Comments about CIR Model as a Part of a Financial Market

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Abstract. We analyze the following problems concerning Cox, Ingersoll & Ross model: Linear risk premiums, pricing defaultable bonds in a structural approach, and asset options pricing with CIR as a short rate. The last two problems are closely related to price bonds in the Longstaff double square root model.

Solutions are given in terms of the Laplace transform and to avoid complicated formulas, we shall give corresponding references.

Key words: Martingales, pricing, local time, Bessel processes.

JEL Classification: E43

Introduction

The Cox, Ingersoll & Ross Model $r(t)$ for interest rates was constructed in 1980. Since then it has been the object of many even recent studies and extensions.

Little is known about how to place it in a financial market: usually there are assumed so called Risk Premiums proportional to $\sqrt{r(t)}$; and linear risk premiums are considered inadmissible cf. Rogers (1995).

However, if one wants to work with CIR Model in Risk Neutral World (RNW) —the only world that can be observed for interest rates (IR) alone—, then it results (in some cases) that linear risk premiums are allowed. In this case the IR in the physical world follow a different model. We shall analyze this question in section 1.

In section 2, we show how to solve the problem of pricing bonds in double square root Longstaff model. The wrong solution was presented by Longstaff (1989) and a simple version was solved by Beaglehole & Tenney (1992).

In the analysis of the original double square root model the local time should appear (ommited by Longstaff). In section 3, we offer a short discussion of problems that are essentially equivalent or similar to the Longstaff one, with CIR as a short rate:

i) Pricing of options on assets.

ii) Pricing of default bonds in Merton's Structural Approach.

In both cases we assume that Asset Prices follow geometric Brownian motion, and are correlated with IR. The problem *ii*) was solved by Wang (1999) assuming independency.

In section 4, we present a simplified approach to pricing bonds if default can occur any time. We only know how to solve one part of our problem.

1 Linear Risk Premiums

We stress that everything we can say about interest rates is deduced from prices of bonds or other interest rate derivatives, and these are priced in so called Risk Neutral World (RNW). In other words, dealing only with interest rates the RW (Real World) is non-existent or at least can not be observed. Therefore in this case the concept of risk premium is dim.

If one wants to consider the RW for interest rates, this RW must be taken from assets.

We proceed with the construction of the RW for IR (interest rates) such that in the RNW the IR follow the CIR model.

For the CIR model in RW, so called linear risk premiums are inappropriate cf. Cox et al. (1985) Rogers (1995).

We will clarify what can be done and what can not in one dimensional financial market driven by Brownian motion, and asset prices that in the RW (under the law P) follow geometric Brownian motion:

$$dS(t) = S(t)\left[\sigma dW(t) + \mu dt\right].$$

Set (discounted prices) $Z_t = \frac{S_t}{\beta_t}$, where

$$\beta_t = e^{\int_0^t r(s)ds},$$

and $r(s)$ is the spot IR in the RW. Now,

$$dZ(t) = Z(t)\left[\sigma dW(t) + (\mu - r(t))dt\right].$$

The RNW is defined as the probability law Q ($Q \sim P$), $t \leq T$ that under Q

$$dZ(t) = \sigma Z(t) dW^*(t).$$

It can be shown that if $r(t)$ is CIR (in real world) then such Q does not exist. An easy argument is based on explosion until $T = 1$ of the process defined by:

$$dx(t) = dW(t) + x^2 dt.$$

This argument was explained to me by Chris Rogers in 1997. Also cf. Revuz & Yor (1998) p. 384.

But what we really want is CIR in the RNW. We prove the following:
Theorem 1 If under P

$$dr(t) = 2\widetilde{\sigma}\sqrt{r(t)}dW(t) + \left[\delta + 2\frac{\mu}{\sigma}\widetilde{\sigma}\sqrt{r(t)} - \left(2\beta\, r(t) + \frac{2\widetilde{\sigma}}{\sigma} r^{\frac{3}{2}}(t)\right)\right] dt, \qquad (1)$$

then for any $T > 0$ exists $Q \sim P$, for the process considered until time T such that under Q the interest rates follow:

$$dr(t) = 2\widetilde{\sigma}\sqrt{r(t)}\, dW^*(t) + (\delta - 2\beta\, r(t))\, dt,\ \widetilde{\sigma}, \delta, \beta > 0$$

Proof. Set $\widetilde{\sigma} = 1 = \sigma$.

