NON HOMOGENEOUS INTEREST RATE STRUCTURE IN A SEMI-MARKOV FRAMEWORK.°

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1. - Introduction.

Homogeneous semi-Markov processes (HSMP) were defined in the fifties (Levy 1954). At beginning their applications were in engineering field, mainly in problem of reliability and maintenance (see Howard 1972). In general this processes are a tool that is useful to study problems linked to the ageing of a system.
Janssen (1966) applied for the first time HSMP in actuarial science. In this field there were many applications of HSMP. Also in this field SMP can be considered a tool that gives good results in real world applications.
Non Homogeneous Semi-Markov Processes (NHSMP) were introduced by Iosifescu Manu (1972) ; these processes were applied in actuarial field see (Hoem 1972, CMIR 1991) and more recently (Haberman, Pitacco 1999, Pitacco, Olivieri 1997).
The Discrete Time Non-Homogeneous Semi-Markov Process (DTNHSMP) were introduced in De Dominicis, Janssen (1984) and a real data actuarial application was presented in this paper that was later on generalized and applied to actuarial problem, see for example (Janssen, Manca 1998).
It is to outline that homogeneous and non-homogeneous SMP are a generalization of the corresponding Markov processes (MP) in the sense that in MP environment the waiting time distribution function (d.f.) in a state before a change is exponential whereas in SMP can be of any type.
In the last twenty years finance made great theoretical advances using stochastic tools starting from the two fundamentals papers respectively of (Black & Scholes 1973) on option pricing and of (Vasicek 1977) on stochastic interest rate. Innovative results were obtained in all financial topics

introducing massively probability tools. In some cases the results generalized the ones given in deterministic environment, for example the ones from (Buhlmann 1992), (Buhlmann 1994) and (Norberg 1995) on the evaluation of stochastic interest rate. In other cases the probabilistic approach gave the possibility to evaluate some financial tools that otherwise should be really difficult to study in a realistic way (see Willmott 1998), mainly for derivative products. Furthermore the opportunity to face in a more correct way some financial tools allowed the introduction of new products, like some special kind of options (see Willmott 1998).
The main part of these results were obtained in homogeneous MP environment. It is to outline that, in authors opinion, the initial time of a financial operation is usually known, whereas the time when the operation will be closed is not known, furthermore changing the initial time of a financial operation usually changes the conditions (as well known this fact is taken in account by means of non homogeneity). This observations imply that it could be really interesting to introduce a stochastic environment also for the time length of financial operations considering also the initial time of operations. This step can be made using NHSMP models. This step was already proposed in some theoretical papers (Svishchuk 1995, Svishchuk, Burdeinyi 1996, Janssen, Manca, De Medici 1996, Janssen, Manca, Volpe di Prignano 1999, Janssen, Manca, Di Biase 1997, 2001) in homogeneous case and (Janssen, Manca 2000, Janssen, Manca, Di Biase 1998) in the non-homogeneous one.
A model useful to construct term structure of implied forward rates in a stochastic environment will be presented in this paper. The stochastic process used to construct the interest rate structure will be the NHSMP, in the paper the financial operations are supposed non-homogeneous in time (see Volpe di Prignano 1985).
It is to precise that a discrete time framework will be used in the paper. There are two reasons to apply discrete time non-homogeneous semi-Markov process (DTNHSMP) instead that the continuous one. The first is that in a previous paper (Janssen, Manca 2001) was proved that discretising the evolution equation of a continuous time NHSMP by means of the simplest quadrature formula the DTNHSMP is obtained and if the time interval of DTNHSMP tends to 0 then is obtained the continuous case. The second is that the to solve DTNHSMP evolution equation is not a difficult task, as the solution of the continuous time can be obtained analytically only in same special cases; in the other cases to find the solution it is necessary to work numerically and the most simple way to obtain the numerical solution of the process evolution equation is the one by which the DTNHSMP is obtained by the continuous time NHSMP.
In the second paragraph will be introduced the theoretical stochastic interest rate model, in the third one will be presented the DTNHSMP in the forth paragraph will be explained how to implement the model by means of DTNHSMP and in the last part will be given an applicative example.

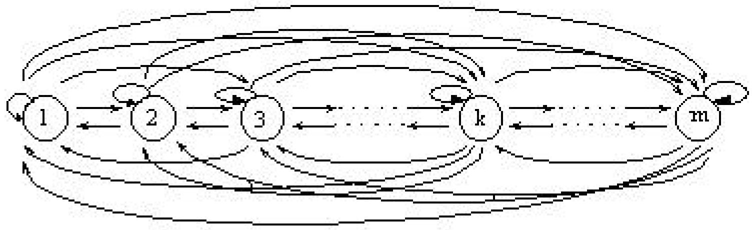
2. A stochastic interest rate model.

