CIR model as a part of a financial market

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Goal: Correction of many errors or imprecisions concerning CIR

Mostly negative results:

Coorect formulas too complicated to be put into practice

$$dW^2(t) = 2W(t)dW(t) + dt$$

$$2W(t)dW(t) + dt = 2\sqrt{W^2(t)}\; d\beta(t) + dt$$

$$\beta(t) = \int_0^t \operatorname{sgn} W(s)dW(s)$$

1) $\sqrt{x^2} = x$

2) $\int_0^{T_y} W^2(s)ds,\ W_0 = x \overset{\text{Law}}{=} \int_0^T W^2(s)ds,\ W_0 = x - y$

3) Linear risk premiums impossible

Wrong problem solved (Correctly)

4) Solution of the problem: pricing asset options with CIR model

(Short rate)

independent of the asset.

1) Linear risk premiums impossible.

What is observed in financial markets?

1. RW.- Real world for assets

2. RNW,. Risk neutral world. If one works with IR alone,

one can not observe RW.

To talk about risk premiums assets needed.

Assumption: One dimensional market.

$$dS(t) = S(t)\left[\sigma dW(t) + \mu dt\right]$$

$$dr(t) = 2\tilde{\sigma}\sqrt{r(t)}dW^*(t) + \left(\delta - 2\beta r(t)\right)dt, \tilde{\sigma}, \delta, \beta > 0$$

$$dZ(t) = Z(t)\left[\sigma dW(t) + \left(\mu - r(t)dt\right)\right]dt$$

$$dZ(t) = \sigma Z(t)dW^*(t)$$

But what we want is CIR on RNW!

Theorem: if in RW

$$dr(t) = 2\tilde{\sigma}\sqrt{r(t)}dW(t) + \left[\delta + 2\frac{\mu}{\sigma}\tilde{\sigma}\sqrt{r(t)} - \left(2\beta r(t) + \frac{2\tilde{\sigma}}{\sigma} r^{\frac{3}{2}}(t)\right)\right]dt$$

Then RNW exists and in RNW we have CIR.
Similar result if

$$dS(t) = S(t)\left[(\lambda+1)r(t) + \mu dt + \sigma dW(t)\right], \text{ for any } \lambda < 0.$$

2) Longstaff model for IR

$$dr(t) = 2\sqrt{r(t)}dW(t) + \left(1 - k\sqrt{r(t)} - 2\lambda r(t)\right)dt, \ \ k, \lambda > 0$$

$r(t) = y^2(t)$, where

$$dy(t) = dW^*(t) - \left(\lambda y(t) + \frac{k}{2}\operatorname{sgn} y(t)\right)dt.$$

Simplified version:

$$r_1(t) = y_1^2(t)$$

and $$dy_1(t) = dW(t) - \left(\text{ë}y(t) + \frac{k}{2} \right) dt$$

Calculations Easy

$$E\left[\exp\left(\int_o^t f(s)W(s)+g(s)dW(s)-\frac{1}{2}\int_0^t \left(f(s)W(s)+g(s)\right)^2 ds\right)\right]=1$$

$$\alpha\ E_x\left[\exp\left(-\left(\frac{\lambda}{2}+1\right)\int_0^t W^2(s)ds-\frac{\lambda k}{2}\int_0^t W(s)ds-\frac{\lambda}{2}W^2(t)-\frac{k}{2}W(t)\right)\right]$$

$$\alpha\ E_x\left[\exp\left(\int_0^t \left(f(s)W(s)+g(s)\right)dW(s)-\frac{1}{2}\int_0^t \left(f(s)W(s)+g(s)\right)^2 ds\right)\right]$$

if and only if in $(0,t)$

$$f'(s)+f^2(s)=\lambda^2+2$$

$$g(s)f(s)+g'(s)=\frac{\lambda k}{2}, \text{ and}$$

$$f(t)=-\lambda$$

$$g(t)=\frac{-k}{2}$$

$$P(0,t) \propto E_x\left[\exp\left(-\left(\frac{\lambda^2}{2}+1\right)\int_0^t W^2(s)ds - \frac{\lambda k}{2}\int_0^t |W(s)|ds - \frac{\lambda}{2}W^2(t) - \frac{k}{2}\left(|W(t)| - L_t^0\right)\right)\right]$$

We can only calculate $P(0,T)$, T exponential time independent of the process.

Very, Very Complicated...

3) Pricing Asset options=Pricing defaultable bonds in the Merton´s structural approach

$$\text{Interest rates} = r_1(t) \oplus r_2(t)$$

$$\text{Asset driven by } W(t)$$

$$r_1 - \text{one dimension CIR driven by } W(t)$$

$$r_2 - \text{CIR}$$

Laplace Transform

$$E\left(\exp(-\lambda \int_0^t r_1(s)ds + \mu W(t)) \right)$$

Equivalent to price bonds in the Longstaff model

Note : The law of $\int_0^t r(s)ds$ is not explicite.

Even in independent case one has to invert Laplace transform.

¿Want to apply CIR in this setting?

Think it twice!