

AN ALTERNATIVE APPROACH TO PORTFOLIO SELECTION

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Abstract.

The absolute deviation of the expected return on a portfolio from its required economic risk capital according to the expected shortfall method identifies with an expected shortfall deviation from the mean return, called portfolio shortfall risk. The natural risk contribution of each portfolio asset to the portfolio shortfall risk is called shortfall risk of the asset. Replacing the variance as a measure of risk in the classical portfolio selection model by the shortfall risk defines mean-shortfall portfolio selection. For some legitimated cases of mean-variance portfolio selection, namely the multivariate elliptical return distributions, both approaches lead to the same conclusions. An important situation, for which the alternative approach appears tractable under more general return distributions, is discussed.

Keywords : economic risk capital, shortfall risk, mean-variance analysis, elliptical distributions, Spearman's correlation coefficient, copula, covariance identity

1. Introduction.

Mean-variance portfolio selection, pioneered by Markowitz, is one of the cornerstones of modern portfolio theory. Divers shortcomings of this approach are known. For example, if one build optioned portfolios using option strategies, the resulting portfolio return distribution may be rather asymmetric and difficult to calculate explicitly (e.g. Bookstaber and Clarke(1983)). As a consequence, Scheuenstuhl and Zagst(1996) do not recommend mean-variance analysis in such a situation. Despite the many recent approaches to the optioned portfolio selection problem, no satisfactory solution has been proposed, which has a universal potential for finance practice (like mean-variance portfolio selection).

In the present paper, an alternative general approach to portfolio selection is considered. It replaces the variance risk measure by a shortfall risk measure, which can be interpreted as absolute deviation of the expected return on a portfolio from its required economic risk capital according to the expected shortfall method (formula (2.4)). In Section 3, it is shown that mean-shortfall and mean-variance portfolio selection are equivalent methods provided the distributions of return belong to the family of elliptical distributions. In Remark 3.1, this result is reinterpreted as an elliptical risk capital model. In Section 4, the alternative approach is applied to the portfolio selection problem for index match funds products, where a slightly modified market shortfall risk measure is used. Under arbitrary location-scale marginal distributions of return, but with a very specific modelling of the dependence structure between the individual asset returns and the market index return, we show that mean-shortfall portfolio selection approximately reduces to mean-variance portfolio selection. However, using more general marginal distributions of return, no such reduction can be expected, as the special case of log-normal returns shows. In a future work, mean-shortfall portfolio selection will be applied to the important optioned portfolio selection problem.

2. ERC, VaR, CVaR and shortfall risk.

Consider a firm confronted with a risky business over some time period, and let the random variable X represent the potential loss or risk the firm incurs at the end of the period. To be able to cover any loss with a high probability, the firm borrows at the beginning of the time period on the capital market the amount ERC_0 , called *economic risk capital*. At the end of the period, the firm has to pay interest on this at the interest rate i_R . To guarantee with certainty the value of the borrowed capital at the end of the period, the firm invests ERC_0 at the risk-free interest rate $i_f < i_R$. The value of the economic risk capital at the end of the period is thus $ERC = ERC_0 \cdot (1 + i_f - i_R)$. The risky business will be successful at the end of the period provided the event $\{X > ERC\}$ occurs only with a small tolerance probability.

There exist several risk management principles applied to evaluate ERC . Two simple methods that have been considered so far are the value-at-risk and the expected shortfall approach (e.g. Arztnr et al.(1997/99), Arztnr(1999), Wirth(1999), Wirth and Hardy(1999), Testuri and Uryasev(2000), Acerbi(2001), Acerbi and Tasche(2001a/b)). According to the value-at-risk method one identifies the economic risk capital with the *value-at-risk* of the loss setting

$$ERC = VaR_a[X] := Q_X(a), \quad (2.1)$$

where $Q_X(u) = \inf \{x | F_X(x) \geq u\}$ is a quantile function of X , with $F_X(x) = \Pr(X \leq x)$ the distribution of X . This quantile represents the maximum possible loss, which is not exceeded with the (high) probability a (called confidence level). According to the expected shortfall method one identifies the economic risk capital with the *conditional value-at-risk* of the loss setting

$$ERC = CVaR_a[X] := E[X | X > VaR_a[X]]. \quad (2.2)$$

This value represents the conditional expected loss given the loss exceeds its value-at-risk. Clearly one has

$$CVaR_a[X] = Q_X(a) + m_X[Q_X(a)] = Q_X(a) + \frac{1}{e} p_X[Q_X(a)], \quad (2.3)$$

where $m_X(x) = E[X - x | X > x]$ is the mean excess function, $p_X(x) = (1 - F_X(x)) \cdot m_X(x)$ is the stop-loss transform, and $e = 1 - a$ is interpreted as loss probability. In Arztnr(1999) the expression (2.3) is called *tail conditional expectation* and abbreviated TailVaR there (for *tail value-at-risk*). Sometimes (2.3) is also named *expected shortfall*, or *mean shortfall*, and *mean excess loss*. Mathematically, VaR and CVaR, which have been defined as functions of random variables, may be viewed as functionals defined on the space of probability distributions associated with these random variables.