In this paragraph we will show how to construct a model that can be used to simulate the stochastic evolution of an implied forward interest rates.
The states of the model will represent all the possible interest rates. In fact a priori it is possible to decide that the interest rate will change in a fixed interval during the time horizon considered. The number of states will be finite more precisely it results:

$$E = \{r_1, r_2, \ldots, r_m\},$$

where the r_i represent all the possible implied stochastic interest rates in increasing order, m gives their number.
The graph of the model is represented in the fig. 1.
As stated before, in the model the number of states is finite. This is from theoretical point of view a great simplification, but in real life problems the interest rate are usually fixed in a discrete range. As specified before the time will move on a discrete time scale.

Fig. 1. *m* states model for interest rate structure model.



The model will work in transient case that means in a bounded time horizon. This can be considered an other simplification but in applicative real life problem the asymptotic behaviour could not be interesting.
The model will reconstruct, for each state, for each starting time s and for each arrival time t, the probability function to be in all the states. So for example if the initial state was r_i then the model will give for each starting time s and for each arrival time t the probability to be in the state $r_j,\ j=1,\ldots,m$. The expected interest rate value for each period of the horizon time will be possible to obtain in this way.

3. The Discrete Time Non-Homogeneous Semi-Markov Process (DTNHSMP)

In this part will be described the DTNHSMP (see De Dominicis, Janssen 1984). First the stochastic process is defined. The notation of Cinlar (1975) will be followed.
$X_n,\ n\in$ N (r.v.) set of states $E=\{1, 2, \ldots, m\}$ representing the state at the n-th transition. $T_n,\ n\in$ N, an other r.v. with set of states equal to N where T_n represents the time of the n-th transition,

$$X_n:\Omega\to E \qquad T_n:\Omega\to \mathrm{N}.$$

The process $(X_n,\ T_n)$ is a non-homogeneous markovian renewal process. The kernel $\mathbf{Q}=[Q_{ij}(s,t)]$ associated to the process is defined in the following way:

$$Q_{ij}(s,t) = \mathrm{P}[X_{n+1}=j, T_{n+1}\le t \mid X_n = I, T_n=s]$$

and it results (Pyke 1961):

$$p_{ij}(s) = \lim_{t\to\infty} Q_{ij}(s,t);\ i,j\in E,\ t\in \mathrm{N}$$

where $\mathbf{P}(s) = [p_{ij}(s)]$ is the transition matrix of the embedded Markov chain in the process. Furthermore it is necessary to introduce the probability that process will have a transition from the state i in a time t-s:

$$S_i(s,t) = \mathrm{P}[T_{n+1}\le t \mid X_n = i,\ T_n=s].$$

Obviously it results that:

$$S_i(s,t) = \sum_{j=1}^{m} Q_{ij}(s,t).$$

Furthermore the following probabilities are considered:

$b_{ij}(s,t) = \mathrm{P}[\, X_{n+1}= j,\ T_{n+1} = t \mid X_n = i,\ T_n= s]$

These probabilities can be given in function of the $Q_{ij}(s,t)$:

$$b_{ij}(s,t)=\begin{cases} Q_{ij}(s,s)=0 & \text{if } \ t=s \\ Q_{ij}(s,t)-Q_{ij}(s,t-1) & \text{if } \ t=s+1,s+2,\ldots \end{cases}$$

Now it is possible to define the probability distribution of the waiting time in each state i, given that the state successively occupied is known:

$G_{ij}(s,t)=\mathrm{P}[\, T_{n+1} \le t \mid X_n = i,\ X_{n+1} = j,\ T_n = s\,]$.

Obviously the related probabilities can be obtained by means of the following formula:

$$G_{ij}(s,t)=\begin{cases} Q_{ij}(s,t)\,/\,p_{ij}(s) & \text{if } \ p_{ij}(s)\neq 0 \\ 1 & \text{if } \ p_{ij}(s)=0 \end{cases}.$$

Now the DTNHSMP $Z = (\, Z_t,\ t\in \mathrm{N})$ can be defined.

It represents, for each considered time, the state occupied by the process. The probabilities to stay in the state j at time t, given that at time s the system was in the state i, are defined in the following way:

$f_{ij}(s,t) = \mathrm{P}[\, Z_t = j \mid Z_s = i\,]$.