Because the law $Q = Q^\beta$ is equivalent to the law Q^0 of the corresponding BESQ^δ process, ($\beta = 0$), and similarly $P = P^\beta \sim P^0$, then it is sufficient to prove the equivalence of P^0 and Q^0.

Applying Itô-Tanaka to $f(x) = |x|^{\frac{3}{2}}$ and occupation times formulas together with the fact that for BESQ^δ, $L_t^a = 0$ for $a \leq 0$ and $\delta > 0$, we have that under Q^0 the exponential local martingale

$$\exp\left\{-\int_0^t X(s)dW^*(s) - \frac{1}{2}\int_0^t X^2(s)ds\right\} =$$

$$\exp\left\{-\frac{X_t^{\frac{3}{2}}}{3} + \frac{X_0^{\frac{3}{2}}}{3} + \frac{1}{2}(\delta + 1)\int_0^t \sqrt{X(s)}ds - \frac{1}{2}\int_0^t X^2(s)ds\right\},$$

is bounded by a constant $k(T)$.

Now easily

$$\eta_t = \mathcal{E}\left[\int_0^\cdot (X(s) - \mu)\, dW(s)\right]_t$$

is a true martingale.

Moreover $\eta_t > 0$, Q^0 almost everywhere. We conclude that $Q^0 \sim P^0$, and $Q \sim P$ on $\mathcal{F}_T$. A similar proof works if in the RW

$$dS(t) = S(t)\left[(\lambda + 1)(r(t) + \mu)\, dt + \sigma dW(t)\right], \text{ for any } \lambda < 0.$$

Namely, that there exist the corresponding model in RW such that in the RNW the IR follow CIR model. We have just proved that in some cases the linear risk premiums for CIR model are admissible, of course in our formulation of the problem.

Important Remark

If one have "extra degrees of freedom", for example:

$$dS(t) = [\sigma_1 dW_1(t) + \sigma_2 dW_2(t) + \mu dt]\, dS(t) \tag{2}$$

and in the RW $r(t) = r_1(t) \oplus r_2(t)$, independent sum of CIR models driven by W_1 and W_2 respectively, that one can drop the drift for discounted prices. This occurs because

$$\mathcal{E}\left(\int_0^{\cdot} r_1(s) dW_2(s)\right)_t$$

is clearly true martingale (simply take the conditional expectation).

Therefore one can drop the drift term applying Girsanov Theorem twice. Clearly the incompatibility of CIR in RW and RNW persists.

Assume now that $r_1 \oplus r_2$ can be reduced to one factor model. This occurs if $\widetilde{\sigma}_i = \widetilde{\sigma}_2, \beta_1 = \beta_2$ in $dr_i(t) = \widetilde{\sigma}\sqrt{r_i(t)} dW_i(t) + (\delta_i - \beta_i r_i(t))\, dt$ by the Pythagoras Theorem, cf. Revuz & Yor(1998).

In this case $dr(t) = \widetilde{\sigma}\sqrt{r(t)} dW^*(t) + (\delta_1 + \delta_2 - \beta r(t))\, dt$ $\widetilde{\sigma} = \sigma_i, \beta = \beta_i$.

Rewriting (2) as $dS(t) = \left(\sqrt{\sigma_1^2 + \sigma_2^2} dW(t) + \mu dt\right) S(t)$ we conclude that in this case there exists an equivalent martingale measure for discounted prices and CIR model in RW. Here W and $W^*$are correlated in a complicated way.

2 Longstaff Model

Note:

The idea of this presentation is not to come to terminal closed formulas, but only to show how to solve the problem of bonds pricing in terms of their Laplace transform.

In (1989) Longstaff constructed the so called double square root model defined in Risk Neutral World by:

$$dr(t) = 2\sqrt{r(t)}\, dW(t) + \left(1 - \kappa\sqrt{r(t)} - 2\lambda r(t)\right) dt,\ \kappa, \lambda > 0$$

Note a similarity with (1)!

