FAIR VALUATION OF THE SURRENDER OPTION EMBEDDED IN A GUARANTEED LIFE INSURANCE PARTICIPATING POLICY

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Participating life insurance **policies** are characterized by the fact that **the insurer's profits are shared with the policyholders**. There are several ways in which this profit-sharing is realized. Usually the shared **profits are credited to the mathematical reserves** at the end of each year, and this implies the **"purchase" of additional insurance**. The **benefits** are then **"adjusted"** in consequence of the "adjustment" of the mathematical reserves.

Participating policies are usually coupled with a minimum interest rate guaranteed.

There is now a great attention towards the options embedded in a participating life insurance contract, in particular on the **"bonus option"**, implied by the participation mechanism, and on the **"surrender option"**, i.e., the policyholder's right to early terminate the contract and to receive the *surrender value*.

In a previous paper we have analysed a life insurance product introduced in Italy at the end of the seventies, the so-called ***rivalutabile***, for which **a special portfolio of investments**, covering at least the mathematical reserves of all the policies with profits issued by the insurance company, **is constituted and kept apart from the other assets**. **At the end of each year a % of the rate of return on this portfolio** ($\Longrightarrow$ *reference portfolio*) in the preceding year **is assigned to the policyholder, provided that it does not fall below the *technical* rate**. We have considered both the case in which the policy is paid by a **single premium** at issuance, and the case in which it is paid by a sequence of **periodical premiums**, and we have obtained a **very simple** closed-form **relation that characterizes "fair" contracts in the Black-Merton-Scholes** (1973) **framework**.

However, **this analysis does not take into account the presence of the surrender option**. In the present paper we partially fill this gap by pricing the **single premium contract**. In another paper, just finished last week, we complete the analysis by pricing also periodical premium contracts.

More in detail, in this paper

- **we define a rule for computing the surrender values**, which introduces an additional contractual parameter in the model.

  By modelling the assets à la **C**ox, **R**oss and **R**ubinstein (1979)

- **we obtain a recursive binomial algorithm for computing the value of the whole contract**.

  Since the value of the corresponding participating contract without the surrender option is expressed in closed-form

- **the value of the surrender opt. is obtained residually**.

The total price is split into the values of three components:

- the *basic contract* (i.e., without profits and without surrender),
- the *bonus option*,
- the *surrender option*.

REMARK: Although these embedded options are not traded separately from the other elements of the contract, we believe that such decomposition can be very useful to an insurance company since it allows it to **understand the incidence of the various components on the premium** and, if necessary, to **identify possible changes in the design of the policy**.

The rest of the presentation is organized as follows:

- we describe the **structure of the contract** and define all the liabilities that the insurer has to face;
- we introduce our **valuation framework**;
- we derive the **fair value of the contract and of all its components** describing, in particular, our recursive algorithm;
- we show some **graphs**.

THE STRUCTURE OF THE CONTRACT

single-premium endowment policy

0 time of issuance

T maturity

benefit paid at the end of the year

C_1 benefit paid at time 1, if the insured dies during the first year

C_t benefit paid at time t, $t = 2, ..., T$

$$\Downarrow$$

C_1 **is given**

C_t **depends on the performance of the ref. portfolio**

PARTICIPATION MECHANISM

i *technical rate* (annual compounded)

g_t rate of return on the reference portfolio during year t

η participation coefficient (between 0 and 1)

δ_t rate of adjustment of the mathematical reserve at time t

$$\delta_t = \max\left\{\frac{\eta g_t - i}{1+i}, 0\right\}, \quad t = 1, ..., T-1$$

Single Pr. Contract $\Longrightarrow C_{t+1} = C_t(1+\delta_t), \quad t = 1, ..., T-1$

$$\Longrightarrow \qquad C_t = C_1 \prod_{k=1}^{t-1}(1+\delta_k), \quad t = 2, ..., T$$

Rationale of the rule for computing the adjustment rates δ_t and interpretation of the technical rate i

Italian insurance companies compute the net single premium as the ***expected value, w.r.t. a suitable mortality distribution, of the initial benefit*** C_1 ***discounted from the random time of payment to time*** 0 ***with the technical rate*** i.

In this case a return at the technical rate i is assigned to the policy since the beginning. Taking into account the adjustment rule (and disregarding the surrender option)

$\Longrightarrow$ **the total return granted to the policy in year t is**

$$(1+i)(1+\delta_t) - 1 = \max\{i, \eta g_t\}$$

$\Longrightarrow$ i **can be interpreted as a min. int. rate guaranteed**.

SURRENDER CONDITIONS

We assume that

- **the surrender decision is taken at the beginning of the year**, just after the announcement of the benefit for the coming year,

- **the surrender value** at the beginning of year $t+1$ (i.e., at time t), denoted by R_t, **depends on the current benefit (C_{t+1}), on one contractual parameter (ρ), and possibly on some other variables** (such as the time to maturity of the policy, the mathematical reserve, etc.):

$$R_t = f(C_{t+1},\, \rho,\, ...), \quad t = 0, ..., T-1$$

THE VALUATION FRAMEWORK

The contract under scrutiny is a typical example of ***contingent-claim***, affected by both the **mortality** and the **financial** risk.

While **the mortality risk determines the moment in which the benefit is due, the financial risk affects the amount of the benefit and the surrender decision**.