They are obtained solving the following evolution equations:

$$f_{ij}(s,t)=d_{ij}(1-S_i(s,t)) + \sum_{b=1}^{m}\sum_{J=1}^{t} b_{ib}(s,J) f_{bj}(J,t) \qquad (3.1)$$

where d_{ij} represents the Kronecker symbol.

4. A Semi-Markov stochastic interest rate approach.

In this part, we introduce a stochastic term structure of implied forward rates.
The structure will be constructed by means of DTNHSMP. The way to apply this process to a general financial problem was presented in (Janssen, Manca 2000).
In this case as specified before the states of the process will be:
$E=\{r_1, r_2, \mathrm{K}\ , r_m\}$,
where the r_i represents all the possible implied stochastic interest rates and m gives the number of the implied interest rates.
In this case the evolution equation of the DTNHSMP will be the following one:

$$f_{ij}(s,t)=d_{ij}(1-S_i(s,t)) + \sum_{b=1}^{m}\sum_{J=1}^{t} b_{ib}(s,J) f_{bj}(J,t)$$

where $f_{ij}(s,t)$ represents the probability that at time t the implied interest rate will be r_j given that the implied interest rate was r_i at time s. This probability is obtained by the sum of two terms. The

first element gives the probability to remain in the starting state without any transition and has sense only if $i = j$ the second term gives the probability to stay at time t in the state j with at least one state transition (the first one in J).
By means of previous positions it is possible to investigate the following r.v. that are considered in (s,h) supposing that the system is in the state i at initial time s.

$$V_i(s,q) = \left(1+\Gamma_i(s,q)\right)^{-1}$$

is the random discount factor related to the period $(q-1,q) \subset (s,h), q = s+1,\ldots,h,$ depending on the one period random rate $\Gamma_i(s,q)$ with values $r_j \in E$ with probability $f_{ij}(s,q)$.

$$A_i(s,h) = \prod_{q=s+1}^{h} V_i(s,q)$$

is the random discount factor related to (s,h).
The independence hypothesis among $V_i(s,q)$ (i.e. the independence among all the possible q-combinations of the h events $E_{ij}(s,n),\ n = s+1,\ldots,h,$ r-grouped, concerning the $V_i(s,n)$ outcome) is assumed.
Computing only the expected values and the variances of the considered r.v., it results for the first ones:

$$E(V_i(s,q)) = \sum_{j=1}^{m} f_{ij}(s,q)(1+r_j)^{-1}$$

$$u_i(s,h) = E(A_i(s,h)) = \prod_{q=1}^{h} E(V_i(s,q))$$

and for the second ones, generalizing the computation of $\sigma^2(XY)$:

$$\sigma^2(V_i(s,q)) = \sum_{j=1}^{m} f_{ij}(s,q)(1+r_j)^{-2} - \left(\sum_{j=1}^{m} f_{ij}(s,q)(1+r_j)^{-1}\right)^2$$

$$\sigma^2\left(A_i(s,h)\right) = \sum_{t=1}^{k} \sum_{q=1}^{\binom{k}{t}} S_{C_q} M_{D_q},\ k = h-s$$

where:

$$S_{C_q} = \prod_{r=1}^{t} \sigma_i^2(s,s+z_r),\ \left(z_1,\ldots,z_t\right) = C_q, \qquad M_{D_q} = \begin{cases} \prod_{r=1}^{k-t} \mu_i^2(s,s+h_r),\ \left(h_1,\ldots,h_{k-t}\right) = D_q, & \text{if } t < k \\ 1, & \text{if } t = k \end{cases}$$

$C_q \in \mathbf{C}(k,t),\ \mathbf{C}(k,t)$ set of all t –combinations of the set $\{1,\ldots,k\}$,
and

$$\begin{cases} \sigma_i^2(s,\lambda) = \sigma^2(V_i(s,\lambda)) \\ \mu_i(s,\lambda) = E(V_i(s,\lambda)) \\ C_q \cup D_q = \{1,\ldots,k\} \end{cases}$$

Once that the $u_i(s,h)$ are known then it is possible to evaluate the mean present value of a given financial operation that begins at time s and ends at time h and in which is known the value at time h. Clearly if the value at time s is known the mean value at time h will be obtained multiplying the initial value by $1/u_i(s,h)$.

The knowledge of the expected value and the variance of $A_i(s,h)$ allows important applications on riskiness control. In particular it allows decision and choices on financial project by using the mean-variance criterion.

