ON RISK RESERVE CONDITIONED BY RUIN

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1. Notation and assumptions

Andersen's risk model comes from the i.i.d. $\{(Y_i, T_i)\}_{i\geqslant 1}$, where

- T_i are the interclaim times,
- Y_i are the amounts of claims

with the probability distribution function (p.d.f.)

$$B_{Y,T}(y,t) = \mathbf{P}\{Y_1 \leqslant y, T_1 \leqslant t\}.$$

These random vectors generate the **risk reserve process**

$$R_u(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \quad t \geqslant 0,$$

where

- $u > 0$ is the initial risk reserve,
- $c = (1+\tau)\mathbf{E}Y_1/\mathbf{E}T_1 > 0$ is the risk premium rate,
- $N(t) = \max\{n : \sum_{i=1}^{n} T_i \leqslant t\}$ (we put $N(t) = 0$ if $T_1 > t$).

Ruin occurs at time s as $R_u(s) < 0$ and the probability that ruin occurs within the time interval $(0, t]$ is

$$\psi(t, u) = \mathbf{P}\Big\{ \inf_{0<s\leqslant t} R_u(s) < 0 \Big\}.$$

Lundberg, F. *Approximerad framställning af sannolikhetsfunktionen. II. Återförsäkring av kollektivrisker.*, Almqwist & Wiksell, Uppsala, 1903.

Cramér, H. *Collective risk theory*, Jubilee volume of Forsäkringsbolaget Skandia, Stockholm, 1955.

Andersen, E. S. *On the collective theory of risk in case of contagion between the claims.*, Trans. XVth International Congress of Actuaries, New York, 1957, vol. II, 219 – 229.

Introduce

$$\begin{aligned}\psi(w; t, u) &= \mathbf{P}\Big\{ R_u(t) \leqslant w, \inf_{0<s\leqslant t} R_u(s) < 0 \Big\} \\ &= \mathbf{P}\Big\{ R_u(t) \leqslant w \mid \inf_{0<s\leqslant t} R_u(s) < 0 \Big\} \psi(t, u).\end{aligned}$$

Evidently, $\psi(+\infty; t, u) = \psi(t, u)$.

2. Approximations for $\psi(t, u)$

For $i = 1, 2, \ldots$ introduce i.i.d. random variables $X_i = Y_i - cT_i$;

$$S_n = \sum_{i=1}^{n} X_i, \quad V_n = \sum_{i=1}^{n} Y_i.$$

For the p.d.f. $B(x, y) = \mathbf{P}\{X_1 \leqslant x, T_1 \leqslant y\}$ and for a positive solution $\varkappa$ of the **Lundberg equation**,

$$\mathbf{E} \exp(\varkappa X_1) = 1,$$

introduce an **associate** p.d.f. by $\bar{B}(dx, dy) = e^{\varkappa x} B(dx, dy)$.

Introduce the associated sequence $\{(\bar{X}_i, \bar{T}_i)\}_{i \geqslant 1}$ of i.i.d. random vectors having the p.d.f. $\bar{B}(x, y)$, and

$$\bar{S}_n = \sum_{i=1}^{n} \bar{X}_i, \quad \bar{U}_n = \sum_{i=1}^{n} \bar{T}_i.$$

Introduce

- the **ladder index** $\mathcal{N} = \inf\{n : \bar{S}_n > 0\}$,
- the **ladder height** $\mathcal{H} = \bar{S}_{\mathcal{N}}$,
- the **ladder time point** $\mathcal{T} = \bar{U}_{\mathcal{N}}$.

Put $\bar{\nu}^{i,j} = \mathbf{E}\bar{X}_1^i\bar{T}_1^j$, $i,j = 0,1\ldots$ and

$$m_1 = \bar{\nu}^{0,1}/\bar{\nu}^{1,0},$$

$$D_1^2 = ((\bar{\nu}^{0,1})^2\bar{\nu}^{2,0} - 2\bar{\nu}^{1,0}\bar{\nu}^{0,1}\bar{\nu}^{1,1} + (\bar{\nu}^{1,0})^2\bar{\nu}^{0,2})/(\bar{\nu}^{1,0})^3,$$

$$C = \frac{1}{\varkappa\bar{\nu}^{1,0}} \exp\Big(- \sum_{n=1}^{\infty} \frac{1}{n}\big[\mathbf{P}(S_n > 0) + \mathbf{P}(\bar{S}_n \leqslant 0)\big]\Big).$$