In this study, for simplicity sake, we set $\sigma = 1$ in the original model $\widetilde{r}(t) = \sigma r(t)$. Clearly:

$$\begin{aligned} r(t) &= y^2(t), \text{ where} \\ dy(t) &= dW^*(t) - \left(\lambda y(t) + \frac{\kappa}{2} \operatorname{sgn} y(t)\right) dt. \end{aligned}$$

In 1992 Beaglehole & Tenney showed that Longstaff's *wrong* formula for Bond Prices in his model gives the correct bond prices in the case of:

$$r_1(t) = y_1^2(t) \text{ and } dy_1(t) = dW(t) - \left(\lambda y(t) + \frac{\kappa}{2}\right) dt.$$

We will show first how to calculate:

$$E_x\left(e^{-\int_0^t r_1(s)ds}\right) \tag{3}$$

by very simple and transparent martingale method.

We start with an obvious fact:

Let $f(s)$ and $g(s)$ are differentiable functions,
then for any t

$$E\left[e^{\int_0^t f(s)W(s)+g(s)dW(s)-\frac{1}{2}\int_0^t (f(s)W(s)+g(s))^2 ds}\right] = 1.$$

In the sequel we shall use the notation $\propto$ for "=" if multiplied by a deterministic function.

By Girsanov Theorem:

$$\begin{aligned}(3) \quad &\propto \quad E_x\left(e^{-\left(\frac{\lambda^2}{2}+1\right)\int_0^t W^2(s)ds - \frac{\lambda\kappa}{2}\int_0^t W(s)ds - \frac{\lambda}{2}W^2(t) - \frac{\kappa}{2}W(t)}\right) \\ &\propto \quad E_x\left(e^{\int_0^t (f(s)W(s)+g(s))dW(s) - \frac{1}{2}\int_0^t (f(s)W(s)+g(s))^2 ds}\right)\end{aligned}$$

if and only if in $(0, t)$

$$\begin{aligned} f'(s) + f^2(s) &= \lambda^2 + 2 \\ g(s)f(s) + g'(s) &= \frac{\lambda\kappa}{2}, \text{ and} \\ f(t) &= -\lambda, \\ g(t) &= -\frac{\kappa}{2}. \end{aligned}$$

Therefore the problem of bonds pricing in the Beaglehole & Tenney model is reduced to entirely elementary calculations.

Note:

The same matching procedure (being simply particular cases) can be used in calculations of:

$$E\left(e^{-\int_0^t X^2(s)}\right), \text{ where}$$

i) X is Ornstein-Uhlenbeck process.

ii) X is Brownian Motion with drift, compare with Yor (1992).

In both cases we use Girsanov theorem twice. Firstly, changing the measure into Brownian Motion measure, and secondly matching two functions.

This matching procedure does not work in the original Longstaff model, it means for calculations of:

$$P(0,t) = E\left(e^{-\int_0^t r(s)ds}\right).$$

An application of Girsanov theorem leads to

$$P(0,t) \propto E_x\left[e^{-\left(\frac{\lambda^2}{2}+1\right)\int_0^t W^2(s)ds-\frac{\kappa\lambda}{2}\int_0^t |W(s)|ds-\frac{\lambda}{2}W^2(t)-\frac{\kappa}{2}\left(|W(t)|-L_t^0\right)}\right].$$

The positive term $+\kappa L_t^0$ makes impossible the direct Feynman-Kac approach to the calculation of the Laplace transform of $P(0,t)$ cf. Karatzas (1991), or equivalently to calculations of $P(0,T)$ where T is an exponential random variable independent of the process.

We shall calculate $P(0,T)$ conditioning with respect to $W(T)$ and L_T as in Yor (1992) proposition 3.2. But in this proposition the process starts at zero, and not at arbitrary x. If one wants to solve the problem reducing first the process to zero, one should know the density of the hitting time of y for the Ornstein-Uhlenbeck process starting at x. This is not an easy problem if $y \neq 0$.

Leblanc et al. (2000) claimed that they solved this problem for general y. But their calculations are erroneous. Neither $\int_0^t W^2(s)ds$ nor the $\mathrm{BES}^{(3)}$ are invariant under translations! They use the translation $E_x\left[\exp\left(\int_0^{T_y} W^2(s)\,ds\right)\right] = E_{x-y}\left[\exp\int_0^{T_0} W^2(s)\,ds\right]$ which of course is incorrect. More discussions in this setting can be found in Göing (1997).