5. An applicative example.

To illustrate numerically the proposed model let us suppose that we are interested in the dynamic evolution of a stochastic interest rate whose possible values are restricted to the following ones:

maximum value:	state 15 = .10
intermediate values:	state 14 = .095
	state 13 = .09
	state 12 = .085
	state 11 = .08
	state 10 = .075
	state 9 = .07
	state 8 = .065
	state 7 = .06
	state 6 = .055
	state 5 = .05
	state 4 = .045
	state 3 = .04
	state 2 = .035
minimum value	state 1 = .03.

We suppose to be in a ten years time horizon. In this way we have 15 state and 11 time period, from time period 0 up to the time period 10.
To get results we constructed a "Mathematica" program able to solve DTNHSMP.
To apply semi-Markov processes is necessary to evaluate:

- **a)** the transition matrix $\mathbf{P}(s)$, embedded Markov chain in DTHSMP;
- **b)** the matrix $\mathbf{G}(s,t)$, whose components are waiting time increasing distribution functions $\forall i,j,s$.

The corresponding values can be obtained by means of observation on real data, in this example we filled up both the matrices with pseudorandom numbers.
The matrix $\mathbf{P}(s)$ is formed by 11 square matrices each one of order 15. It results that:
$P_{ij}(s) > 0 \ ,i,j \in \{1,\ldots,15\},\ s \in \{0,\ldots,10\}$.

This matrix represents the transition probabilities at time s. Filling up this matrix we supposed that, $\forall s$, the transition probabilities are bigger in the three mean diagonals and they decrease moving away from them.
The matrix $\mathbf{G}(s,t)$ is formed by 11×11 square matrices each one of order 15. It results that:

$$G_{ij}(s,t)\begin{cases} >0 \ , i, j \in \{1,\ldots,15\}, \ s<t \\ =0, \qquad\qquad\qquad s\geq t \end{cases} \tag{5.1}$$

To take in account we are working in the transient case, with a 10 year horizon time, we consider all these distribution functions trimmed at the last period.
After the construction of the embedded Markov chain and the distribution functions we were able to apply the DTNHSMP.
In the first step we obtain the matrix $\mathbf{Q}(s,t)$, after this we can compute the matrix $\mathbf{B}(s,t)$, the next step is to obtain the matrix $\mathbf{S}(s,t)$ at last we can solve the evolution equation (3.1).
Once solved the (3.1) we obtain the following probability distributions:

$$P_{i,1}(s,t),\ldots,P_{i,15}(s,t) \text{ with } i\in\{1,\ldots,15\} \text{ and } s,t\in\{0,\ldots,10\}. \tag{5.2}$$

Now it is possible to compute for each i, t of the (5.2) the interest rate mean values and related standard deviations.
In the following four pages are presented the results related to the described example. More precisely Tables 1.1, 1.2 and 1.3 contain the mean interest rates. The Table 1.1 contains the mean interest rates related at starting time 0, the Table 1.2 those related at starting time 4 and the Table 1.3 those related at starting time 8. Each column of each table represents the starting state (value of interest rate) respectively at time 0, 4 and 8. The rows represent the values of the mean interest rates at time t. The elements of the Tables 2.1, 2.2 and 2.3 give the values of uni-periodical discount factors. The elements of the Tables 3.1, 3.2 and 3.3 give the mean discount factors from the time s up to the time t. Clearly each table, regarding the starting time s, corresponds to the one of the previous pages. At last the Tables 4.1, 4.2 and 4.3 respectively give the variance of the elements of the Tables 3.1, 3.2 and 3.3.

Conclusions

In the paper a new approach, by means of NHSMP, to the construction of stochastic interest rate structure has been presented. The assumptions that are necessary to apply the model are the ones for the SMP process to real problems. This means that we can apply this model to get a “physical measure” fitting the real data without the more restrictive assumptions of “risk neutral measure” in view to really explain one given interest rate market. Finally we think that this more general approach can be used in other financial applications.

Table 1.1: Successive uni-periodical mean interest rates starting form any initial state at time 0.