Theorem. *Suppose that in the Andersen's risk model with $\tau > 0$ the characteristic function $\beta_{T,Y}(t_1,t_2)$ is absolutely integrable and $D_1 > 0$. Then*

$$\lim_{u\to\infty} \sup_{t\geqslant 0} \left| e^{\varkappa u}\psi(t,u) - C\Phi_{(m_1u, D_1^2u)}(t)\right| = 0.$$

von Bahr, B. *Ruin probabilities expressed in terms of ladder height distributions*, *SAJ*, 1974, vol. 57, 190 – 204.

Theorem 1. *Suppose that in the Andersen's risk model with $\tau > 0$ the characteristic function $\beta_{T,Y}(t_1,t_2)$ is absolutely integrable, $D_1 > 0$ and $\mathbf{E}T_1^3 < \infty$. Then, as $u \to \infty$,*

$$\sup_{t\geqslant 0} \Big| e^{\varkappa u}\psi(t,u) - C\Big(\Phi_{(m_1u, D_1^2u)}(t) - Q_1(t(u)) \times \varphi_{(m_1u,D_1^2u)}(t)\Big)\Big| = \overline{o}\big(u^{-1/2}\big),$$

where $t(u) = (t - mu)/(D_1u^{1/2})$,

$$Q_1(t) = \frac{1}{6}\chi_{(3,0)}(t^2-1) - \Big(\frac{\mathbf{E}\mathcal{H}\mathcal{T}}{\mathbf{E}\mathcal{H}} - \frac{\mathbf{E}\mathcal{H}^2\mathbf{E}\mathcal{T}}{(\mathbf{E}\mathcal{H})^2}\Big) + \Big(\frac{\mathbf{E}\mathcal{T}}{\varkappa\mathbf{E}\mathcal{H}} - \frac{\mathbf{E}\mathcal{T}e^{-\varkappa\mathcal{H}}}{1-\mathbf{E}e^{-\varkappa\mathcal{H}}}\Big),$$

and

$$\chi_{(3,0)} = \Big(\mathbf{E}\big(\bar\nu^{1,0}\bar T_1 - \bar\nu^{0,1}\bar X_1\big)^3(\bar\nu^{1,0})^{-4} -$$

$$- 3\mathbf{E}\big(\bar\nu^{1,0}\bar T_1 - \bar\nu^{0,1}\bar X_1\big)^2\mathbf{E}\big[\bar X_1(\bar\nu^{1,0}\bar T_1 - \bar\nu^{0,1}\bar X_1)\big](\bar\nu^{1,0})^{-5}\Big)D_1^{-2}.$$

Malinovskii, V. K. *Corrected normal approximation for the probability of ruin within finite time*, *SAJ*, 1994, 161 – 174.

Malinovskii, V. K. *Approximations and upper bounds on probabilities of large deviations in the problem of ruin within finite time*, *SAJ*, 1996, 124 – 147.

3. Approximations for $\psi(w,t,u)$

Put $\nu^{i,j} = \mathbf{E}Y_1^i T_1^j$, $\bar\nu^{i,j} = \mathbf{E}\bar X_1^i \bar T_1^j$, $i,j = 0,1\dots$ and introduce

$$m_1 = \bar\nu^{0,1}/\bar\nu^{1,0}, \quad m_2 = c - \nu^{1,0}/\nu^{0,1},$$

$$D_1^2 = ((\bar\nu^{0,1})^2\bar\nu^{2,0} - 2\bar\nu^{1,0}\bar\nu^{0,1}\bar\nu^{1,1} + (\bar\nu^{1,0})^2\bar\nu^{0,2})/(\bar\nu^{1,0})^3,$$

$$D_2^2 = ((\nu^{0,1})^2\nu^{2,0} - 2\nu^{1,0}\nu^{0,1}\nu^{1,1} + (\nu^{1,0})^2\nu^{0,2})/(\nu^{0,1})^3,$$

$$C = \frac{1}{\varkappa\bar\nu^{1,0}} \exp\Big(- \sum_{n=1}^{\infty} \frac{1}{n}\big[\mathbf{P}(S_n > 0) + \mathbf{P}(\bar S_n \leqslant 0)\big]\Big).$$

