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NON HOMOGENEOUS INTEREST RATE STRUCTURE IN A SEMI-MARKOV FRAMEWORK

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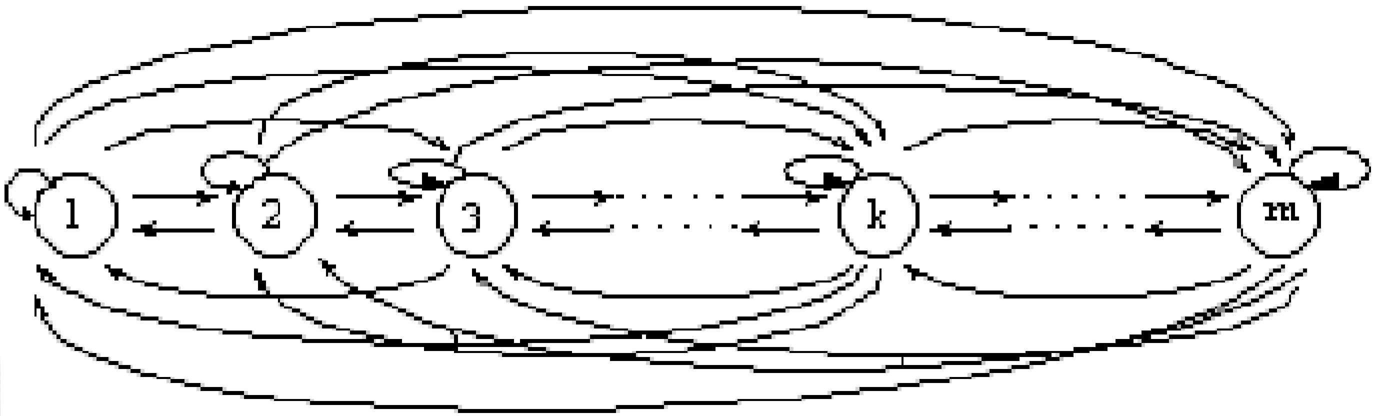
A New Millennium. A New Challenge for Actuaries

A stochastic interest rate model.



The Discrete Time non-Homogeneous SMP

- (X_n, T_n) non-homogeneous markovian renewal process
- $Q_{ij}(s,t) = P[X_{n+1} = j, T_{n+1} \le t \mid X_n = i, T_n = s]$
- $p_{ij}(s) = \lim_{t \to \infty} Q_{ij}(s,t)$; $i, j \in E$, $s, t \in N$
- $S_i(s,t) = P[T_{n+1} \le t \mid X_n = i, T_n = s]$. $S_i(s,t) = \sum Q_{ij}(s,t)$.

The Discrete Time non-Homogeneous SMP

$$b_{ij}(\mathrm{s},t) = \mathrm{P}[\, X_{n+1} = j,\ T_{n+1} = t \mid X_n = i,\ T_n = \mathrm{s}]$$

$$G_{ij}(\mathrm{s},t) = \mathrm{P}[\, T_{n+1} \leq t \mid X_n = i,\ X_{n+1} = j,\ T_n = \mathrm{s}].$$

$$\phi_{ij}(s,t) = \mathrm{P}[\, Z_t = j \mid \mathrm{Z}_s = i \,]$$

$$\phi_{ij}(s,t) = \delta_{ij}(1 - S_i(s,t)) + \sum_{\beta=1}^{m} \sum_{\vartheta=1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t)$$

Semi-Markov stochastic interest rate

$E = \{\rho_1, \rho_2, \ldots, \rho_m\}$ System states (interest rates)

- Semi-Markov transition probabilities $\phi_{ij}(s,t)$
- probability to remain in the starting state $\delta_{ij}(1 - S_i(s,t))$
- probability to stay at *t* in the state *j* with at least one transition $\sum_{\beta=1}^{m} \sum_{\vartheta=1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t)$

Stochastic discount factors

- Uni-periodical random discount factor

$$V_i(s,\theta) = \left(1 + \Gamma_i(s,\theta)\right)^{-1}$$

- random discount factor related to

$$A_i(s,h) = \prod_{\theta=s+1}^{h} V_i(s,\theta)$$

- independence hypothesis among uni-periodical discount factor

Mean discount factor computation

$$E(V_i(s,\theta)) = \sum_{j=1}^{m} \phi_{ij}(s,\theta)(1+\rho_j)^{-1}$$

$$\upsilon_i(s,h) = E(A_i(s,h)) = \prod_{\theta=1}^{h} E(V_i(s,\theta))$$

Variance computation

$$\sigma^2(V_i(s,\theta)) = \sum_{j=1}^{m} \phi_{ij}(s,\theta)(1+\rho_j)^{-2} - \left(\sum_{j=1}^{m} \phi_{ij}(s,\theta)(1+\rho_j)^{-1} \right)^2$$

$$\sigma^2\left(A_i(s,h)\right) = \sum_{\tau=1}^{k} \sum_{q=1}^{\binom{k}{\tau}} S_{C_q} M_{D_q}, \; k = h - s$$

Variance computation

$$S_{C_q} = \prod_{r=1}^{\tau} \sigma_i^2(s, s+\zeta_r), \; (\zeta_1, \ldots, \zeta_\tau) = C_q,$$

$$M_{D_q} = \begin{cases} \prod_{r=1}^{k-\tau} \mu_i^2(s, s+\eta_r), \; (\eta_1, \ldots, \eta_{k-\tau}) = D_q, & \text{if } \tau < k \\ 1, & \text{if } \tau = k \end{cases}$$

$C_q \in \mathbf{C}(k,\tau)$, $\mathbf{C}(k,\tau)$ τ–combinations of the set $\{1, \ldots, k\}$

$$C_q \cup D_q = \{1, \ldots, k\}$$

An applicative example (1)

state 1 = .03, state 2 = .035, state 3 = .04,
state 4 = .045, state 5 = .05, state 6 = .055,
state 7 = .06, state 8 = .065, state 9 = .07,
state 10 = .075, state 11 = .08, state 12 = .085,
state 13 = .09, state 14 = .095, state 15 = .10.