Writing $P(0,T) = E_x\left(e^{-A(T)}\right)$, and T_0 the hitting time of zero by $W(t)$ we have:

$$P(0,T) = E_x\left(e^{-A(T)}; T_0 \geq T\right) + E_x\left(e^{-A(T)}; T_0 < T\right) = I + II.$$

Assume for example that $x > 0$.

Because in the first term local time and absolute value do not appear, it is easy to obtain the analytical expression for "I" by changing the initial point, the law into Ornstein-Uhlenbeck one, the parameter of the exponential distribution, and using the corresponding formula from Borodin & Salminen (1996) p. 412.

Now, by an elementary argument

$$\begin{aligned} E_x\left(e^{-A(T)}, T_0 < T\right) &= E_x\left(e^{-A(T_0)}, T_0 < T\right) E_0\left(e^{-A(\tau)}\right) \\ &= E_x\left(e^{-\widetilde{\widetilde{A}}(T_0)},\ T_0 < T\right) E_0\left(e^{-A(\tau)}\right), \text{ where} \\ \widetilde{\widetilde{A}}(t) &= -\left(\frac{\lambda^2}{2}+1\right)\int_0^t W^2(s)ds - \frac{\lambda\kappa}{2}\int_0^t W(s)ds, \end{aligned}$$

and τ is another exponential time (the same parameter) independent of the process.

Therefore, the first expectation in the product can be expressed as:

$$\frac{E_x\left(e^{-\widetilde{\widetilde{A}}(T)}, T > T_0\right)}{E_0\left(e^{-\widetilde{\widetilde{A}}(\tau)}\right)}$$

and we can use the former procedure.

Now we shall calculate $E_0\left(e^{-\widetilde{A}(\tau)}\right)\left(\tau \sim \exp\frac{\theta^2}{2}\right)$ where

$$\widetilde{A}(t) = -\left(\frac{\lambda^2}{2} + 1\right)\int_0^t W^2(s)ds - \frac{\lambda\kappa}{2}\int_0^t |W(s)|\, ds.$$

By the proposition 3.2 from Yor (1992),

$$E_0\left(e^{-\widetilde{A}(\tau)} \mid l_\tau = l, W_\tau = a\right) \propto E\left[e^{-\widetilde{A}(\tau_l) - \frac{\theta^2}{2}\tau_l}\right] E_a\left(e^{-\widetilde{A}_{T_0} - \frac{\theta^2}{2}T_0}\right)$$

and τ_l is the inverse local time at zero.

Therefore we have to calculate:

$$\int_0^\infty dl\, e^{\frac{\kappa}{2}l} E_0\left[e^{-\theta^2\frac{\tau_l}{2} - \widetilde{A}(\tau_l)}\right]\int_{-\infty}^\infty E_a\left(e^{-\theta^2\frac{T_0}{2} - \widetilde{A}(T_0)}\right) e^{-\frac{\lambda}{2}a^2 - \frac{\kappa}{2}|a|}da.$$

Calculations of $E_a\left(e^{-\theta^2\frac{T_0}{2} - \widetilde{A}(T_0)}\right)$ by the same argument as calculations of I, reduce the problem to the formula 2.0.1 page 429 from Borodin & Salminen. On the other hand this formula represents the solution of the equation:

$$\frac{1}{2}v''(a) = \left(\frac{\theta^2}{2} + f(a)\right)v(a), 0 \leq v(a) \leq 1\ ,\ v(0) = 1, \tag{4}$$

where

$$f(a) = \left(\frac{\lambda^2}{2} + 1\right)a^2 + \frac{\lambda\kappa}{2}|a|\,.$$

For further calculations we will need another solution $h(m)$ of the equation (4) written as

$$h(m) = mu(m),\ \text{where } h(m) = v(m)\int_0^m \frac{1}{v^2(s)}ds, \tag{5}$$

cf. Jeanblanc et al. (1996).