For the Normal distribution and density functions $\Phi_{(\mu,\sigma^2)}(z)$ and $\varphi_{(\mu,\sigma^2)}(z)$ introduce

$$g(z) = z + \varphi_{(0,1)}(z)\Phi^{-1}_{(0,1)}(z).$$

Theorem 2. *Suppose that in the Andersen's risk model with $\tau > 0$ the characteristic function $\beta_{Y,T}(t_1,t_2)$ is absolutely integrable and $0 < D_1, D_2 < \infty$. Then, as $u \to \infty$,*

$$\lim_{u\to\infty} \sup_{t\geqslant 0, w\in\mathbf{R}} \Big| e^{\varkappa u}\psi(w;t,u) - C\int_0^t \varphi_{(m_1 u, D_1^2 u)}(z) \times \Phi_{(m_2[t-z], D_2^2[t-z])}(w)\,dz\Big| = 0.$$

Theorem 3. *Under the conditions of Theorem 2*

$$\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big] = m_2 D_1 \sqrt{u} g\left(\frac{t - m_1 u}{D_1\sqrt{u}}\right)(1 + \overline{o}(1)),$$
$$\mathbf{D}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big] = D_2^2 D_1 \sqrt{u} g\left(\frac{t - m_1 u}{D_1\sqrt{u}}\right)(1 + \overline{o}(1)),$$

as $u \to \infty$.

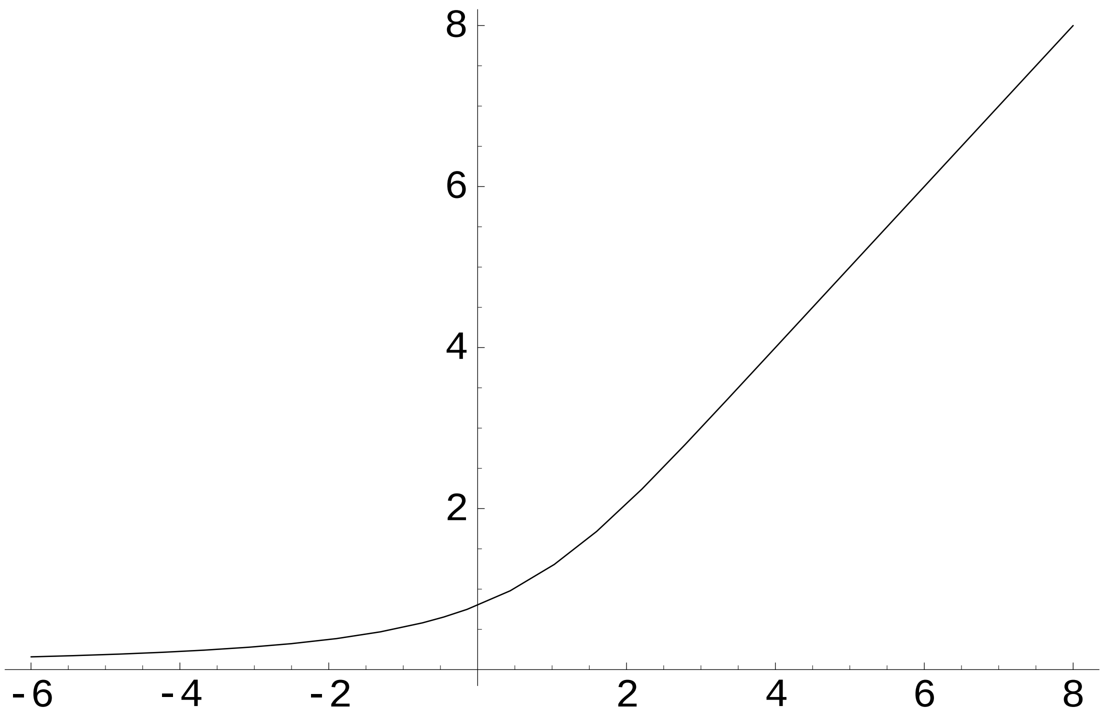

Fig. 2. Function $g(z) = z + \varphi_{(0,1)}(z)\Phi^{-1}_{(0,1)}(z)$.

