}(t)}(\mathbf{u} + \mathbf{c}(t) - \mathbf{x} - \mathbf{w})\, d\mathbf{w}, \tag{2.2}$$

where $b_{\mathbf{w}}$, $\mathbf{w} \in \mathbb{R}^{\mathrm{r},+}$, *is a real function (with a* r*-dimensional index) defined by*

$$0 = \int_{\mathbf{w}=\mathbf{0}}^{\mathbf{s}} b_{\mathbf{s}-\mathbf{w}}\, dF_{\mathbf{S}(v_{\mathbf{s}})}(\mathbf{w}), \quad \mathbf{s} \in \mathbb{R}^{\mathrm{r},+}. \tag{2.3}$$

For numerical calculations claim amounts are discretized as in Castañer et al. (2011).

3 Limited Stop-Loss reinsurance application

As a specific example, we now explicitly formulae the model for a limited stop-loss reinsurance contract. Let $U_1(t)$ and $U_2(t)$ be the surplus process of the cedent and the reinsurer, respectively. The global claim amounts during the successive periods $(t, t+1]$ form a sequence of independent random variables X_t, possibly with different distributions. Suppose that for each X_t both companies agree to fix a retention level $d_t > 0$ and a limiting level $m_t > d_t$ to the cedent. So, the risk covered by the cedent is

$$X_{1,t} = \min(X_t, d_t) + \max(X_t - m_t, 0),$$

and for the reinsurer,

$$X_{2,t} = X_t - X_{1,t} = \min\{m_t - d_t, \max(X_t - d_t, 0)\}.$$

The corresponding surpluses are then given by

$$U_l(t) = u_l + c_l(t) - S_l(t), \quad l = 1, 2,$$

where $c_l(t) = c_{l,1} + \ldots + c_{l,t}$ and $S_l(t) = X_{l,1} + \ldots + X_{l,t}$. Note that $X_{1,t}$ and $X_{2,t}$ being interdependent, this is also true, of course, for $U_1(t)$ and $U_2(t)$. The ruin time $T = \min(T_1, T_2)$ is the first instant when the cedent (at time T_1) or the reinsurer (at time T_2) become insolvent.

In the present application we consider only the joint probability of ruin of cedent and reinsurer, that is

$$\psi(t) = 1 - \phi(t, \mathbf{0}).$$

We consider that claim amounts (X_t), deductible (d_t) and limit (m_t) do not change from one period to another. Insurer and reinsurer use the expected value principle to calculate the premiums with safety loading factors θ_1 and θ_2 respectively, $c_2 = E[X_2]\,(1 + \theta_2)$ and $c_1 = E[X]\,(1 + \theta_1) - c_2$.

The first question is how to calculate the stop-loss premium, $\pi_X(d)$. In general

$$\pi_X(d) = \int_d^\infty (X - d)\, dF_X(x).$$

Then the premium of this limited stop-loss can be calculated from the premium of a stop-loss non limited,

$$E[X_2] = \pi_X(d) - \pi_X(m).$$

Explicit expressions of $\pi_X(d)$ are only available for some distributions of X (e.g. exponential, lognormal, gamma, normal and normal-power). If X follows a compound distribution and the Panjer recursion can be applied, it is possible also to easily calculate the stop-loss premium (see Kaas et al. (2008), Mikosch (2004)).

Let us suppose in this application that the claim amount in one period can be approached by a translated gamma distribution. Let G be a gamma random variable with (positive) parameters (α,β); its distribution function is given by

$$Ga(x; \alpha, \beta) = P(G \leq x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} e^{-\beta y} dy, \quad x > 0.$$

Then, the total claim amount X is approximated by a translated gamma random variable $G + x_0$ such that the first three central moments coincide. This implies that

$$\begin{aligned} \alpha &= 4\sigma^6(X)/\{E[X - E(X)]^3\}^2, \\ \beta &= 2\sigma^2(X)/E[X - E(X)]^3, \\ x_0 &= E(X) - \alpha/\beta. \end{aligned} \tag{3.4}$$

In this case the stop-loss premium is

$$\pi_X(d) = \frac{\alpha}{\beta}\left(1 - Ga(d - x_0; \alpha + 1, \beta)\right) - (d - x_0)\left(1 - Ga(d - x_0; \alpha, \beta)\right).$$