It is important to observe that both ERC functionals satisfy two important risk-preference criteria in the economics of insurance (see Denuit et al.(1999) for a recent review). They are consistent with the risk preferences of profit-seeking decision makers respectively profit-seeking risk averse decision makers. To see this, recall two partial orders of riskiness.

Definitions 2.1. A risk X is less dangerous than a risk Y in the *stochastic order*, written $X \leq_{st} Y$, if $Q_X(u) \leq Q_Y(u)$ for all $u \in [0,1]$. A risk X is less dangerous than a risk Y in the *stop-loss order*, written $X \leq_{sl} Y$, if $p_X(x) \leq p_Y(x)$ for all x .

To compare economic risk capitals using criteria, which do not depend on the choice of the loss tolerance level, let us use two further partial orders of riskiness.

Definitions 2.2. A loss X is less dangerous than a loss Y in the *VaR order*, written $X \leq_{VaR} Y$, if the value-at-risk quantities satisfy $VaR_a[X] \leq VaR_a[Y]$, for all $a \in [0,1]$. A loss X is less dangerous than a loss Y in the *CVaR order*, written $X \leq_{CVaR} Y$, if the conditional value-at-risk quantities satisfy $CVaR_a[X] \leq CVaR_a[Y]$, for all $a \in [0,1]$.

The value-at-risk and expected shortfall methods are consistent with ordering of risks in the sense that profit-seeking (risk averse) decision makers require higher VaR (CVaR) by increasing risk, where risk is compared using the stochastic order $\leq_{st}$ (stop-loss order $\leq_{sl}$). Reciprocally, increasing VaR (CVaR) is always coupled with higher risk. These ordering properties are contained in the following result.

Theorem 2.1. If X and Y are two loss random variables, then $X \leq_{VaR} Y \Leftrightarrow X \leq_{st} Y$ and $X \leq_{CVaR} Y \Leftrightarrow X \leq_{sl} Y$.

Proof. This has been shown in Hürlimann(2001a), Theorem 1.1. $\diamond$

Finally, it is important to observe that, except for a world of elliptical linear portfolio losses (Embrechts et al.(1998), Fundamental Theorem of Risk Management), the VaR functional has several shortcomings. It is not subadditive and not scalar multiplicative, and it cannot discriminate between risk-averse and risk-taking portfolios (examples 1 to 3 in Wirth(1999)). If subadditivity holds, merging two risks does not create extra risk. If a firm must meet a requirement of extra economic risk capital that did not satisfy this property, the firm might separate in two subunits requiring less capital, a matter of concern for the supervising authority. In situations where no diversification occurs capital requirement depends on the size of the risk as expressed by the scalar multiplicative property. In contrast to this, the CVaR functional, which is subadditive and scalar multiplicative, is a coherent risk measure in the sense of Artzner et al.(1997) and appears thus more suitable in general applications. A recent work devoted to the evaluation of economic risk capital in life-insurance using the VaR and CVaR approaches is Ballmann and Hürlimann(2001).

Relevant in risk management is often not CVaR itself, but its deviation from the expected loss, that is the quantity

$$SFR_a[X] = CVaR_a[X] - E[X]. \quad (2.4)$$

This convenient and natural relative CVaR measure, called *shortfall risk* in the following, plays the role of the variance in a general portfolio selection model, which goes beyond the classical mean-variance portfolio theory by Markowitz(1952/59/87/94).

3. Equivalence of mean-shortfall and mean-variance analysis.

Given a collection of n assets with vector of random returns $\vec{R} = (R_1, \dots, R_n)$, the main goal of portfolio selection is the determination of an optimal portfolio with respect to some meaningful criterion. If $\vec{w} = (w_1, \dots, w_n)$ represent the fractions of the portfolio held in each asset, then $R = \vec{w}^T \cdot \vec{R}$ describes the portfolio return, which should be optimised in some way.

There exist many different approaches, which have been proposed for portfolio selection. In the present Section, the classical mean-variance approach is compared with the alternative mean-shortfall approach.