We assume, in fact, that

- **financial** and **insurance markets** are **perfectly competitive**, **frictionless** (no taxes, no transaction costs such as, e.g., expenses and relative loadings of the insurance premiums, short-sale allowed), and **free of arbitrage opportunities**.
- all the **agents** are **rational** and **non-satiated**, and share the **same information**.

$\Longrightarrow$ In this framework, **the surrender decision can only be the consequence of a rational choice**, taken after comparison, at any time, between the total value of the policy (including the option of surrendering it in the future) and the surrender value.

OTHER ASSUMPTIONS:

- **stochastic independence** between the insured's lifetime and the financial variables

- mortality probabilities are extracted from a ***"risk-neutral"* mortality measure** (i.e., all insurance prices are computed as expected values with respect to this specific measure)

- mortality probabilities depend on the **age of the insured** (we denote by x this age at time 0).

THE FINANCIAL SET-UP

We assume that **the rate of return on risk-free assets is deterministic and constant**, and denote by r the (annual compounded) riskless rate.

$\Longrightarrow$ The financial risk which affects the policy is generated by a stochastic evolution of the rates of return on the reference portfolio.

In this connection, we assume that

- it is a **well-diversified portfolio**, split into *units*,
- **yields** are immediately **reinvested** and shared among units
- the **reinvested yields increase** only **the unit-price** of the portfolio but **not the total number of units** (that changes when new investments or withdrawals are made).

$\Longrightarrow$ **The rates of return on the reference portfolio are completely determined by the evolution of its unit price**

Denoting by G_τ this unit-price at time τ (≥ 0)

$$\Longrightarrow \quad g_t = \frac{G_t}{G_{t-1}} - 1, \quad t = 1, ..., T-1$$

For describing the stochastic evolution of G_τ, we choose the discrete model by **Cox, Ross and Rubinstein** (1979) (**CRR**), universally acknowledged for its important properties.

REMARK: CRR may be seen either as an **"exact" model** under which "exact" values for both European and American-style contingent-claims can be computed, or as an **approximation of the Black and Scholes** (1973) **and Merton** (1973) **model** to which it asymptotically converges.

More in detail, we

- divide each policy year into N subperiods of equal length,
- let $\Delta = 1/N$,
- fix a volatility parameter $\sigma > \sqrt{\Delta}\,\ln(1+r)$,
- set $u = \exp(\sigma\sqrt{\Delta})$ and $d = 1/u$.

Then we assume that

- G_τ **can be observed at the discrete times** $\tau = t + h\Delta$ ($t = 0, 1, ...; h = 0, 1, ..., N-1$),

- conditionally on all relevant information available at time τ, $G_{\tau+\Delta}$ **can take only two possible values**: uG_τ (**"up" value**) and dG_τ (**"down" value**).

$\Longrightarrow$ In this discrete setting, **absence of arbitrage is equivalent to the existence of a** risk-neutral **probability measure under which all financial prices, discounted by means of the risk-free rate, are martingales**.

Under this risk-neutral measure, the probability of $\{G_{\tau+\Delta} = uG_\tau\}$ conditioned on all information available at time τ (that is, in particular, on the knowledge of the value taken by G_τ), is given by

$$q = \frac{(1+r)^\Delta - d}{u - d}$$

while

$$1 - q = \frac{u - (1+r)^\Delta}{u - d}$$

represents the (conditioned) probability of $\{G_{\tau+\Delta} = dG_\tau\}$.

REMARK: In order to prevent arbitrage opportunities, we have fixed σ in such a way that $d < (1+r)^{\Delta} < u$, which implies a strictly positive value for both q and $1-q$.

The above assumptions imply that g_t, $t=1,2,...,T-1$

- are i.i.d.

- take one of the following $N+1$ possible values:

$$\gamma_j = u^{N-j} d^j - 1, \quad j = 0, 1, ..., N$$

with (risk-neutral) probability

$$Q_j = \binom{N}{j} q^{N-j} (1-q)^j, \quad j = 0, 1, ..., N$$

Moreover

- also the adjustment rates of the benefit, δ_t, t=1, 2, ..., T−1, are i.i.d.
- they take one of the following n+1 possible values:

$$\mu_j = \begin{cases} \frac{\eta\gamma_j - i}{1+i} & j = 0, 1, ..., n-1 \quad \text{(with prob. } Q_j) \\ 0 & j = n \quad \text{(with prob. } 1-\sum_{k=0}^{n-1} Q_k) \end{cases}$$

where

$$n = \left\lfloor \frac{N}{2} + 1 - \frac{ln(1 + i/\eta)}{2ln(u)} \right\rfloor$$

(with $\lfloor y \rfloor$ the integer part of a real number y) represents the **minimum number of "downs" such that a call option on the rate of return on the reference portfolio in a given year with exercise price i/η does not expire in the money**.

THE VALUE OF THE WHOLE CONTRACT: U^W

The stochastic evolution of the benefit $\{C_t, t{=}1,2,...,T\}$ **can be represented by means of an** $(n{+}1)$**-nomial tree**:

- in the root we represent the initial benefit C_1 (given);
- each node has $n{+}1$ branches that connect it to $n{+}1$ subsequent nodes;
- in the nodes "at time t" we represent the possible values of C_{t+1}.