If X is continuous it will be discretized using a span of h, $X^{(h)}$. The parameters of stop-loss treaty must be also expressed in the new units, $D = \left\lfloor \frac{d}{h} \right\rfloor$ and $M = \left\lfloor \frac{m}{h} \right\rfloor$. With the limited stop-loss the bivariate discret distribution $P[X_1 = x_1 h, X_2 = x_2 h]$ is

$$P[X_1 = x_1 h, X_2 = x_2 h] = \begin{cases} P[X^{(h)} = 0], & x_1 = x_2 = 0, \\ P[X^{(h)} = x_1 h], & 0 < x_1 \leq D, x_2 = 0, \\ P[X^{(h)} = (x_2 + D)\,h], & x_1 = D, 0 < x_2 \leq M - D, \\ P[X^{(h)} = (x_1 + M - D)\,h], & x_1 > D, x_2 = M - D, \end{cases}$$

and the other combinations are zero. This can be expressed in matrix form as follows,

$x_1 \backslash x_2$	0	$\cdots$	x_2	$\cdots$	$(M-D)$
0	$P[X = 0h]$	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$
x_1	$P[X = x_1 h]$	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$
D	$P[X = Dh]$	$\cdots$	$P[X = (x_2 + D)\,h]$	$\cdots$	$P[X = Mh]$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
x_1	0	$\cdots$	0	$\cdots$	$P[X = (x_1 + M - D)\,h]$
$\vdots$	$\vdots$		$\vdots$		$\vdots$

Let us consider a numerical example. In this example, X is modeled by a translated gamma random variable with parameters ($\alpha = 8/9$, $\beta = 2/3$, $x_0 = -1/3$). These values are obtained from (3.4) if a compound Poisson distribution with parameter $\lambda = 1$ and i.i.d. exponential individual claim amounts with mean 1 is approximated by a translated gamma. Table 1 includes the periodic premiums of insurer and reinsurer (net and with safety loading) for different values of deductible and limit and different safety loadings.

d	m	$E[X_2]$	c_2	c_1	θ
0.8	1	0.077457	0.085202	0.964798	0.045802
0.8	1.5	0.228983	0.251881	0.798119	0.035151
0.8	2	0.335434	0.368978	0.681022	0.024763
0.8	3	0.463465	0.509812	0.540188	0.006809
1	2	0.257977	0.283775	0.766225	0.032617
1	3	0.386009	0.424610	0.625390	0.018566
2	3	0.128031	0.140834	0.909166	0.042658
2	4	0.192114	0.211325	0.838675	0.038110
2	100	0.257324	0.283056	0.766944	0.032676

Table 1: Premiums of insurer and reinsurer with $\theta_1 = 0.05$, $\theta_2 = 0.1$

Following with the same example, now in Table 2 we show for diferent initial levels of reserves of the insurer and the reinsurer both the individual ruin probability for each one and the joint ruin probability, taking a time horizon $t = 2$ when $d = 0.8$, $m = 1.5$ and considering a discretized span $h = 0.01$.