0.03	0.035	0.04	0.045	0.05	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
0.0313117	0.0362482	0.0410653	0.0462766	0.0508937	0.0552552	0.0600835	0.0649177	0.0693448	0.0749862	0.0794244	0.0834017	0.0887046	0.0934663	0.0976678
0.0325927	0.0374371	0.0424734	0.0473747	0.0516856	0.0562533	0.0602967	0.0642571	0.0693575	0.0745601	0.0785613	0.0826425	0.0876005	0.0923466	0.0951265
0.0340818	0.0390862	0.0438421	0.048314	0.0520527	0.056978	0.0603204	0.0643015	0.0688756	0.0739198	0.0775237	0.0816355	0.0857487	0.0898041	0.0929551
0.0358741	0.0408771	0.0459381	0.0490302	0.0531727	0.057275	0.0605327	0.06447	0.0689909	0.0740286	0.0767965	0.0809452	0.0842229	0.0885653	0.0912346
0.0380755	0.0425181	0.0467187	0.0497572	0.0541028	0.0581358	0.0606889	0.064556	0.0686434	0.0726681	0.0754422	0.0801404	0.0827545	0.0864757	0.0893618
0.0401347	0.0447594	0.047995	0.0513721	0.0554276	0.0587872	0.0609608	0.0645653	0.0681594	0.0714897	0.0746732	0.0784592	0.0807399	0.0843726	0.0870258
0.0424678	0.0469699	0.0494644	0.0524156	0.056173	0.0597425	0.0615751	0.0646024	0.0676103	0.0711308	0.0741005	0.0770235	0.0792597	0.08269	0.0837965
0.0451926	0.0491101	0.0515963	0.0540177	0.0573675	0.0607736	0.0616994	0.06482	0.0670139	0.0698163	0.0728642	0.0756477	0.0777013	0.0803733	0.0815721
0.0482788	0.0520942	0.0536129	0.0557446	0.0587244	0.0615508	0.0620236	0.0650969	0.0666341	0.068929	0.0713703	0.073507	0.0760236	0.0776508	0.0781019
0.0519793	0.0547817	0.0560115	0.0582907	0.0598774	0.0621408	0.062675	0.0643683	0.0657689	0.0669042	0.0691672	0.0705498	0.0723702	0.0739991	0.0740946

Table 1.2: Successive uni-periodical mean interest rates starting form any initial state at time 4.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.03	0.035	0.04	0.045	0.05	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
0.033043	0.0368161	0.0413054	0.046916	0.0509838	0.0555684	0.0609238	0.0644413	0.0697617	0.0738096	0.0782969	0.0837931	0.0875415	0.0929274	0.0972504
0.0362772	0.0389844	0.04326	0.0479117	0.052456	0.0568673	0.0615507	0.0644452	0.0692061	0.0730365	0.0779259	0.0818967	0.0844548	0.0906465	0.0941515
0.0401253	0.0413231	0.0454411	0.0495253	0.0539465	0.0579253	0.0620243	0.0646337	0.0685485	0.0717218	0.0766322	0.0802873	0.0828746	0.0858626	0.0908084
0.0445293	0.0450569	0.0481518	0.0507887	0.0553373	0.0596829	0.062692	0.0648214	0.068126	0.0710142	0.0750498	0.0787232	0.0806149	0.083747	0.0876651
0.0487061	0.0483461	0.0510052	0.0527337	0.0565825	0.0605929	0.0627621	0.0650231	0.0675952	0.0698315	0.0733716	0.0757558	0.0775936	0.0801469	0.0839004
0.0525402	0.0527473	0.054603	0.055652	0.0582154	0.0618618	0.0628699	0.06372	0.0665083	0.0681983	0.0705325	0.0733529	0.0735601	0.07493	0.0777694

Table 1.3: Successive uni-periodical mean interest rates starting form any initial state at time 8.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.03	0.035	0.04	0.045	0.05	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
0.0372843	0.0415861	0.0442579	0.0491011	0.0526493	0.0581422	0.059891	0.0649148	0.0696864	0.0737518	0.0777112	0.0809003	0.0838619	0.0874934	0.0899269
0.0479944	0.0498598	0.0502006	0.0541221	0.0561315	0.0584075	0.0620889	0.0645637	0.0667158	0.0708113	0.0720421	0.0759025	0.0772346	0.0781211	0.0803677

Table 2.1: Successive uni-periodical mean discount factors starting form any initial state at time 0.