Mean-variance approach

Let $\vec{m} = (m_1, \dots, m_n)$ be the vector of expected returns, and let $C = (s_{ij})$ be the covariance matrix between the returns. The portfolio variance is described by the quantity $s_R^2 = \vec{w}^T \cdot C \cdot \vec{w}$. Further, let $\vec{e} = (1, \dots, 1)$ be the unit vector. In its simplest form (short sales allowed but no riskless lending and borrowing) the portfolio selection problem consists to minimise the portfolio variance by given expected return :

$$\min \left\{ \frac{1}{2} \vec{w}^T \cdot C \cdot \vec{w} \right\} \text{ under the constraints } \vec{w}^T \cdot \vec{m} = m_R, \quad \vec{w}^T \cdot \vec{e} = 1. \quad (3.1)$$

Mean-shortfall approach

The portfolio variance as a measure of risk represents the expected square deviation from the mean return. If only adverse returns are relevant, an alternative measure of risk is the expected shortfall deviation from the mean return, which has found in Section 2 an economic risk capital interpretation and justification. Denote the *shortfall risk* at the confidence level α of the portfolio return by

$$r_\alpha[R] = E[R] - E[R | R \leq VaR_\alpha[R]]. \quad (3.2)$$

Requiring the additive property, it is natural to define the risk contribution of an asset to the portfolio shortfall risk by

$$r_\alpha[R_i | R] = E[R_i] - E[R_i | R \leq VaR_\alpha[R]], \quad i = 1, \dots, n. \quad (3.3)$$

These quantities are called *asset shortfall risks* and summarised into the shortfall risk vector $\vec{r}_\alpha[\vec{R}] = (r_\alpha[R_1 | R], \dots, r_\alpha[R_n | R])$. Then the simplest mean-shortfall portfolio selection problem consists to minimise the portfolio shortfall risk by given expected return :

$$\min \left\{ \vec{w}^T \cdot \vec{r}_\alpha[\vec{R}] \right\} \text{ under the constraints } \vec{w}^T \cdot \vec{m} = m_R, \quad \vec{w}^T \cdot \vec{e} = 1. \quad (3.4)$$

It is remarkable that in several important situations the mean-shortfall approach is equivalent to the mean-variance Markowitz approach.

Example 3.1.

Suppose $\overset{P}{R} = (R_1, \dots, R_n)$ has a multivariate normal distribution with mean $\overset{P}{m}$ and positive definite covariance matrix C . From Theorem 3.1 below one knows that

$$r_a[R_i|R] = \frac{\text{Cov}[R_i, R]}{s_R^2} \cdot r_a[R], \quad i = 1, \dots, n. \quad (3.5)$$

On the other hand, for a normal distribution with mean m_R and variance s_R^2 , one has

$$r_a[R] = \frac{f(z_a)}{1-a} \cdot s_R, \quad \Phi(z_a) = a, \quad (3.6)$$

where $f(x) = \Phi'(x)$ and $\Phi(x)$ is the standard normal distribution. It follows that

$$\overset{P}{W}^T \cdot \overset{P}{F}_a[\overset{P}{R}] = r_a[R] = \frac{f(z_a)}{1-a} \cdot s_R. \quad (3.7)$$

Therefore, for any fixed confidence level, mean-shortfall portfolio selection is equivalent to mean-variance portfolio selection.

More generally, it is known that the mean-variance approach is a legitimated theory under the expected utility model (maximisation of the expected utility of final wealth) if the distributions of return belong to the family of elliptical distributions (Chamberlain(1983)). As shown by Ross(1978), an even broader class of distributions implies the mean-variance capital asset pricing model. In the elliptical situation, mean-shortfall portfolio selection is also equivalent to mean-variance portfolio selection. Indeed, by Theorem 3.1 one has as in (3.7) that $\overset{P}{W}^T \cdot \overset{P}{F}_a[\overset{P}{R}] = r_a[R]$, and the result follows by the proof of Theorem 1 in Embrechts et al.(1998) because $r_a[R]$ is a positive homogenous and translation invariant measure. This main result for multivariate elliptical distributions generalizes the corresponding result for a multivariate normal distribution, which has been shown independently in a less elegant way by Rockafellar and Uryasev(2000), Proposition 4.1.