$\Longrightarrow$ **The possible trajectories that the stochastic process of the benefit can follow from time** 0 **to time** t $(t = 1,2,...,T{-}1)$ **are** $(n{+}1)^t$, but not all these trajectories lead to different nodes. The tree is, in fact, *recombining*, **and the different nodes** (or *states of nature*) **at time** t **are** $\binom{n+t}{n}$.

In the same tree we represent

- the **surrender values** R_t, $t{=}0, 1, ..., T{-}1$
- the **values of the whole contract**
- the **"*continuation*" values**.

REMARK: The last two values will be computed by means of a *backward recursive procedure* operating from time $T{-}1$ to time 0.

We denote by

$\{W_t,\ t{=}0, 1, ..., T{-}1\}$ the stochastic process with components the values of the whole contract at the beginning of year $t{+}1$ (time t),

$\{V_t,\ t{=}0, 1, ..., T{-}1\}$ the stochastic process with components the continuation values at the beginning of year $t{+}1$ (time t).

$$\Longrightarrow \quad U^W = W_0$$

- **At time** $T-1$

In each node (if the insured is alive) **the continuation value is** given by

$$V_{T-1} = (1+r)^{-1} C_T$$

since the benefit C_T is due with certainty at time T.

$\Longrightarrow$ **The value of the whole contract is**

$$W_{T-1} = max\{V_{T-1},\, R_{T-1}\}$$

since the (rational and non-satiated) policyholder chooses between continuation and surrender in order to maximize his(her) profit.

- **At time** $t < T-1$

Assume to be in a given node K (with the insured still alive).

Now, in order to catch the link between values at time t and values at time $t+1$, we denote by

$$C_{t+1}^K, \quad R_t^K, \quad W_t^K, \quad V_t^K$$

the benefit, the surrender value, the value of the whole contract, the continuation value, at time t in the node K, and by

$$W_{t+1}^{K(j)}, \quad V_{t+1}^{K(j)} \quad (j=0, 1, ..., n)$$

the value of the whole contract and the continuation value at time $t+1$ in each node following K.

REMARK: Continuation $\Rightarrow$ to **receive**, at time $t+1$, **the benefit** C_{t+1}^K if the insured dies within 1 year, or to **be entitled to a contract with** total **random value** W_{t+1}, if the ins. survives.

$\Longrightarrow$ **The continuation value at time** t (in the node K) **is** given by the risk-neutral expectation of these payoffs, discounted for 1 year with the risk-free rate:

$$V_t^K = (1+r)^{-1}\left\{ q_{x+t}C_{t+1}^K + p_{x+t}\left[\sum_{j=0}^{n-1} W_{t+1}^{K(j)} Q_j + \right.\right.$$
$$\left.\left. + \; W_{t+1}^{K(n)}\left(1-\sum_{j=0}^{n-1} Q_j\right)\right]\right\}, \quad t=0,1,...,T-2$$

$\Longrightarrow$ **The value of the whole contract is**

$$W_t^K = max\{V_t^K, R_t^K\}, \quad t=0,1,...,T-2$$

THE VALUE OF THE NON-SURRENDABLE PARTICIPATING CONTRACT: U^P

We define "non-surrendable participating contract" an *endowment policy* with *stochastic* benefit C_t and **without the surr. option**.

To compute its value we need, first of all, to compute the **market price at time** 0 **of the benefit** C_t**, supposed to be due with certainty at time** t ($t = 1, 2, ..., T$). We denote this price by $\pi(C_t)$. While

$$\pi(C_1) = C_1(1+r)^{-1}$$

for $t > 1$

$$\Longrightarrow \quad \pi(C_t) = E^Q[(1+r)^{-t}C_t] = E^Q\left[(1+r)^{-t}C_1\prod_{k=1}^{t-1}(1+\delta_k)\right]$$

where E^Q denotes expectation wrt the financial risk-neutral measure.

Stochastic Independence of $\delta_k,\ k = 1, 2, ..., T-1$

$$\Longrightarrow \quad \pi(C_t) = C_1(1+r)^{-t} \prod_{k=1}^{t-1} E^Q[1+\delta_k]$$

Identical Distribution of $\delta_k,\ k = 1, 2, ..., T-1$

$$\Longrightarrow \quad \pi(C_t) = C_1(1+r)^{-t} \left(1 + \sum_{j=0}^{n-1} \mu_j Q_j\right)^{t-1}, \quad t = 2, 3, ..., T$$

Letting $\mu = E^Q[\delta_k] = \sum_{j=0}^{n-1} \mu_j Q_j$ and $\lambda = \frac{r-\mu}{1+\mu}$

$$\Longrightarrow \quad \pi(C_t) = \frac{C_1}{1+\mu}(1+\lambda)^{-t}, \quad t = 2, 3, ..., T$$

$\Longrightarrow$ The fair value of the non-surr. part. contract is given by

$$
\begin{aligned}
U^P &= \sum_{t=1}^{T-1} \pi(C_t)\, {}_{t-1/1}q_x + \pi(C_T)\, {}_{T-1}p_x \\
&= \frac{C_1}{1+\mu} \left[\sum_{t=1}^{T-1} (1+\lambda)^{-t}\, {}_{t-1/1}q_x + (1+\lambda)^{-T}\, {}_{T-1}p_x \right] \\
&= \frac{C_1}{1+\mu} A^{(\lambda)}_{x:\overline{T}|}
\end{aligned}
$$