0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
0.969639	0.96502	0.960555	0.95577	0.951571	0.947638	0.943322	0.93904	0.935152	0.930245	0.92642	0.923019	0.918523	0.914523	0.911022
0.968436	0.963914	0.959257	0.954768	0.950854	0.946743	0.943132	0.939623	0.935141	0.930613	0.927161	0.923666	0.919455	0.91546	0.913137
0.967041	0.962384	0.957999	0.953913	0.950523	0.946093	0.943111	0.939583	0.935563	0.931168	0.928054	0.924526	0.921023	0.917596	0.914951
0.965368	0.960728	0.95608	0.953261	0.949512	0.945828	0.942922	0.939435	0.935462	0.931074	0.928681	0.925116	0.92232	0.91864	0.916393
0.963321	0.959216	0.955366	0.952601	0.948674	0.945058	0.942783	0.939359	0.935766	0.932255	0.92985	0.925806	0.92357	0.920407	0.917969
0.961414	0.957158	0.954203	0.951138	0.947483	0.944477	0.942542	0.939351	0.93619	0.93328	0.930515	0.927249	0.925292	0.922192	0.919941
0.959262	0.955137	0.952867	0.950195	0.946815	0.943625	0.941997	0.939318	0.936671	0.933593	0.931012	0.928485	0.926561	0.923625	0.922682
0.956761	0.953189	0.950935	0.948751	0.945745	0.942708	0.941886	0.939126	0.937195	0.93474	0.932084	0.929672	0.927901	0.925606	0.92458
0.953945	0.950485	0.949115	0.947199	0.944533	0.942018	0.941599	0.938882	0.937529	0.935516	0.933384	0.931526	0.929348	0.927944	0.927556
0.950589	0.948063	0.946959	0.94492	0.943505	0.941495	0.941021	0.939524	0.93829	0.937291	0.935307	0.934099	0.932514	0.931099	0.931017

Table 2.2: Successive uni-periodical mean discount factors starting form any initial state at time 4.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
0.968014	0.964491	0.960333	0.955186	0.951489	0.947357	0.942575	0.93946	0.934788	0.931264	0.927388	0.922685	0.919505	0.914974	0.911369
0.964993	0.962478	0.958534	0.954279	0.950159	0.946193	0.942018	0.939457	0.935273	0.931935	0.927708	0.924303	0.922122	0.916887	0.91395
0.961423	0.960317	0.956534	0.952812	0.948815	0.945246	0.941598	0.93929	0.935849	0.933078	0.928822	0.92568	0.923468	0.920927	0.916751
0.957369	0.956886	0.95406	0.951666	0.947564	0.943679	0.941006	0.939125	0.936219	0.933694	0.930189	0.927022	0.925399	0.922725	0.919401
0.953556	0.953883	0.95147	0.949908	0.946448	0.942869	0.940944	0.938947	0.936685	0.934727	0.931644	0.929579	0.927994	0.9258	0.922594
0.950082	0.949896	0.948224	0.947282	0.944987	0.941742	0.940849	0.940097	0.937639	0.936156	0.934115	0.93166	0.93148	0.930293	0.927842

Table 2.3: Successive uni-periodical mean discount factors starting form any initial state at time 8.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
0.964056	0.960074	0.957618	0.953197	0.949984	0.945053	0.943493	0.939042	0.934853	0.931314	0.927892	0.925155	0.922627	0.919546	0.917493
0.954204	0.952508	0.952199	0.948657	0.946852	0.944816	0.941541	0.939352	0.937457	0.933871	0.932799	0.929452	0.928303	0.92754	0.925611

Table 3.1: Mean discount factors from 0 and time maturity *t*.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
0.941397	0.932386	0.92361	0.914613	0.906258	0.898235	0.889926	0.881727	0.873974	0.865344	0.857796	0.850708	0.842681	0.835181	0.828202
0.911683	0.89874	0.88598	0.873243	0.86172	0.850397	0.839318	0.828491	0.817289	0.8053	0.795315	0.78577	0.774808	0.764575	0.756262
0.881635	0.864933	0.848768	0.832998	0.819084	0.804555	0.79157	0.778436	0.764625	0.74987	0.738095	0.726465	0.713616	0.701571	0.691942
0.851103	0.830966	0.81149	0.794064	0.77773	0.760971	0.746389	0.73129	0.715277	0.698185	0.685455	0.672065	0.658182	0.644491	0.634091
0.819885	0.797075	0.77527	0.756427	0.737812	0.719162	0.703684	0.686944	0.669332	0.650886	0.63737	0.622201	0.607878	0.593194	0.582076
0.788249	0.762927	0.739765	0.719466	0.699065	0.679232	0.663251	0.645281	0.626622	0.607459	0.593083	0.576935	0.562464	0.547039	0.535476
0.756137	0.7287	0.704898	0.683633	0.661885	0.64094	0.62478	0.606124	0.586939	0.567119	0.552167	0.535676	0.521158	0.505259	0.494074
0.723443	0.694589	0.670312	0.648597	0.625974	0.60422	0.588472	0.569227	0.550076	0.530109	0.514666	0.498003	0.483583	0.467671	0.456811
0.690125	0.660197	0.636203	0.614351	0.591253	0.569186	0.554104	0.534437	0.515712	0.495925	0.480381	0.463903	0.449416	0.433973	0.423718