Elementary calculations show that $u(0) = 1,\ u(m) > 1$, for $m > 0$

The final part (the most interesting from the point of view of stochastic analysis) is the calculation of

$$\int_0^\infty dl\, e^{\frac{\kappa l}{2}} E_0 \left(e^{-\theta^2 \frac{\tau_l}{2} - \widetilde{A}(\tau_l)} \right)$$

(note that of course the integral does exist). There are two possibilities of calculations of this integral:

i) Calculations in terms of Ray-Knight theorem.

By occupation time and Ray-Knight theorems, cf. Revuz & Yor (1994), we have

$$\begin{aligned}
\int_0^{\tau_l} ds f(W(s)) &= \int_{-\infty}^{+\infty} f(x) L_{\tau_l}^x dx \\
&= \int_{-\infty}^{+\infty} \left(\left(\frac{\lambda^2}{2} + 1 \right) x^2 + \frac{\lambda\kappa}{2} |x| \right) L_{\tau_l}^x dx \\
&= \int_0^\infty \left(\left(\frac{\lambda^2}{2} + 1 \right) x^2 + \frac{\lambda\kappa}{2} x \right) (X_1(x) + X_2(x))\, dx \\
&= \int_0^{+\infty} g(x) \left(X_1(x) + X_2(x) \right) dx
\end{aligned}$$

where $X_1,\ X_2$ are two independent squared Bessel processes of dimension zero starting at l. Putting $\frac{\theta^2 \tau_l}{2}$ inside the integral and applying Pitman & Yor formula for squared Bessel processes we have that:

$$E_0 \left(e^{-\frac{\theta^2 \tau_l}{2} - \widetilde{A}(\tau_l)} \right) = e^{l v^+(0)},$$

being v^+ right hand derivative at zero, of the function v defined by formula (4).

Therefore the solution can be written as

$$\int_0^\infty \exp\left(\frac{\kappa l}{2} \right) \exp\left(l v^+(0) \right) = \frac{1}{-v^+(0) - \frac{\kappa}{2}} .$$

ii) The second way of calculations is given in terms of the excursion theory and this will lead to more explicite formula. We follow closely the general approach from Yor (1994). Results easily from the multiplicative formula for excursions that

$$\frac{\theta^2}{2} \int e^{\kappa l} E_0 \left(e^{-\theta^2 \frac{\tau_l}{2} - \widetilde{A}(\tau_l)} \right) dl = \frac{\theta^2}{2(D_\theta - \frac{\kappa}{2})} ,$$

where

$$D_\theta(f) = \int \mathbf{n}(d\varepsilon) \left[1 - e^{-\frac{\theta^2}{2} V - \int_0^V ds f(\varepsilon_s)} \right]$$

cf. Yor (1994) pages 69 and 75.

We know a priori that $D_\theta > \frac{\kappa}{2}$.

Now D_θ is given by:

$$\int_0^\infty \frac{dm}{m^2}\left(1-\frac{1}{u(m)}\right)^2.$$

This, because of Williams representation of excursions, $\frac{1}{m^2}$ is the 'law' of the maximum and conditioning we have to calculate

$$E_0^{(3)}\left(e^{-\frac{\theta^2}{2}T_m-\int_0^{T_m} dt\, f(R(t))}\right), \text{where}$$

$R(t)$ is BES^3 process starting at zero, T_m is the hitting time of m, and therefore $u(m)$ is given by (5).

3 Related Problems

In this section we review briefly another problems concerning CIR. These problems are closely related to the Longstaff model.

a. *Default bonds in the structural Merton approach.* For discussion we refer to the paper by Wang(1999), who solved the problem in the case of:

   i. Default occuring at the time t (the horizon).

   ii. The value on the firm follows geometric Brownian motion *independent* of the CIR interest rates.

   In our solution we do not assume independency. We solve the problem in this setting if we know how to price options on assets with CIR as a short rate.

b. *Options on assets with CIR as a short rate.* Assume that an asset follows geometric Brownian motion driven by $W(t)$, and interest rates follow $r_1(t) \oplus r_2(t)$, where $\oplus$ stands for the independent sum.

Here $r_2(t)$ is CIR, and $r_1(t)$ is one dimensional CIR model driven by $W(t)$.

