# A RISK MEASURE BEYOND COHERENCE

Shaun S. Wang, PhD, FCAS
SCOR

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# Risk Measure

- To decide capital requirement for a given risk portfolio
- Impacts “return-on-capital” calculation
- Different perspectives
  - management & shareholder
  - regulator & policy-holder

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# Value-at-Risk

- $\text{VaR}(\alpha) = \text{Min}\ \{x \mid F(x) \geq \alpha\}$
  - $\alpha$ threshold, e.g., 99% or 95%.
  - A ruin concept to actuaries
- Widely used in banking industry for setting capital requirements
- Management concerns mostly survival probability

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# Coherence Rules

Artzner, Delbaen, Eber, and Heath (99) proposed consistency rules from *regulatory perspective*:

1. $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
2. If $X \leq Y$, $\rho(X) \leq \rho(Y)$.
3. For $b>0$, $\rho(bX) = b\rho(X)$.
4. For constant $c$, $\rho(X+c) = \rho(X) + c$.

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# Tail-VAR or CTE

- Artzner et al showed that VaR is not coherent
- They introduced Tail-VaR or Conditional Tail Expectation (CTE)

$$\mathrm{CTE}(\alpha) = \mathrm{VaR}(\alpha) + \frac{\Pr\{X > \mathrm{VaR}(\alpha)\}}{1-\alpha} \cdot \mathrm{E}\left[X - \mathrm{VaR}(\alpha) \mid X > \mathrm{VaR}(\alpha)\right]$$

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# Myth about Tail-VaR

- Researchers/ practitioners have shown tremendous interests in Tail-VaR !!
- Being talked about *so* frequently *that* it seemed to be **the only** coherent measure.
- *Did you know the truth?* Many other risk-measures are coherent (and beyond).

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# Distortion Measure

- Let $g$:[0,1]→[0,1] be increasing with $g(0)=0$ and $g(1)=1$.
- Let $F(x)$ be the CDF for the shortfall
- Define distortion: $F^*(x) = g[F(x)]$.
- Define risk-measure: $\rho(X) = E^*[X]$.
- Theorem: $\rho(X)$ is coherent when "$g$" is continuous.

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# Example

- VaR non-coherent since "g" discontinuous

$$g(u) = \begin{cases} 0, & \text{when} \quad u < \alpha, \\ 1, & \text{when} \quad u \geq \alpha, \end{cases}$$

- Tail-VaR coherent since "g" continuous. But "g" non-differentiable, nor 1-to-1.

$$g(u) = \begin{cases} 0, & \text{when} \quad u < \alpha, \\ \dfrac{u-\alpha}{1-\alpha}, & \text{when} \quad u \geq \alpha, \end{cases}$$

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# A Smooth Distortion

- Introduce $g(u)=\Phi[\Phi^{-1}(u)-\lambda]$
- "g" is differentiable and 1-to-1
- The unique distortion that recovers CAPM & Black-Scholes (Wang, 2000 JRI)
- A new result of Buhlmann (1980 ASTIN) equilibrium pricing model.

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# A New Risk Measure

- *For threshold* $\alpha$, *let* $\lambda = \Phi^{-1}(\alpha)$ be the $100\alpha$-th *standard normal* percentile.
- *Let* $F^*(x) = \Phi[\Phi^{-1}(F(x)) - \lambda]$.
- *Define risk measure as the mean value under F**

$$WT(\alpha) = E^*[X].$$

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# WT-measure of Normal Risk

- When $F$ is normal$(\mu,\sigma^2)$, $F^*(x) = \Phi[\Phi^{-1}(F(x)) - \lambda]$ is normal$(\mu+\lambda\sigma,\sigma^2)$
- WT($\alpha$) identical to VaR($\alpha$), the 100$\alpha$-th percentile.
- This fact gives a basis for selecting the threshold $\alpha$.

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# WT-measure of lognormal risk

- If ln(X) ~ normal($\mu$,$\sigma^2$), under distortion $g(u) = \Phi[\Phi^{-1}(u) - \lambda]$ we get ln(X*) ~ normal($\mu+\lambda\sigma$,$\sigma^2$)
- WT($\alpha$)= exp($\mu+\lambda\sigma+\sigma^2/2$) with $\lambda = \Phi^{-1}(\alpha)$.
- WT($\alpha$) corresponds to percentile $\Phi(\lambda+\sigma/2)$, higher than $\alpha=\Phi(\lambda)$.

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# Ex. 1

| Portfolio A | |
|---|---|
| Loss x | f(x) |
| 0 | 0.6 |
| 1 | 0.375 |
| 5 | 0.025 |

| Portfolio B | |
|---|---|
| Loss x | f(x) |
| 0 | 0.6 |
| 1 | 0.39 |
| 11 | 0.01 |

| Portfolio | Mean | CTE(0.95) | WT(0.95) |
|---|---|---|---|
| A | 0.5 | 3.00 | 2.42 |
| B | 0.5 | 3.00 | 3.40 |

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# Ex. 2

| | Case A | Case B |
|---|---|---|
| **Scenario** | ***loss $*** | ***loss $*** |
| #1 | 1 | 0 |
| #2 | 2 | 0 |
| … | … | … |
| #9 | 9 | 0 |
| *#10* | *10* | *10* |
| | | |
| | | |
| **CTE(0.95)** | **10** | **10** |
| **WT(0.95)** | **9.12** | **6.42** |

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# Conclusion

- There are many *coherent* risk-measures other than Tail-VaR
- WT-measure is a good alternative to Tail-VaR
- It is better to look at the whole distribution of short-fall.

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# Feedback ?

Swang@scor.com

Shaunwang@attbi.com

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