Table 3.2: Mean discount factors from 4 and time maturity *t*.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
0.939819	0.931876	0.923397	0.914054	0.90618	0.897969	0.889221	0.882122	0.873633	0.866292	0.858693	0.850401	0.843583	0.835593	0.828517
0.906919	0.89691	0.885107	0.872262	0.861015	0.849651	0.837663	0.828715	0.817086	0.807328	0.796616	0.786028	0.777886	0.766144	0.757224
0.871932	0.861318	0.846635	0.831102	0.816944	0.80313	0.788742	0.778404	0.764669	0.7533	0.739915	0.72761	0.718353	0.705563	0.694186
0.834761	0.824183	0.807741	0.790932	0.774107	0.757896	0.742211	0.731019	0.715898	0.703352	0.688261	0.674511	0.664763	0.65104	0.638235
0.795991	0.786174	0.768542	0.751312	0.732652	0.714597	0.698379	0.686388	0.670571	0.657441	0.641214	0.627011	0.616896	0.602733	0.588832

Table 3.3: Mean discount factors from 8 and time maturity *t*.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0.970874	0.966184	0.961538	0.956938	0.952381	0.947867	0.943396	0.938967	0.934579	0.930233	0.925926	0.921659	0.917431	0.913242	0.909091
0.935977	0.927608	0.920786	0.91215	0.904747	0.895784	0.890088	0.88173	0.873695	0.866339	0.85916	0.852677	0.846447	0.839768	0.834084

Table 4.1: Variance related to Table 3.1.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.0000287573	0.000039411	0.000039304	0.0000344795	0.0000236922	0.0000284407	0.0000201337	0.0000195909	0.0000330319	0.0000167389	0.0000265367	0.000041721	0.0000400718	0.0000416054	0.0000583956
0.0000831111	0.000109726	0.00010389	0.0000984895	0.0000743398	0.0000824357	0.0000619775	0.0000599455	0.0000714548	0.0000534354	0.0000665284	0.0000939029	0.0000866628	0.0000883955	0.000141574
0.000163436	0.000210485	0.000188984	0.000176184	0.000128232	0.000156849	0.00010932	0.000106669	0.000113457	0.0000977411	0.000119693	0.00014803	0.00014126	0.000157889	0.000224309
0.000259706	0.000330413	0.000292023	0.000259244	0.00020325	0.000233158	0.000162909	0.000158724	0.000161069	0.000143988	0.000171383	0.000198221	0.000196385	0.000219686	0.000297631
0.000370844	0.000459168	0.000393116	0.000343043	0.00028336	0.000308583	0.000216519	0.000210211	0.000209665	0.000192727	0.000229366	0.000242947	0.000246132	0.000276228	0.000359701
0.00048914	0.000588358	0.000490668	0.000432922	0.000361198	0.000378	0.000270114	0.000260789	0.000255696	0.00023953	0.000278336	0.000285978	0.000292014	0.000327668	0.000411662
0.000610489	0.000712153	0.000582437	0.000518857	0.000433445	0.000446034	0.000324204	0.000303488	0.000298706	0.000282726	0.000320123	0.000324112	0.000330882	0.000367936	0.000454353
0.000731036	0.000820931	0.000665339	0.000596731	0.000502872	0.000503171	0.000372525	0.000343497	0.000343102	0.000323659	0.000355731	0.000357453	0.000359394	0.000398803	0.000481798
0.000840324	0.000914567	0.000737255	0.000663639	0.000562743	0.000551147	0.00041876	0.000380549	0.000380552	0.00035757	0.000384985	0.000385503	0.000379507	0.000420524	0.000499425
0.00093535	0.000987243	0.00079881	0.000717034	0.000614671	0.000590955	0.00046062	0.000419155	0.000415487	0.000390765	0.000411515	0.000410414	0.000401376	0.000437659	0.000508468