Theorem 3.1. Suppose $\overset{P}{R} = (R_1, \dots, R_n)$ has a multivariate elliptical density function with mean $\overset{P}{m}$ and positive definite covariance matrix C

$$f(\overset{P}{x}) = \frac{1}{\sqrt{\det(C)}} g[(\overset{P}{x} - \overset{P}{m})^T \cdot C^{-1} \cdot (\overset{P}{x} - \overset{P}{m})], \quad (3.8)$$

where $g : [0, \infty) \rightarrow [0, \infty)$ is some appropriate function. Then one has

$$r_a[R_i|R] = \frac{\text{Cov}[R_i, R]}{s_R^2} \cdot r_a[R], \quad i = 1, \dots, n. \quad (3.9)$$

Proof. The necessary background on elliptical distributions is found in Fang, Kotz and Ng(1987). The properties of elliptical distributions imply that the conditional distribution of R_i given R is again elliptical with conditional mean

$$E[R_i|R] = m_i - \frac{\text{Cov}[R_i, R]}{s_R^2} \cdot (R - m_R), \quad i = 1, \dots, n, \quad (3.10)$$

which implies immediately (3.9). $\diamond$

Remark 3.1.

In the context of Section 2, Theorem 3.1 yields a simple covariance principle for allocating risk capital in an elliptical economy. Let $G = G_1 + \dots + G_n$ be the gain of a risky business with sub-unit gains $G_i, i = 1, \dots, n$, and let $X = -G, X_i = -G_i, i = 1, \dots, n$, be the corresponding losses. Since $CVaR_a[X] + E[G] = r_a[G]$ it is natural to allocate risk capital according to the additive rule $CVaR_a[X_i] + E[G_i] = r_a[G_i|G]$, which yields the *elliptical risk capital model*

$$CVaR_a[X_i] = \frac{\text{Cov}[G_i, G]}{\text{Var}[G]} \cdot [CVaR_a[X] + E[G]] - E[G_i], \quad i = 1, \dots, n. \quad (3.11)$$

Rewritten in terms of losses using (2.3) one obtains the explicit formula

$$CVaR_a[X_i] = E[X_i] + \frac{\text{Cov}[X_i, X]}{\text{Var}[X]} \cdot [Q_X(1-e) + m_X[Q_X(1-e)] - E[X]], \quad i = 1, \dots, n. \quad (3.12)$$

Example 3.2.

It appears instructive to illustrate our results with a non-trivial but tractable multivariate elliptical distribution, which finds wide interest in both Insurance and Finance. The mixture of a normal with inverted gamma variance yields the Pearson type VII distribution or generalised Student t (e.g. Hogg and Klugman(1984), pp.52-53, Heilmann(1989), example 3.7, Kotz et al.(1995), Section 28.6). It has been proposed to model financial returns by Praetz(1972) (see also Blattberg and Gonedes(1974), Kon(1984), Taylor(1992), Section 2.8, Hürlimann(2001b)). Another recent actuarial application is found in Hürlimann(1995). The multivariate density of a random vector $\overset{P}{R} = (R_1, \dots, R_n)$ defined by

$$f(\overset{P}{x}) = \frac{[1 + (\overset{P}{x} - \overset{P}{m})^T \cdot C^{-1} \cdot (\overset{P}{x} - \overset{P}{m})]^{-(b + \frac{a}{2})}}{B(b, \frac{a}{2}) \cdot \sqrt{\det(C)}}, \quad (3.13)$$

$$\overset{P}{m} = (m_i), C = (c_{ij}), B(b, \frac{a}{2}) = \frac{\sqrt{p} \cdot \Gamma(b)}{\Gamma(b + \frac{1}{2})}, b > 0,$$

has location-scaled transformed Pearson VII marginal densities

$$f_i(x) = B(b, \frac{1}{2})^{-1} \frac{1}{c_{ii}} \cdot \left[1 + \left(\frac{x - m_i}{c_{ii}} \right)^2 \right]^{-(b + \frac{1}{2})}, \quad i = 1, \dots, n. \quad (3.14)$$

If $b > 1$ the variance $s_i^2 = \text{Var}[R_i]$ exists, and one has $c_{ii} = \sqrt{2(b-1)} \cdot s_i$. If $b = \frac{u}{2}, u = 1, 2, 3, \dots$, one recovers a location-scale transformed Student t with u degrees of freedom. In particular $b = \frac{1}{2}$ is a Cauchy and $b = 1$ is a Bowers distribution (for the latter see Hürlimann(1993/95/97/98) among others). If $b \rightarrow \infty$ the random variable $\sqrt{2(b-1)} \cdot (R_i - m_i) \cdot c_{ii}^{-1}$ converges to a standard normal random variable. On the other hand, any linear combination $R = \overset{\text{P}}{W}^T \cdot \overset{\text{P}}{R}$ has density