THE VALUE OF THE SURRENDER OPTION: S

$$
\Longrightarrow \quad S = U^W - U^P = W_0 - \frac{C_1}{1+\mu} A^{(\lambda)}_{x:\overline{T}|}
$$

THE VALUE OF THE BASIC CONTRACT: U^B

We define "basic contract" a *standard endowment policy* with constant benefit C_1 (**without profits** and **without the surrender option**).

$$\Downarrow$$

$$U^B = C_1 A^{(r)}_{x:\overline{T}|} = C_1 \left[\sum_{t=1}^{T-1} (1+r)^{-t} \; {}_{t-1/1}q_x \; + \; (1+r)^{-T} \; {}_{T-1}p_x \right]$$

THE VALUE OF THE BONUS OPTION: B

$$\Longrightarrow \quad B = U^P - U^B = C_1 \left[\frac{A^{(\lambda)}_{x:\overline{T}|}}{1+\mu} - \; A^{(r)}_{x:\overline{T}|} \right]$$

SOME NUMERICAL RESULTS

We present now some **graphs** showing the **behaviour of the value of the whole contract and its components** w.r.t. some parameters. To produce them we have

- extracted the mortality probabilities from S.I.F. 1991,
- fixed C_1=1, T=5, N=250,
- considered different values for the remaining parameters.

REMARK: Our choice for N implies a **daily change in the unit price of the reference portfolio** since there are about 250 trading days in a year, and guarantees a **very good approximation to the Black-Merton-Scholes** (1973) **model**. However, this high number of steps in each year requires a **large amount of CPU time**; that is why we have not fixed a high value for T.

We have assumed that the **surrender value** is given by the **current benefit discounted from maturity to the surrender date** with the (annual compounded) rate ρ:

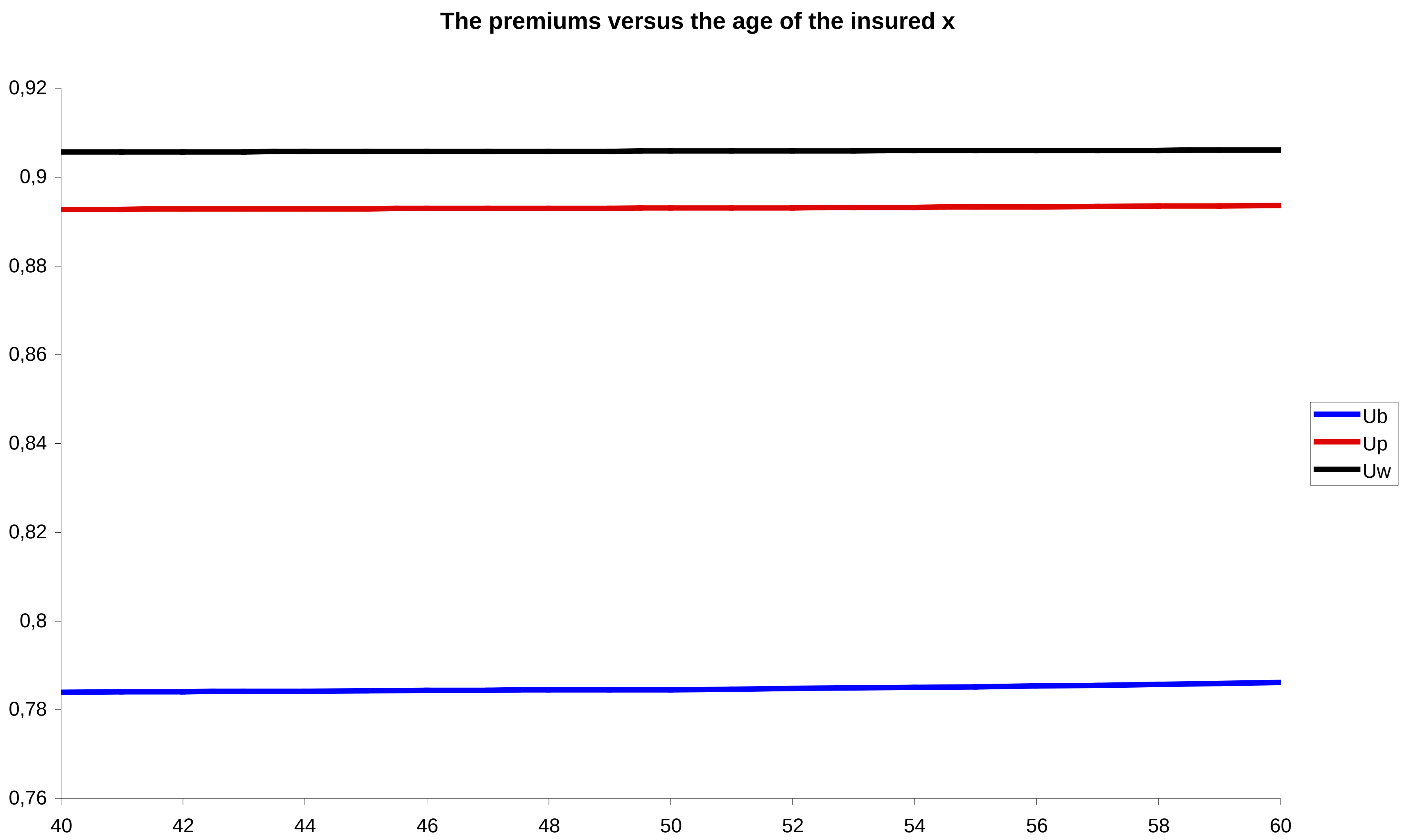
$$R_t = C_{t+1}(1+\rho)^{-(T-t)}, \quad t = 0, 1, ..., T-1.$$