Table 4.2: Variance related to Table 3.2.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.000104746	0.0000643331	0.000052377	0.0000534596	0.0000341503	0.0000252458	0.000039374	0.0000267392	0.0000366234	0.0000358625	0.0000478441	0.0000365939	0.0000628833	0.000055322	0.000075146
0.000268427	0.000171323	0.000149711	0.000127662	0.000103277	0.000083911	0.000106971	0.0000828106	0.000108702	0.000102357	0.000104226	0.0000979361	0.000157403	0.000141206	0.000187841
0.000481661	0.000305157	0.000268936	0.000223092	0.000207635	0.000166876	0.000187384	0.00015831	0.000191681	0.0001878	0.000184607	0.000166042	0.00024399	0.000262896	0.000303506
0.000721843	0.000478306	0.000401899	0.000319717	0.000316154	0.000263746	0.000278795	0.000246671	0.000272467	0.000274565	0.000272263	0.000247601	0.000327893	0.000365577	0.000406334
0.000945114	0.000652858	0.000543894	0.000428623	0.000425047	0.000361328	0.000366964	0.000330139	0.000353188	0.000357788	0.000358983	0.000332461	0.000408537	0.000458478	0.000490514
0.00112121	0.000807698	0.000681007	0.000544092	0.000537888	0.000461073	0.000463951	0.000423724	0.000434883	0.000438028	0.000444466	0.000410725	0.00048146	0.000536885	0.000561136

Table 4.3: Variance related to Table 3.3.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.000186702	0.000139632	0.000131333	0.000109766	0.000134277	0.000114848	0.0000898892	0.000110108	0.0000879102	0.0000975102	0.00010082	0.0000798316	0.000151825	0.000169933	0.000214272
0.000471402	0.000374145	0.000333565	0.000321976	0.000319235	0.000283408	0.000248659	0.000286982	0.000271867	0.000277936	0.000309279	0.000270026	0.000357532	0.000415783	0.000446678

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NON HOMOGENEOUS INTEREST RATE STRUCTURE IN A SEMI-MARKOV FRAMEWORK.°

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Summary

In this paper, we present a new model useful to construct the term structure of implied forward rates in a stochastic environment.. The stochastic process used to construct the interest rate structure will be the Non-Homogeneous Semi-Markov Process (NHSMP) as the financial operations are supposed non-homogeneous in time. It is to precise that a discrete time framework will be used in the paper. There are two reasons to apply discrete time non-homogeneous semi-Markov process (DTNHSMP) instead that the continuous one. The first is that in a previous paper was proved that discretising the evolution equation of a continuous time NHSMP by means of the simplest quadrature formula the DTNHSMP is obtained and if the time interval of DTNHSMP tends to 0 then is obtained the continuous case. The second is that the to solve DTNHSMP evolution equation is not a difficult task, as the solution of the continuous time can be obtained analytically only in same special cases; in the other cases to find the solution it is necessary to work numerically and the most simple way to obtain the numerical solution of the process evolution equation is the one by which the DTNHSMP is obtained by the continuous time NHSMP.
The assumptions that are necessary to apply the model are the ones for the SMP process to real problems. This means that we can apply this model to get a "physical measure" fitting the real data without the more restrictive assumptions of "risk neutral measure" in view to really explain one given interest rate market. Morever, it is clear that it is this physical measure which is useful for the actuarial applications.
Finally, we think that this more general approach can be used in other financial applications.

Résumé

Cet article présente un nouveau modèle pour la structure à terme des taux d'intérêt forward dans un environnement stochastique.
Ce modèle se base sur les processus semi-markoviens non-homogènes (en abrégé PSMNH) et suppose que les opérations financières sont non-homogènes dans le temps, avec une échelle des temps discrète.
Cette restriction est justifiée pour deux raisons : la première, par le fait que nous avons montré précédemment que la discrétisation temporelle des équations intégrales des probabilités de transition pour les PSMNH par la méthode de quadrature conduit à des solutions convergant vers celle en temps continu; la deuxième est que la résolution des équations en temps discret est faisable sans trop de difficultés de façon générale alors qu' une forme explicite de la solution en temps continu n'existe que pour des cas très particuliers.
Ce modèle a pour ambition de travailler avec la mesure probabilité "physique" gérant la structure des taux qui pourra être ajustée par le modèle développé et non la mesure de probabilité "risque neutre" servant uniquement à l'évaluation des produits dérivés sous l'hypothèse de l'AOA.
De plus, c'est cette mesure" physique" qui est utile pour les applications actuarielles.
Ainsi, le but de cette modélisation est d'approcher la "vraie" structure des taux et donner lieu à de nouvelles appliactions en finance.