$$f_R(x) = B(b, \frac{1}{2})^{-1} \frac{1}{c} \cdot \left[1 + \left(\frac{x - m}{c} \right)^2 \right]^{-(b + \frac{1}{2})}, \quad (3.15)$$

with $m = \overset{\text{P}}{W}^T \cdot \overset{\text{P}}{m}$ and $c = \overset{\text{P}}{W}^T \cdot C \cdot \overset{\text{P}}{W}$. By the proof of Theorem 3.1 one has

$$r_a[R_i|R] = \left(\sum_{j=1}^n w_j \cdot \frac{c_{ij}}{c} \right) \cdot r_a[R], \quad i = 1, \dots, n, \quad (3.16)$$

where $r_a[R]$ remains to be calculated. It is convenient to define $I(x) = (E[R] - E[R|R \leq x]) \cdot F_R(x)$. A partial integration and rearrangement yields

$$I(x) = (x - m) \cdot \bar{F}_R(x) + p_R(x), \quad (3.17)$$

where $\bar{F}_R(x) = 1 - F_R(x)$ is the survival function and $p_R(x) = E[(R - x)_+]$ the stop-loss transform. It is not difficult to show that (e.g. Hürlimann(2001b))

$$p_R(x) = \frac{c^2 + (x - m)^2}{2a - 1} \cdot f_R(x) - (x - m) \cdot \bar{F}_R(x), \quad (3.18)$$

from which one gets

$$r_a[R] = \frac{c^2 + (x - m)^2}{2b - 1} \cdot \frac{f_R(u_a)}{1 - a}, \quad \text{with } F_R(u_a) = 1 - a. \quad (3.19)$$

Since $c^2 = 2(b-1) \cdot s_R^2$ for $b > 1$, the formula (3.19) converges to (3.6) for $b \rightarrow \infty$ as should be. The special case $b = 1$ has interesting applications. This is due to the fact that Bowers' distribution, which is extremal with respect to the stop-loss order, yields the maximum stop-loss transform by given mean and variance (consult the references).

4. Mean-shortfall portfolio selection for index match funds.

An index match fund product, as introduced on the financial market by the Credit Suisse Group during 1999, should achieve a return close to the performance of a financial market index. This situation can be modelled as follows. Consider the random variables

- R^M : the return of the market index with marginal distribution $F^M(x)$
 R_i : the return of asset number i in the index family with marginal distribution $F_i(x), i = 1, \dots, n$.

Then an optimal vector $\overset{\mu}{w} = (w_1, \dots, w_n)$ of weights should be chosen such that the portfolio return $R = \overset{\mu}{w}^T \cdot \overset{\mu}{R}$ approximately matches the market return, that is $R \approx R^M$. Instead of $r_a[R_i|R]$ as shortfall risk of asset i , we propose to use the slightly modified *market shortfall risk* vector $\overset{\mu}{r}_a[\overset{\mu}{R}|R^M] = (r_a[R_1|R^M], \dots, r_a[R_n|R^M])$ with

$$r_a[R_i|R^M] = E[R_i] - E[R_i|R^M \leq VaR_a[R^M]] \quad i = 1, \dots, n, \quad (4.1)$$

and to minimise the overall portfolio measure

$$r_a[R; R^M] := \overset{\mu}{w}^T \cdot \overset{\mu}{r}_a[\overset{\mu}{R}|R^M]. \quad (4.2)$$

It is interesting to note that in this context mean-shortfall portfolio selection reduces approximately to mean-variance portfolio selection for a general but very specific modelling of the dependence structure between the individual asset returns and the market index return (comments after Theorem 4.3).

Some preliminary results on bivariate distributions are required. Consider the one-parameter copula function defined for a parameter $q \in [0, 1]$ by

$$C_q(u, v) = \begin{cases} [u + q(1-u)] \cdot v, & v \leq u, \\ [v + q(1-v)] \cdot u, & v > u, \end{cases} \quad (4.3)$$

and for $q \in [-1, 0]$ by

$$C_q(u, v) = \begin{cases} (1+q) \cdot uv, & u+v < 1, \\ uv + q \cdot (1-u) \cdot (1-v), & u+v \geq 1. \end{cases} \quad (4.4)$$