Remark. In the particular case of the Poisson-Exponential model

$$c = \lambda(1+\tau)/\mu$$

and

$$\varkappa = \mu\tau/(1+\tau),$$

$$m_1 = \mu/(\lambda\tau(1+\tau)), \quad m_2 = \tau\lambda/\mu,$$

$$D_1^2 = 2\mu/(\lambda^2\tau^3) \quad D_2^2 = 2\lambda/\mu^2,$$

$$C = 1/(1+\tau).$$

In particular, the approximation for the expectation

$$\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big]$$

at time point $t = m_1 u$ in this case is

$$\sqrt{2u}g(0)\,/\sqrt{\mu\tau}. \tag{1}$$

4. Corrected approximations and a numerical example

For the ladder index $\mathcal{N} = \inf\{n : \bar{S}_n > 0\}$, the ladder height $\mathcal{H} = \bar{S}_{\mathcal{N}}$, the ladder time point $\mathcal{T} = \bar{U}_{\mathcal{N}}$ and $\mathcal{W} = \mathbf{E}(\mathcal{T}\mathbf{E}\mathcal{H} - \mathcal{H}\mathbf{E}\mathcal{T})$ introduce

$$\theta_1 = \frac{\mathbf{E}\mathcal{H}}{1 - \mathbf{E}e^{-\varkappa\mathcal{H}}} - \frac{1}{\varkappa}, \quad \theta_2 = \frac{\mathbf{E}\mathcal{T}}{1 - \mathbf{E}e^{-\varkappa\mathcal{H}}} - \frac{\mathbf{E}\mathcal{T}e^{-\varkappa\mathcal{H}}}{1 - \mathbf{E}e^{-\varkappa\mathcal{H}}},$$
$$\theta_3 = \frac{1}{\varkappa} - \frac{\mathbf{E}\mathcal{H}e^{-\varkappa\mathcal{H}}}{1 - \mathbf{E}e^{-\varkappa\mathcal{H}}}, \quad k_1 = \mathbf{E}\mathcal{W}^2, \quad k_2 = \frac{\mathbf{E}\mathcal{W}^3}{6k_1},$$
$$k_3 = \mathbf{E}\mathcal{T}\mathbf{D}\mathcal{H} - \mathbf{E}\mathcal{H}\mathbf{Cov}(\mathcal{H}, \mathcal{T}).$$

The following approximation elaborates the first relation of Theorem 2.

Theorem 4. *Suppose that in the collective risk model with $\tau > 0$ the characteristic function $\beta_{Y,T}(t_1, t_2)$ is absolutely integrable, $0 < D_1, D_2 < \infty$ and $\mathbf{E}T_1^3 < \infty$. Then, as $u \to \infty$,*

$$\sup_{t \geqslant 0} \Big| \mathbf{E}\big[R_u(t) \mid \inf_{0 < s \leqslant t} R_u(s) < 0\big]\psi(t,u)$$
$$- \nu^{1,0}\Big(1 - \frac{\nu^{0,2}}{2(\nu^{0,1})^2}\Big)\psi(t,u)$$
$$- Ce^{-\varkappa u}\tau\frac{\nu^{1,0}}{\nu^{0,1}}D_1\sqrt{u}\Big[\Big(\frac{t - m_1 u}{D_1\sqrt{u}}\Big)\Phi_{(0,1)}\Big(\frac{t - m_1 u}{D_1\sqrt{u}}\Big) + \varphi_{(0,1)}\Big(\frac{t - m_1 u}{D_1\sqrt{u}}\Big)\Big]$$