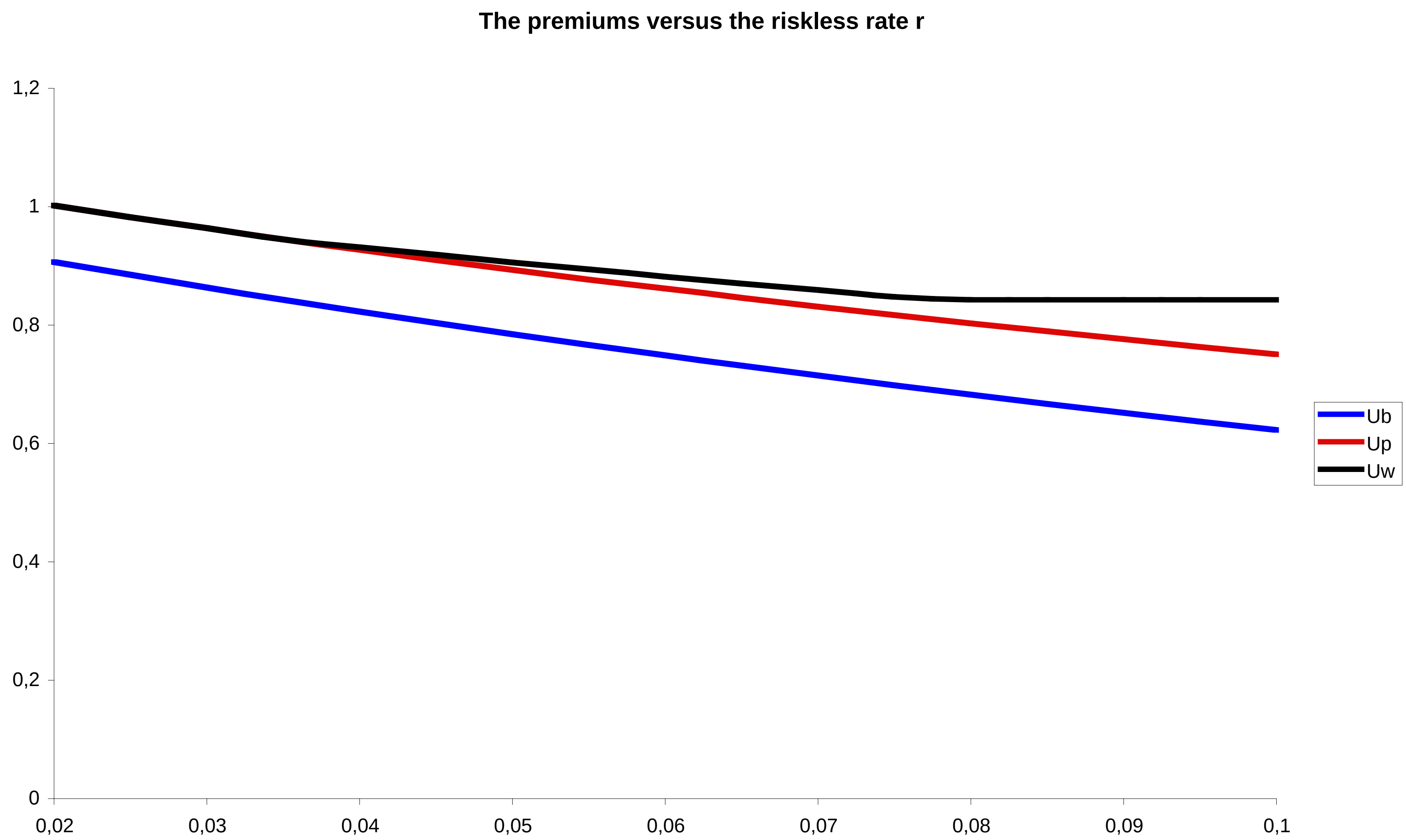
We have fixed the following **basic set of values for the parameters** x, r, i, η, σ, ρ, and then we have **moved each parameter one at a time**:

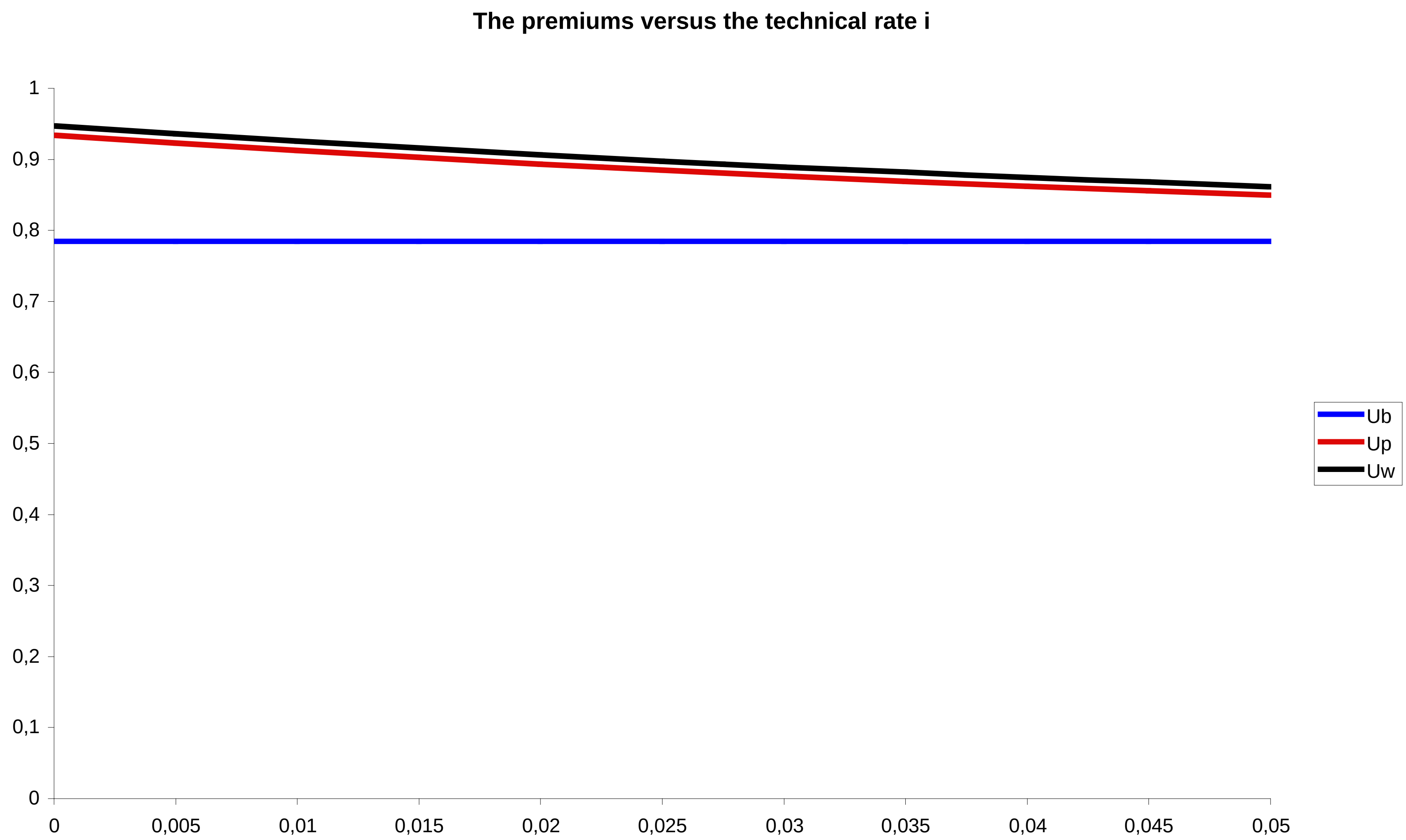
$$x = 50,\ r = 0.05,\ i = 0.02,\ \eta = 0.5,\ \sigma = 0.15,\ \rho = 0.035.$$

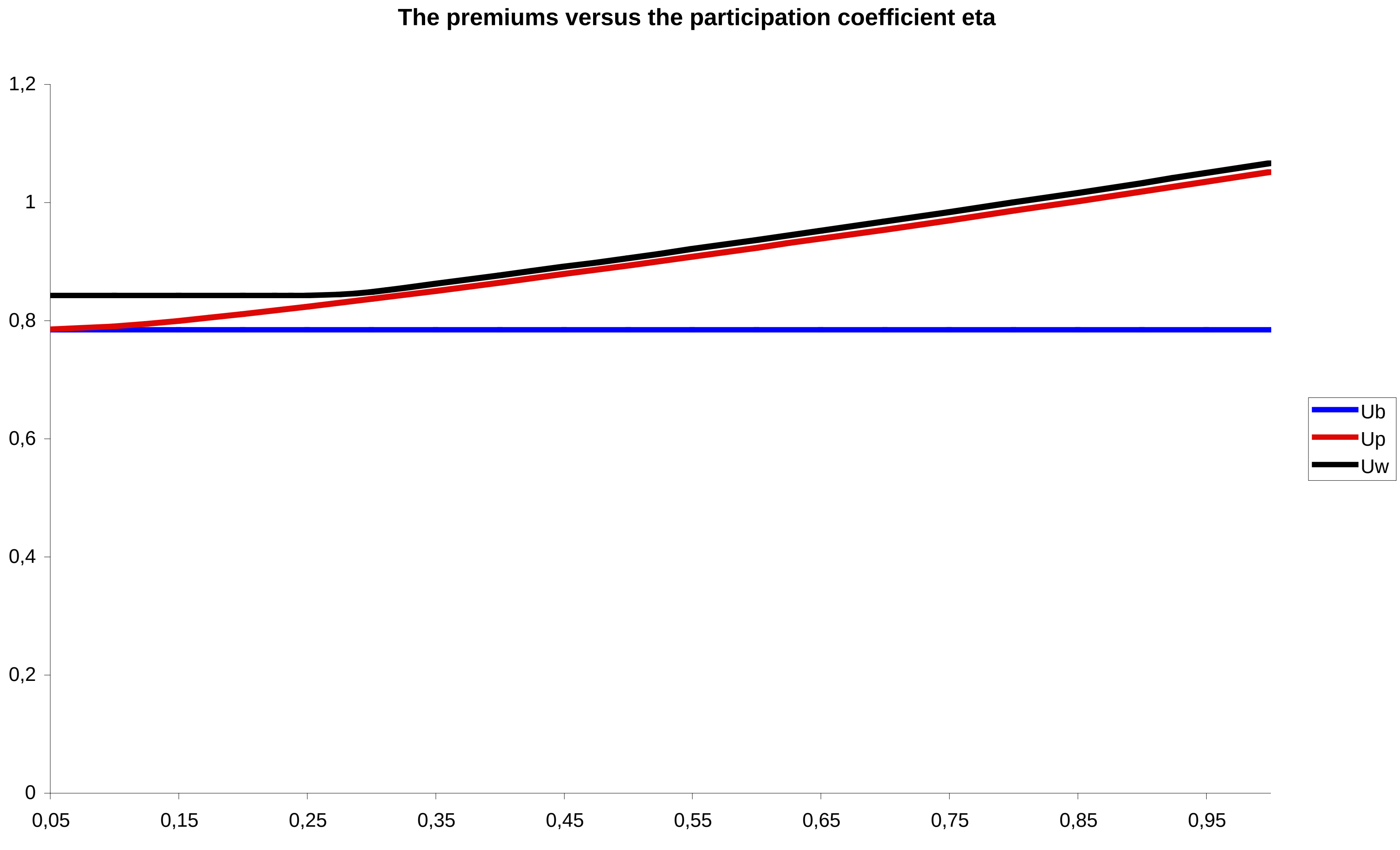
With these parameters we have **obtained the following results**:

$$U^B = 0.7845,\ B = 0.1084,\ U^P = 0.8930,\ S = 0.0128,\ U^W = 0.9058.$$


The premiums versus the age of the insured x
0,92
0,9
0,88
0,86
0,84
0,82
0,8
0,78
0,76
40
42
44
46
48
50
52
54
56
58
60
Ub
Up
Uw

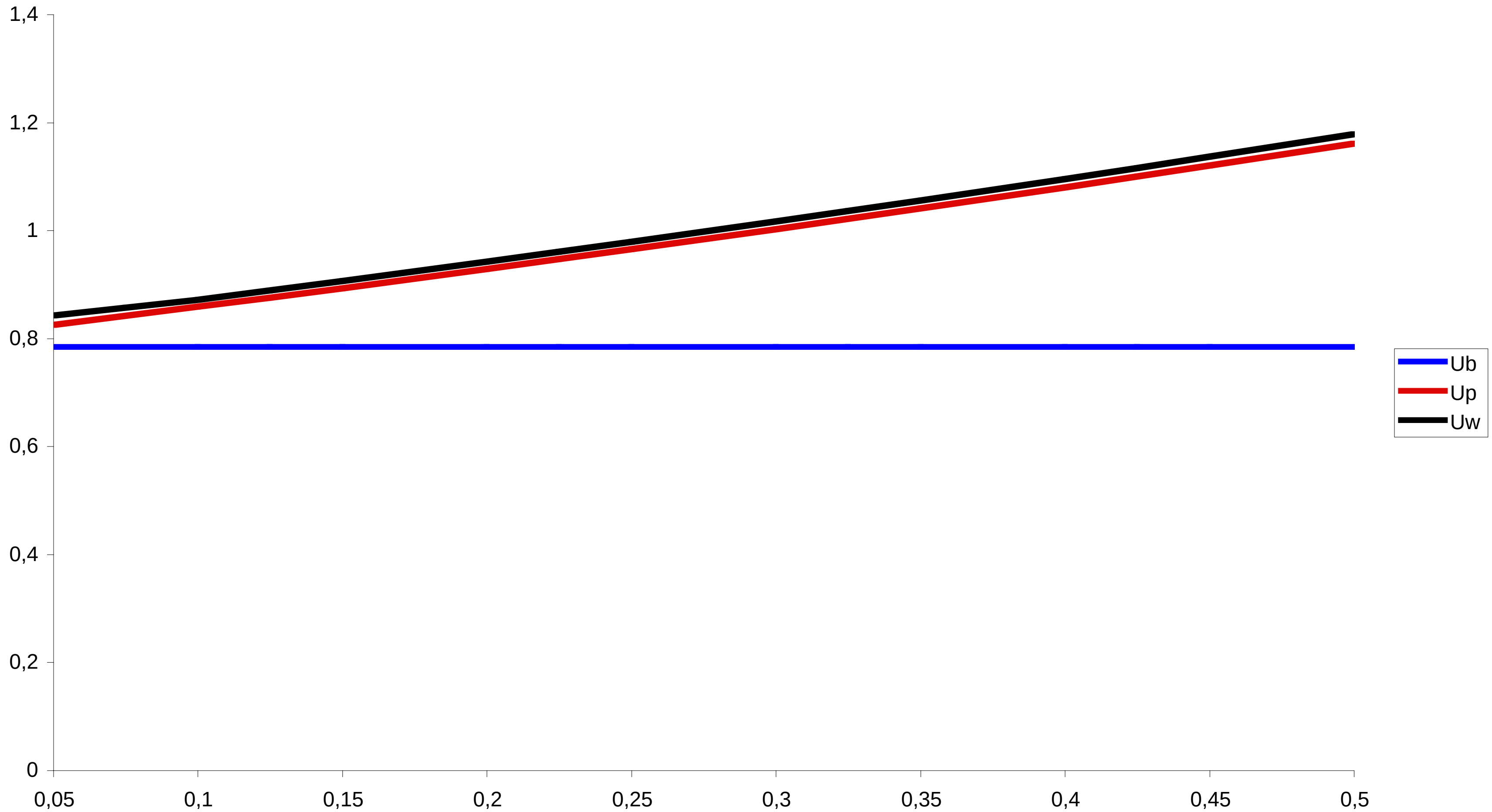

The premiums versus the riskless rate r
1,2
1
0,8
0,6
0,4
0,2
0
0,02
0,03
0,04
0,05
0,06
0,07
0,08
0,09
0,1
Ub
Up
Uw

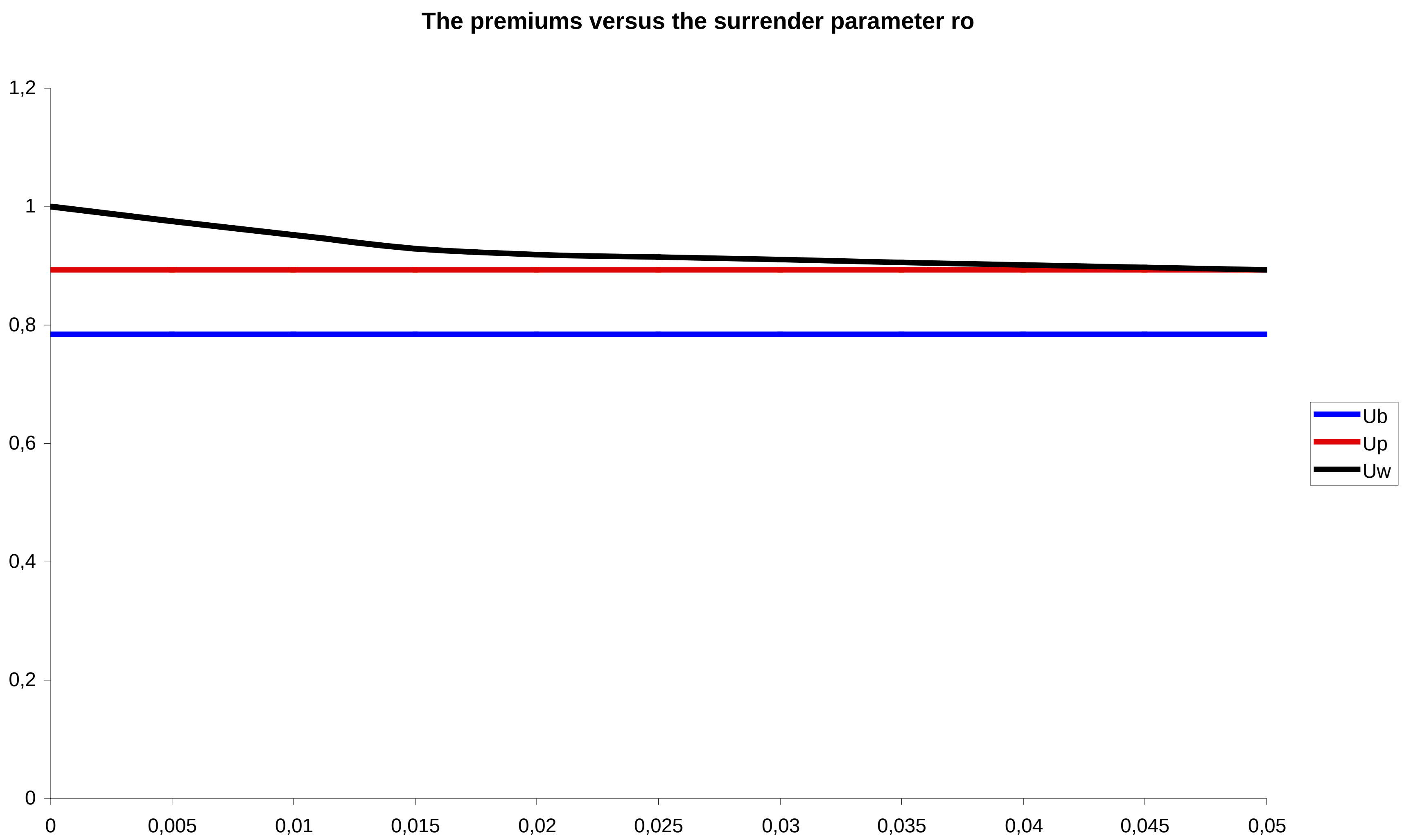

The premiums versus the technical rate i
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
0
0,005
0,01
0,015
0,02
0,025
0,03
0,035
0,04
0,045
0,05
Ub
Up
Uw



The premiums versus the participation coefficient eta
1,2
1
0,8
0,6
0,4
0,2
0
0,05
0,15
0,25
0,35
0,45
0,55
0,65
0,75
0,85
0,95
Ub
Up
Uw

The premiums versus the volatility parameter sigma




The premiums versus the surrender parameter ro
1,2
1
0,8
0,6
0,4
0,2
0
0
0,005
0,01
0,015
0,02
0,025
0,03
0,035
0,04
0,045
0,05
Ub
Up
Uw