For $q \in [0, 1]$ this copula is family B11 in Joe(1997), p.148. It represents a mixture of perfect dependence and independence. If X and Y are uniform(0,1), $Y = X$ with probability q and Y is independent of X with probability $1-q$, then (X, Y) has the linear Spearman copula. This distribution has been first considered by Konijn(1959) and motivated in Cohen(1960) along Cohen's kappa statistic (see Hutchinson and Lai(1990), Section 10.9). For the extended copula, the chosen nomenclature *linear* refers to the piecewise linear sections of this copula, and *Spearman* refers to the fact that the *grade correlation coefficient* r_s by Spearman(1904) coincides with the parameter q . This follows from the calculation

$$r_s = 12 \cdot \int_0^1 \int_0^1 [C_q(u, v) - uv] du dv = q, \quad (4.5)$$

where a proof of the integral representation is given in Nelsen(1991). The linear Spearman copula, which leads to the so-called *linear Spearman bivariate distribution*, has a singular component, which according to Joe should limit its field of applicability. Despite of this it has many interesting and important properties and is suitable for analytical computation. It appears important to note that the linear Spearman copula leads to a simple tail dependence structure, which is of interest when extreme values are involved. The *coefficient of (upper) tail dependence* of a couple (X, Y) is defined by

$$l = l(X, Y) = \lim_{a \rightarrow 1^-} \Pr(Y > Q_Y(a) | X > Q_X(a)), \quad (4.6)$$

provided a limit l in $[0, 1]$ exists. If $l \in (0, 1]$ then the couple (X, Y) is called *asymptotically dependent* (in the upper tail) while if $l = 0$ one speaks of *asymptotic independence*. Tail dependence is an asymptotic property of the copula. Its calculation follows easily from the relation

$$\begin{aligned} & \Pr(Y > Q_Y(a) | X > Q_X(a)) \\ &= \frac{1 - \Pr(X \leq Q_X(a)) - \Pr(Y \leq Q_Y(a)) + \Pr(X \leq Q_X(a), Y \leq Q_Y(a))}{1 - \Pr(X \leq Q_X(a))}. \end{aligned} \quad (4.7)$$

For a linear Spearman couple one obtains

$$l(X, Y) = \lim_{a \rightarrow 1^-} \frac{1 - 2a + C_q(a, a)}{1 - a} = \lim_{a \rightarrow 1^-} (1 - a + qa) = q. \quad (4.8)$$

Therefore, unless X and Y are independent, a linear Spearman couple is always asymptotically dependent. This is a desirable property in insurance and financial modelling, where data tend to be dependent in their extreme values. In contrast to this, the ubiquitous Gaussian copula yields always asymptotic independence, unless perfect correlation holds (e.g. Embrechts et al.(1998), Section 4.4).

Let (X, Y) be a random vector with absolutely continuous margins $F(x)$ and $G(y)$. Then the copula representation

$$H_q(x, y) = C_q[F(x), G(y)] \quad (4.9)$$

defines the so-called *linear Spearman bivariate distribution*. It satisfies the *monotone quadrant dependence* structure introduced by Lehmann(1966). For $q \geq 0$ the family is *positive quadrant dependent* such that $H_q(x, y) \geq F(x)G(y)$, and for $q \leq 0$ it is *negative quadrant dependent* such that $H_q(x, y) \leq F(x)G(y)$.

To model the dependence structure between the individual asset returns and the market index return, assume that the random pairs $(R_i, R^M), i = 1, \dots, n$, follow linear Spearman bivariate distributions

$$H_i(x, y) = C_{q_i}[F_i(x), F^M(y)], \quad i = 1, \dots, n, \quad (4.10)$$

where $q_i = r_s(R_i, R^M)$ is Spearman's grade correlation of the pair (R_i, R^M) . In finance markets, it is natural to assume positive quadrant dependence, that is $q_i \geq 0$ (similar results for $q_i \leq 0$ can also be derived). The considered model will be called *linear Spearman bivariate model of asset returns*. The following result holds.

Theorem 4.1. The market shortfall risk of the linear Spearman bivariate model of asset returns is determined by the formula

$$r_a[R_i | R^M] = q_i \{E[R_i] - E[F_i^{-1}[G(R^M)]]R^M \leq VaR_a[R^M]\} \quad i = 1, \dots, n. \quad (4.11)$$

Proof. Let (X, Y) be a linear Spearman bivariate random couple with margins $F(x)$, $G(y)$ and joint distribution $H(x, y) = C_q[F(x), G(y)]$, $q \geq 0$. Consider the shortfall risk measure of X on Y defined by

$$r_a[X|Y] = E[X] - E[X|Y \leq u_a], \quad u_a = VaR_a[Y]. \quad (4.12)$$

In terms of the copula function, the conditional distribution of X given Y is determined by

$$F(x|y) = \frac{\partial C_q}{\partial v}[F(x), G(y)] = \begin{cases} F(x) + q\bar{F}(x), & x \geq F^{-1}[G(y)], \\ (1-q)F(x), & x < F^{-1}[G(y)]. \end{cases} \quad (4.13)$$