- Ten years time horizon

An applicative example (2)

- necessary to evaluate:

$\mathbf{P}(s)$ embedded Markov chain in DTHSMP

$\mathbf{G}(s,t)$ waiting time increasing d. f.

- Matrices filled up by pseudorandom numbers

The algorithm

- First step matrix $\mathbf{Q}(s,t)$
- Second step matrix $\mathbf{B}(s,t)$
- Third step matrix $\mathbf{S}(s,t)$
- Last step solves the evolution equation

Mean interest rates starting at time 0

0.03	0.05	0.075	0.01
0.0300000	0.0500000	0.0750000	0.1000000
0.0313117	0.0508937	0.0749862	0.0976678
0.0325927	0.0516856	0.0745601	0.0951265
0.0340818	0.0520527	0.0739198	0.0929551
0.0358741	0.0531727	0.0740286	0.0912346
0.0380755	0.0541028	0.0726681	0.0893618
0.0401347	0.0554276	0.0714897	0.0870258
0.0424678	0.0561730	0.0711308	0.0837965
0.0451926	0.0573675	0.0698162	0.0815721
0.0482788	0.0587244	0.0689290	0.0781019
0.0519793	0.0598774	0.0669042	0.0740946

Uni-periodical discount factors starting at time 0

0.03	0.05	0.075	0.01
0.970874	0.952381	0.930233	0.909091
0.969639	0.951571	0.930245	0.911022
0.968437	0.950854	0.930613	0.913137
0.967041	0.950523	0.931168	0.914951
0.965368	0.949512	0.931074	0.916393
0.963321	0.948674	0.932255	0.917969
0.961414	0.947483	0.933280	0.919941
0.959262	0.946815	0.933593	0.922682
0.956761	0.945745	0.934740	0.924580
0.953945	0.944533	0.935516	0.927556
0.950589	0.943505	0.937291	0.931017

Mean discount factors from 0 to *t*

0.03	0.05	0.075	0.01
1	1	1	1
0.9708738	0.9523810	0.9302326	0.9090909
0.9413970	0.9062582	0.8653437	0.8282022
0.9116829	0.8617196	0.8053004	0.7562617
0.8816351	0.8190841	0.7498702	0.6919421
0.8511026	0.7777301	0.6981845	0.6340911
0.8198851	0.7378123	0.6508859	0.5820757
0.7882490	0.6990648	0.6074588	0.5354755
0.7561375	0.6618848	0.5671192	0.4940739
0.7234432	0.6259742	0.5301090	0.4568108
0.6901248	0.5912532	0.4959253	0.4237177

Variance of mean discount factors

0.03	0.05	0.075	0.01
0	0	0	0
0.000028757	0.000023692	0.000016739	0.000058396
0.000083111	0.000074340	0.000053435	0.000141574
0.000163436	0.000128232	0.000097741	0.000224309
0.000259706	0.000203250	0.000143988	0.000297631
0.000370844	0.000283360	0.000192727	0.000359701
0.000489140	0.000361198	0.000239530	0.000411662
0.000610489	0.000433445	0.000282726	0.000454353
0.000731036	0.000502872	0.000323659	0.000481798
0.000840324	0.000562743	0.000357570	0.000499425
0.000935350	0.000614671	0.000390765	0.000508468

Mean interest rates starting at time 4

0.03	0.05	0.075	0.01
0	0	0	0
0	0	0	0
0	0	0	0
0.0300000	0.0500000	0.0750000	0.1000000
0.0326482	0.0510495	0.0742183	0.0971066
0.0353426	0.0523266	0.0740331	0.0944975
0.0375709	0.0539074	0.0730120	0.0909115
0.0406571	0.0553341	0.0721907	0.0887579
0.0443856	0.0562587	0.0709398	0.0859190
0.0472434	0.0577495	0.0703910	0.0824350
0.0516196	0.0594015	0.0689016	0.0771291

Uni-periodical discount factors starting at time 4

0.03	0.05	0.075	0.01
0	0	0	0
0	0	0	0
0	0	0	0
0.9708738	0.9523810	0.9302325	0.9090909
0.9683840	0.9514299	0.9309094	0.9114885
0.9658640	0.9502753	0.9310700	0.9136613
0.9637896	0.9488500	0.9319560	0.9166646
0.9609313	0.9475673	0.9326699	0.9184779
0.9575007	0.9467377	0.9337593	0.9208790
0.9548879	0.9454034	0.9342380	0.9238430
0.9509142	0.9439291	0.9355398	0.9283938

Mean discount factors from 4 to *t*

0.03	0.05	0.075	0.01
1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1
0.9708737	0.9523810	0.9302326	0.9090909
0.9401786	0.9061238	0.8659623	0.8286259
0.9080847	0.8610670	0.8062715	0.7570834
0.8752025	0.8170234	0.7514096	0.6939916
0.8410095	0.7741847	0.7008171	0.6374159
0.8052672	0.7329498	0.6543945	0.5869829
0.7689399	0.6929332	0.6113602	0.5422801

Conclusions

- stochastic interest rate by means of SMP
- "physical measure" fitting real data
- "risk neutral measure" more restrictive assumptions
- more general approach other financial applications

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