$$
\begin{aligned}
&- Ce^{-\varkappa u}\tau\frac{\nu^{1,0}}{\nu^{0,1}}\Phi_{(0,1)}\Big(\frac{t-m_1u}{D_1\sqrt{u}}\Big)\Big(\frac{t-m_1u}{D_1\sqrt{u}}\Big)\Big(\frac{k_3}{2(\mathbf{E}\mathcal{H})^2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + 3\frac{k_2}{\mathbf{E}\mathcal{H}} - \theta_1\frac{\mathbf{E}\mathcal{T}}{\mathbf{E}\mathcal{H}} + \theta_2\Big)\\
&- Ce^{-\varkappa u}\tau\frac{\nu^{1,0}}{\nu^{0,1}}\Phi_{(0,1)}\Big(\frac{t-m_1u}{D_1\sqrt{u}}\Big)\Big(\theta_2 - \theta_1\frac{\mathbf{E}\mathcal{T}}{\mathbf{E}\mathcal{H}} - \frac{k_3}{(\mathbf{E}\mathcal{H})^2} - \theta_3\frac{\mathbf{E}T_1}{\tau\mathbf{E}Y_1}\Big)\\
&- Ce^{-\varkappa u}\tau\frac{\nu^{1,0}}{\nu^{0,1}}\varphi_{(0,1)}\Big(\frac{t-m_1u}{D_1\sqrt{u}}\Big)\Big(\frac{t-m_1u}{D_1\sqrt{u}}\Big)\\
&\qquad\qquad\qquad\qquad \times\Big(\theta_1\frac{\mathbf{E}\mathcal{T}}{\mathbf{E}\mathcal{H}} - \theta_2 - 2\frac{k_2}{\mathbf{E}\mathcal{H}}\Big)\Bigg| = \overline{o}(e^{-\varkappa u}).
\end{aligned}
$$

Numerical example. Assume that the (i.i.d.) amounts of claims $\{Y_i\}_{i\geqslant 1}$ and the (i.i.d.) inter-occurrence times $\{T_i\}_{i\geqslant 1}$ are mutually independent and exponential with parameters $\mu > 0$ and $\lambda > 0$ respectively (the Poisson-Exponential model).

By Theorem 4, corrected approximation for

$$\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big]\psi(t,u)$$

at the time point $t = m_1 u$ is:

$$Ce^{-\varkappa u}\Phi_{(0,1)}(0)\Big(\frac{\sqrt{2u}}{\sqrt{\mu\tau}}g(0) - \frac{3+3\tau+\tau^2}{\mu(1+\tau)}\Big).$$

By Theorem 1, the approximation for $\psi(t,u)$ at the time point $t = m_1 u$ is:

$$Ce^{-\varkappa u}\Phi_{(0,1)}(0)\Big(1 - Q_1(0)\frac{\lambda\tau^{3/2}}{\sqrt{2\mu u}}g(0)\Big),$$

where

$$Q_1(0) = \frac{2+\tau^2}{\lambda\tau(1+\tau)} - \frac{\tau+2}{2\lambda^2\tau^2}.$$

These approximations yield a corrected approximation for the conditional expectation

$$\mathbf{E}\big[R_u(t) \mid \inf_{0<s\leqslant t} R_u(s) < 0\big]$$

at the time point $t = m_1 u$:

$$\Big(\frac{\sqrt{2u}}{\sqrt{\mu\tau}} g(0) - \frac{3+3\tau+\tau^2}{\mu(1+\tau)}\Big) \Big/ \Big(1 - Q_1(0)\frac{\lambda\tau^{3/2}}{\sqrt{2\mu u}} g(0)\Big). \tag{2}$$

Compare the approximation (1) and the corrected approximation (2) to the results of **direct simulation**. For this, simulate N risk reserve trajectories and calculate the mean value of the risk reserve at the time point $t = m_1 u$ over all those trajectories which fall below zero at least once within time $t = m_1 u$.

TABLE 1: $\lambda = \mu = 1$, $t = 99\,502$, $u = 500$, $\tau = 0.005$, $N = 10\,000$

	Simulation runs							
	1	2	3	4	5	6	7	8
# of trajectories which fall below zero	287	327	325	315	296	278	286	311
Empirical mean conditioned by ruin	209	224	242	220	214	207	195	222
Approximation (1) for the mean	357							
Corrected approximation (2) for the mean	261							

The data in this table demonstrates a reasonably good accuracy.

TABLE 2: $\lambda = \mu = 1$, $t = 499\,500$, $u = 500$, $\tau = 0.001$, $N = 1000$

	Simulation runs							
	1	2	3	4	5	6	7	8
# of trajectories which fall below zero	189	213	190	222	184	227	396	397
Empirical mean conditioned by ruin	326	369	346	339	368	358	310	335
Approximation (1) for the mean	798							
Corrected approximation (2) for the mean	442							

The poorer accuracy in this table is due to a smaller τ which brings this case within the scope of the problem of $\tau \to 0$, as $u \to \infty$ (see e.g., **Malinovskii, V. K.** *Probabilities of ruin when the safety loading tends to zero*, *AAP*, 2000, vol. 32, 885 – 923).