$u_1 \backslash u_2$		0	0.25	0.5	0.75	1
0	Insurer	$(0.5143, 0.5177)$	$(0.5143, 0.5177)$	$(0.5143, 0.5177)$	$(0.5143, 0.5177)$	$(0.5143, 0.5177)$
	Reinsurer	$(0.5360, 0.5392)$	$(0.3209, 0.3234)$	$(0.1072, 0.1087)$	$(0.0773, 0.0784)$	$(0, 0)$
	Joint	$(0.5845, 0.5878)$	$(0.5143, 0.5177)$	$(0.5143, 0.5177)$	$(0.5143, 0.5177)$	$(0.5143, 0.5177)$
0.1	Insurer	$(0.3698, 0.3724)$	$(0.3698, 0.3724)$	$(0.3698, 0.3724)$	$(0.3698, 0.3724)$	$(0.3698, 0.3724)$
	Reinsurer	$(0.5360, 0.5392)$	$(0.3209, 0.3234)$	$(0.1072, 0.1087)$	$(0.0773, 0.0784)$	$(0, 0)$
	Joint	$(0.5360, 0.5392)$	$(0.4169, 0.4199)$	$(0.3756, 0.3783)$	$(0.3712, 0.3738)$	$(0.3698, 0.3724)$
0.25	Insurer	$(0.3384, 0.3409)$	$(0.3384, 0.3409)$	$(0.3384, 0.3409)$	$(0.3384, 0.3409)$	$(0.3384, 0.3409)$
	Reinsurer	$(0.5360, 0.5392)$	$(0.3209, 0.3234)$	$(0.1072, 0.1087)$	$(0.0773, 0.0784)$	$(0, 0)$
	Joint	$(0.5360, 0.5392)$	$(0.4074, 0.4103)$	$(0.3488, 0.3514)$	$(0.3418, 0.3443)$	$(0.3384, 0.3409)$
0.5	Insurer	$(0.2920, 0.2941)$	$(0.2920, 0.2941)$	$(0.2920, 0.2941)$	$(0.2920, 0.2941)$	$(0.2920, 0.2941)$
	Reinsurer	$(0.5360, 0.5392)$	$(0.3209, 0.3234)$	$(0.1072, 0.1087)$	$(0.0773, 0.0784)$	$(0, 0)$
	Joint	$(0.5360, 0.5392)$	$(0.3936, 0.3964)$	$(0.3100, 0.3124)$	$(0.2994, 0.3017)$	$(0.2920, 0.2941)$
0.75	Insurer	$(0.2518, 0.2537)$	$(0.2518, 0.2537)$	$(0.2518, 0.2537)$	$(0.2518, 0.2537)$	$(0.2518, 0.2537)$
	Reinsurer	$(0.5360, 0.5392)$	$(0.3209, 0.3234)$	$(0.1072, 0.1087)$	$(0.0773, 0.0784)$	$(0, 0)$
	Joint	$(0.5360, 0.5392)$	$(0.3820, 0.3848)$	$(0.2776, 0.2799)$	$(0.2639, 0.2660)$	$(0.2518, 0.2537)$
1	Insurer	$(0.2171, 0.2188)$	$(0.2171, 0.2188)$	$(0.2171, 0.2188)$	$(0.2171, 0.2188)$	$(0.2171, 0.2188)$
	Reinsurer	$(0.5360, 0.5392)$	$(0.3209, 0.3234)$	$(0.1072, 0.1087)$	$(0.0773, 0.0784)$	$(0, 0)$
	Joint	$(0.5360, 0.5392)$	$(0.3723, 0.3751)$	$(0.2504, 0.2526)$	$(0.2341, 0.2361)$	$(0.2171, 0.2188)$

Table 2: Probabilities of ruin with $t = 2$, $d = 0.8$, $m = 1.5$, $\theta_1 = 0.05$, $\theta_2 = 0.1$, $h = 0.01$ where the stop-loss premium is 0.228983

In Figures 1 and 2 we plot the data obtained in Table 2 in order to better analyze the behaviour of the ruin probabilities. In Figure 1 all the data are presented jointly and Figure 2 shows the individual and joint ruin probabilities for both the insurer and the reinsurer.

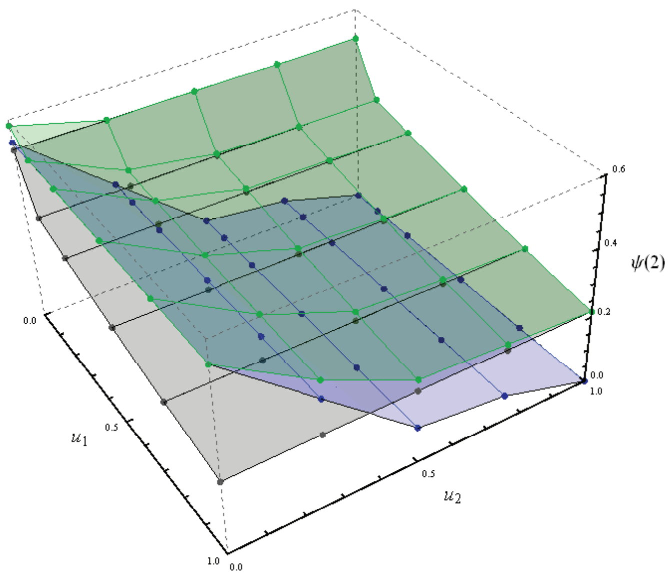

Figure 1: Probabilities of ruin (upper bound) with $t = 2$ as a function of initial reserves of reinsurer (u_2) and insurer (u_1): Reinsurer (blue), Insurer (black) and joint probability (green)