The analytical solution of pricing options is equivalent to the knowledge of the joint law of $\int_0^t r_1(s)\,ds$ and $W(t)$. To calculate the Laplace transform of

$$E\left(e^{-\lambda\int_0^t r_1(s)ds+\mu W(t)}\right)$$

we use Girsanov theorem and the problem is equivalent to price bonds in the Longstaff model. Note that even in Wang's case, one has to invert Laplace Transform!

4 Dynamical approach to default

In this section we present a very simple approach to pricing bonds if a default occurs when the value of the firm falls below a given level anytime between 0 and t.

We are interested in the computation of

$$\widetilde{P} = E\left(\exp\left(-\frac{1}{2}\int_0^1 W^2(s)\,ds\right) I\left\{\underset{s\in[0,1]}{\forall}\beta(s)+\varepsilon s \geq -d\right\}\right),$$

where $\beta(s) = \int_0^s \operatorname{sgn} W(u)\, dW(u)$, $\varepsilon = \pm 1$.

The motivation is that $W^2(s)$ is BESQ[1] driven by $\beta(s)$. We do not know how to calculate this expectation if $\varepsilon = +1$.

On the other hand, if $\varepsilon = -1$ we are able to compute this expectation applying *brute force* of conditional expectations. It is easy to see that in this case if default occurs, it occurs also after g_1, the last zero of $W(t)$ before 1.

Given $g_1 = u$,

$$\widetilde{P} = E\left(e^{-\frac{1}{2}\int_0^u W^2(s)ds} \mid W(u)=0\right) *$$

$$\int E\left[e^{-\frac{1}{2}\int_u^1 W^2(s)ds}\, I\left\{\underset{(u,1]}{\forall} W(s) \geq s+u-d+l\right\} \mid W(s)>0,\, s\in(u,1]\right] *$$

$$f_{L_u|g_1}(l \mid u)dl,$$

being L the local time at zero.

The first term is explicite, and the conditional density is known, cf. Revuz & Yor (1998). Condition now with respect to $W(1) = a$, and invert time.

We have to calculate

$$E_a\left(\exp\left(-\frac{1}{2}\int_0^{T_0} W^2(s)ds\right) I\left\{\underset{s\in[0,\tilde{u}]}{\forall} W(s) \geq -s-\widetilde{d} \mid T_0=\widetilde{u}\right\}\right) \qquad (6)$$

where $\widetilde{d} = d-1-u-l$, $\widetilde{u} = 1-u$, and T_0 is the hitting time of zero (If $-\widetilde{d} < 0$ default does not occur).

Now change the law into the one of Ornstein-Uhlenbeck process $X(s)$. It remains to calculate

$$E_a\left(I\left\{\underset{s\in[0,\tilde{u}]}{\forall} X(s) > -s-\widetilde{d} \mid T_0=\widetilde{u}\right\}\right). \qquad (7)$$

Let T be the hitting time of the line $-s-\widetilde{d}$ by $X(s)$ starting at a. A manipulation of densities for $s<\widetilde{u}$

$$\begin{aligned} f_{T|T_0}^{(a)}\left(s \mid \widetilde{u}\right) &= \frac{f_{T_0|T}^{(a)}\left(\widetilde{u} \mid s\right) f_T^{(a)}\left(s\right)}{f_{T_0}^{(a)}\left(\widetilde{u}\right)} \\ &= \frac{f_{T_0}^{\left(-s-\tilde{d}\right)}\left(\widetilde{u}-s\right) f_T^{(a)}\left(s\right)}{f_{T_0}^{(a)}\left(\widetilde{u}\right)} \end{aligned}$$

allows to express (6) using known terms.

For the first hitting time of a linear barrier by Ornstein-Uhlenbeck process see for example Shepp (1969).

5 Final Remarks

We have analyzed problems concerning CIR Model for interest rates placed in a financial market and correlated with asset prices.

We were particularly interested in the joint law of $W(t)$ and $\int_0^t r(s)ds$ where $r(s)$ was driven by $W(t)$.

Finding its Laplace transform is equivalent to pricing bonds in Longstaff's Double Square Root Model. The main difficulty was the appearance of a local time (omitted by Longstaff) and this forced calculation of bonds expiring in exponential time. Because solutions are given by complicated formulas, they can not be put into practice.

Avoiding local time was the spirit of the simplified model by Beaglehole & Tenney.

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