Since $F(x|y)$ is non-increasing in y , one notes by passing that X is *positively regression dependent* on Y after Lehmann(1966), or in more recent terminology X is *stochastically increasing* in Y (e.g. Joe(1997)). In general, the regression function of X on Y is given by

$$E[X|y] = \int_0^\infty [1 - F(x|y)]dx - \int_{-\infty}^0 F(x|y)dx. \quad (4.14)$$

For the linear Spearman bivariate model with $q \geq 0$, one obtains

$$E[X|y] = E[X] - q \cdot (E[X] - F^{-1}[G(y)]), \quad (4.15)$$

from which it follows that

$$r_a[X|Y] = q \cdot (E[X] - E[F^{-1}[G(y)] | Y \leq u_a]). \quad (4.16)$$

Setting $(X, Y) = (R_i, R^M)$ shows the desired formula. $\diamond$

A general covariance identity will be helpful.

Theorem 4.2. Let (X, Y) be a linear Spearman bivariate random couple with margins $F(x)$, $G(y)$ and joint distribution $H(x, y) = C_q[F(x), G(y)]$, $q \geq 0$. Assume Y satisfies the following regularity assumption

$$\begin{aligned} \lim_{y \rightarrow -\infty} y(y)G(y) &= 0, \\ \lim_{y \rightarrow -\infty} y(y) \left(E[X]G(y) - \int_{-\infty}^y F^{-1}[G(y)]dG(y) \right) &= 0, \end{aligned} \quad (\text{RA})$$

where $y(y)$ is some differentiable function. Then one has the covariance identity

$$\text{Cov}[X, y(Y)] = q \cdot E[(F^{-1}[G(Y)] - E[X]) \cdot y(Y)] \quad (4.17)$$

Proof. Applying a well-known formula by Hoeffding(1940) and Lehmann(1966), Lemma 2, one obtains

$$\begin{aligned} \text{Cov}[X, y(Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [H(x, y) - F(x)G(y)] y'(y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(y) [F(x|Y \leq y) - F(x)] y'(y) dx dy \\ &= \int_{-\infty}^{\infty} G(y) \left[\int_{-\infty}^0 F(x|Y \leq y) dx - \int_{-\infty}^0 F(x) dx + \int_0^{\infty} \bar{F}(x) dx - \int_0^{\infty} [1 - F(x|Y \leq y)] dx \right] y'(y) dy \\ &= \int_{-\infty}^{\infty} (E[X] - E[X|Y \leq y]) G(y) y'(y) dy. \end{aligned}$$

Inserting the formula $E[X] - E[X|Y \leq y] = q \cdot (E[X] - E[F^{-1}[G(Y)]|Y \leq y])$ derived from (4.15), using further partial integration and the assumption (RA), one obtains the desired identity as follows :

$$\begin{aligned} \text{Cov}[X, y(Y)] &= q \cdot \int_{-\infty}^{\infty} \left(E[X]G(y) - \int_{-\infty}^y F^{-1}[G(y)]dG(y) \right) y'(y) dy \\ &= q \cdot y(y) \left[E[X]G(y) - \int_{-\infty}^y F^{-1}[G(y)]dG(y) \right]_{-\infty}^{\infty} + q \cdot \int_{-\infty}^{\infty} y(y) (F^{-1}[G(y)] - E[X]) dG(y) \\ &= q \cdot E[(F^{-1}[G(Y)] - E[X]) \cdot y(Y)]. \quad \diamond \end{aligned}$$

We are ready for our main result on location-scale asset models.

Theorem 4.3. Let $(R_i, R^M), i=1, \dots, n$, be a linear Spearman bivariate model of asset returns with location-scale margins $F_i(x) = D\left(\frac{x-m_i}{c_i}\right)$, $G(y) = D\left(\frac{y-m_M}{c_M}\right)$, $m_i = E[R_i]$, $m_M = E[R^M]$, c_i, c_M the scale parameters. Suppose the variances $s_i^2 = \text{Var}[R_i]$, $s_M^2 = \text{Var}[R^M]$ exist, and assume $y(y) = y$ satisfies the regularity assumption (RA) of Theorem 4.2. Then the market shortfall risk is determined by the formula

$$r_a[R_i|R^M] = \frac{\text{Cov}[R_i, R^M]}{s_R^2} \cdot r_a[R^M], \quad i=1, \dots, n. \quad (4.18)$$