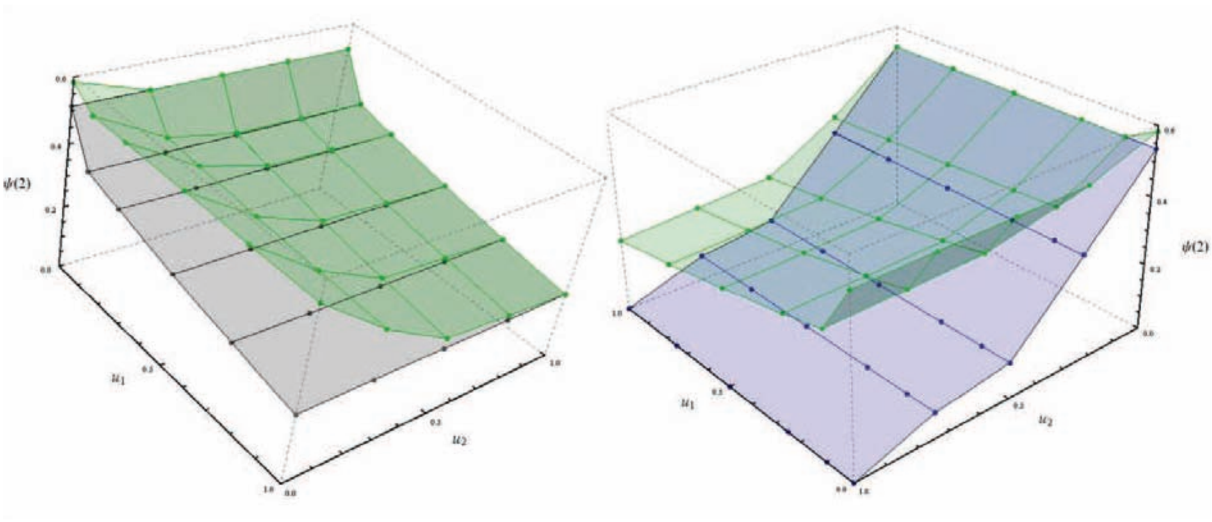

Figure 2: Probabilities of ruin (upper bound) with $t = 2$ as a function of initial reserves of reinsurer (u_2) and insurer (u_1): Left Graph(Insurer (black) and joint probability (green)) and Right Graph (Reinsurer (blue) and joint probability (green))

An interesting question related with reinsurance and joint survival of the cedent and reinsurer is analized in Ignatov et al. (2004). In that paper they propose to use the joint survival of the cedent and the reinsurer and the probability of survival of the reinsurer given the cedent's survival as two criteria for finding the optimal excess of loss treaty. As a first approach to this question in the stop-loss treaties, we consider here one of the two optimal reinsurance problems that can be defined when we use the joint survival probability as optimality criterion

$$\max_{c_2} P\left(T_1 > t, T_2 > t\right).$$

So, we are interested in finding the constant premium (c_2) that the reinsurer must earn in order to maximize the joint survival probability given that the other variables are fixed.

Let us consider a horizon of $t = 2$. In Table 3 we show the premium that reinsurer must earn and the maximum joint survival probability that can be attained in that case, for different values of the initial reserves of the cedent and the reinsurer.

$u_1 \backslash u_2$		0	0.25	0.5	0.75
0	c_2	0.25	0.25	0.2	0.01
	Joint survival	0.463902	0.576772	0.618062	0.654511
0.25	c_2	0.37	0.36	0.2	0.01
	Joint survival	0.558611	0.597721	0.659904	0.687004
0.5	c_2	0.5	0.45	0.2	0.07
	Joint survival	0.580630	0.646240	0.694899	0.715798
0.75	c_2	0.56	0.45	0.2	0.13
	Joint survival	0.600992	0.688630	0.724202	0.742955

Table 3: Optimal joint survival (upper bound) with $t = 2$, $d = 0.8$, $m = 1.5$, $\theta_1 = 0.05$, $h = 0.01$

The maximum joint survival probability increases with the initial reserves of insurer and/or reinsurer. As regards the optimal reinsurer premium, Table 3 shows that for a fixed value of u_1 the optimal premium decreases with respect to u_2, and on the other hand for a fixed value of u_2 the optimal premium is increasing or constant with respect to u_1.

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