Proof. The location-scale assumption implies the relation

$$F^{-1}[G(y)] = m_i + \frac{c_i}{c_M} \cdot (y - m_M).$$

Inserting the formula of Theorem 4.1, one obtains immediately

$$r_a[R_i | R^M] = q_i \cdot \frac{c_i}{c_M} \cdot r_a[R^M], \quad i = 1, \dots, n.$$

On the other hand, inserting the same expression into the covariance identity (4.17) with $y(y) = y$, one obtains

$$\text{Cov}[R_i, R^M] = q_i \cdot \frac{c_i}{c_M} \cdot s_M^2, \quad i = 1, \dots, n,$$

which implies the desired result. $\diamond$

Some comments are in order. In Financial Economics, the quantity $b_i^M = \frac{\text{Cov}[R_i, R^M]}{s_M^2}$ is called *beta factor* of the asset number i , and represents the *market risk* of the asset (e.g. Sharpe(1985)). Under the assumptions of Theorem 4.3, the market shortfall risk of a linear portfolio $R = \mathbf{w}^T \cdot \mathbf{R}$ is proportional to the linear weighted sum of the beta factors

$$r_a[R; R^M] = \left(\sum_{i=1}^n w_i \cdot b_i^M \right) \cdot r_a[R^M], \quad (4.19)$$

which for an optimal portfolio selection should be minimised under the constraints $\mathbf{w}^T \cdot \mathbf{m} = m_R$, $\mathbf{w}^T \cdot \mathbf{e} = 1$. This linear problem can be solved in an elementary way. On the other hand, the portfolio manager of an index match fund, whose benchmark is the market return, will approximately hold the market portfolio with return $R \approx R^M$. In this situation, one has approximately $b_i^M \approx \frac{\text{Cov}[R_i, R^M]}{s_M^2}$ and $r_a[R; R^M] \approx s_R^2 \cdot \frac{r_a[R^M]}{s_M^2}$. This identifies mean-shortfall portfolio selection of an index match fund approximately with mean-variance portfolio selection. In particular, the latter appears useful for some non-elliptical distributions of asset returns.

To conclude, let us show that the above mean-shortfall portfolio selection model differs in general from mean-variance portfolio selection.

Example 4.1.

Assume the linear Spearman bivariate model of asset returns $(R_i, R^M), i = 1, \dots, n$, has log-location-scale margins such that $F_i(x) = D\left(\frac{\ln(x) - m_i}{s_i}\right)$, $G(y) = D\left(\frac{\ln(y) - m_M}{s_M}\right)$. Then one has

$$F^{-1}[G(y)] = \exp \left[m_i + \frac{s_i}{s_M} (\ln(y) - m_M) \right], \quad (4.20)$$

and Theorem 4.1 yields the relationship

$$r_a[R_i|R^M] = q_i \cdot \left\{ E[R_i] - \exp\left(m_i - \frac{s_i}{s_M} \cdot m_M\right) \cdot E\left[(R^M)^{\frac{s_i}{s_M}} | R^M \leq u_a\right] \right\}, \quad u_a = \text{VaR}_a[R^M]. \quad (4.21)$$

In the important special case of log-normal asset margins, with $D(x) = \Phi(x)$ the standard normal distribution, one obtains without difficulty the formula

$$r_a[R_i|R^M] = q_i \cdot E[R_i] \cdot \left[1 - \frac{\Phi\left(\frac{\ln(u_a) - m_M - s_i}{s_M}\right)}{\Phi\left(\frac{\ln(u_a) - m_M}{s_M}\right)} \right]. \quad (4.22)$$

On the other hand, Theorem 4.2 yields the covariance formula

$$\text{Cov}[R_i, R^M] = q_i \cdot E[R_i] \cdot E[R^M] \cdot (e^{s_i s_M} - 1). \quad (4.23)$$

Comparing (4.22) and (4.23) one obtains

$$r_a[R_i|R^M] = \text{Cov}[R_i, R^M] \cdot \left[1 - \frac{\Phi\left(\frac{\ln(u_a) - m_M - s_i}{s_M}\right)}{\Phi\left(\frac{\ln(u_a) - m_M}{s_M}\right)} \right] \cdot \frac{1}{E[R^M] \cdot (e^{s_i s_M} - 1)}, \quad (4.24)$$

which shows that the factor multiplying the covariance depends on the individual asset characteristics. As a consequence, the market shortfall risk of the portfolio is not exactly proportional to the variance (or standard deviation) of the portfolio return, and mean-shortfall portfolio selection differs (at least slightly) from mean-variance portfolio selection.

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