NUMBER OF IBNR CLAIMS AND MULTIVARIATE COMPOUND POISSON DISTRIBUTION

Louis G. Doray, Canada

Abstract

We consider the distribution of the number of incurred but not reported (IBNR) claims under a discrete time period model when the number of claims incurred in each accident period follows a compound Poisson model and there is a random delay until reporting of a claim to the insurance company. This discrete delay has a probability mass function depending on a vector of unknown parameters. We find the joint probability generating function (pgf) of the number of claims reported in the periods following an accident period, and the pgf of the number of IBNR claims; we derive the marginal distribution of the number of claims reported in each period after the accident period. We discuss the identifiability problems which occur when a non-parametric distribution is assumed for the reporting lag, or when its cumulative distribution function or its survival function has a certain form.

NOMBRE DE SINISTRES IBNR ET DISTRIBUTON DE POISSON COMPOSÉE MULTIVARIÉE

Louis G. Doray, Canada

Résumé

Nous considérons la distribution du nombre de sinsitres survenus mais non-signalés (SMNS) avec un modèle en temps discret lorsque le nombre de sinstres survenus durant une période d'accident suit un modèle de Poisson composé et qu'il y a un délai aléatoire jusqu'à la signalisation du sinistre à la compagnie d'assurance. Ce délai discret a une fonction de probabilité qui dépend d'un vecteur de paramètres inconnus. Nous trouvons la fonction génératrice des probabilités (fgp) conjointe du nombre de sinistres signalés dans les périodes suivant une période d'accident, et la fgp du nombre de sinistres SMNS; nous dérivons la distribution marginale du nombre de sinistres signalés dans chaque période après la période d'accident. Nous discutons des problèmes d'identifiabilité qui surviennent sous l'hypothèse d'une distribution non-paramétrique pour le délai de signalisation, ou lorsque sa fonction de répartition ou de survie prend une certaine forme.

1 Introduction and Notation

The problem of estimating the number of claims incurred but not reported (IBNR) at a certain date by an insurer, when some information is available for each individual claim, such as the date of occurence of the accident and the date of reporting of the claim, has been extensively studied.

Jewell (1989, 1990) has assumed a homogeneous Poisson process for the number of claims incurred in an accident period and has developed estimators for the number of IBNR claims at a certain date. Hesselager and Witting (1988) and Neuhaus (1992) have assumed that the delay probabilities for the observations can vary between occurence years. Hesselager (1995) has assumed a non-homogeneous Poisson process for the incurral of claims and has studied a credibility estimator to predict the number of claims incurred in future periods.

In this paper, we will extend the above models by asuming that the number of claims in an accident period follows a compound Poisson distribution.

Let X_l denote the time of accident l; associated with each claim l (arising from accident l), is a random time until reporting of that claim to the insurance company, the random variable W_l, assumed independent of X_l, with probability density function (pdf) $f(w;\theta)$ and cumulative distribution function (cdf) $F(w;\theta)$, where θ is a vector of parameters. Accident l is therefore reported at time

$$Y_l = X_l + W_l.$$

Claim l is either reported in the observation period $(0,t]$ if $Y_l \leq t$ or it is an IBNR claim at time t if $Y_l > t$. All reported claims in the observation interval $(0,t]$ are observed by the insurance company.

We consider the following discrete time period model. Accident l occurs in accident period $i \in \{1,2,\ldots,T\}$, the exposure period. The reporting lag, assumed independent of the incurral process, is a discrete random variable with probability function p_j. We suppose that there exists a maximum possible value m, for the reporting lag, so that $j = 0, 1, 2, \ldots, m$. The observation period is the set $\{1, 2, \ldots, k\}$, where k is greater than T.

Let the random variables N_i represent the number of claims incurred in accident period i and R_{ij}, the number of claims incurred in accident period i which are reported j periods later, in period $i+j$, $j = 0, 1, 2, \ldots, m$. The random variable $U_i = \sum_{j=k-i+1}^{m} R_{ij}$ will denote the number of IBNR claims at the end of the observation interval for accident period i, and $R_{i\bullet} = \sum_{j=0}^{k-i} R_{ij}$, the total number of claims reported by the end of the observation period for accident period i, so that $N_i = R_{i\bullet} + U_i$. A capital letter will denote a random variable, while a lower-case letter will represent the realized value of the corresponding random variable, which is either observed or not. We assume a compound Poisson distribution for N_i, and look at the joint distribution of the R_{ij}'s and the distribution of U_i.

The paper is organized as follows. In section 2, we study the joint probability generating function (pgf) of the number of claims reported in the intervals following an accident period and the pgf of the number of IBNR claims. From the joint pgf, we derive the marginal distribution of the number of claims reported in each period after the accident period. We also find the pgf of the total number of IBNR claims corresponding to independent accident periods. In section 3, we discuss estimation of the parameters under the assumption of a Poisson process, and the identifiability problems which occur when a non-parametric distribution is assumed for the reporting lag, or when its cdf or survival function has a certain form. We also motivate the use of modified discrete distributions for the reporting lag. In section 4, we consider the model obtained when the number of claims in each period follows a negative binomial distribution and derive the distribution of the number of unreported claims; we also discuss the identifiability problems encountered in estimating the parameters.

2 The pgf of the claims number

We assume that the random variable N_i follows a compound Poisson distribution $(\lambda_i, P_i(z))$ with pgf

$$P_{N_i}(z) = E(z^{N_i}) = \exp\left\{\lambda_i\left[P_i(z) - 1\right]\right\}, \tag{1}$$

where $P_i(z)$ is the pgf of the compounding distribution. Proposition 1 gives the pgf of the joint distribution of the random variables R_{ij}'s, $(j = 0, 1, \ldots, m-1)$.

Proposition 1: If N_i follows a compound Poisson distribution $(\lambda_i, P_i(z))$, the random variables $(R_{i0}, \ldots, R_{i,m-1})$ have joint pgf

$$P_{R_{i0},\ldots,R_{i,m-1}}(z_0, \ldots, z_{m-1}) = \exp\{\lambda_i[P_i(p_0 z_0 + \ldots + p_{m-1}z_{m-1} + p_m) - 1]\}$$

Proof:

$$\begin{aligned}
P_{R_{i0},\ldots,R_{i,m-1}}(z_0, \ldots, z_{m-1}) &= E(z_0^{R_{i0}} \times \ldots \times z_{m-1}^{R_{i,m-1}}) \\
&= E_{N_i}\left[E\left((z_0^{R_{i0}} \times \ldots \times z_{m-1}^{R_{i,m-1}}) \mid N_i\right)\right] \\
&\quad \text{(by conditioning on } N_i\text{)} \\
&= E_{N_i}\left[(p_0 z_0 + \ldots + p_{m-1}z_{m-1} + p_m)^{N_i}\right] \quad \text{(given } N_i\text{,} \\
&\quad R_{i0}, \ldots, R_{i,m-1} \text{ follow a multinomial distribution)} \\
&= \exp\left\{\lambda_i\left[P_i\left(p_0 z_0 + \ldots + p_{m-1}z_{m-1} + p_m\right) - 1\right]\right\}, \text{ by (1).}
\end{aligned}$$

□

The marginal pgf of R_{ij} is obtained by evaluating $P_{R_{i0},\ldots,R_{i,m-1}}(z_0, \ldots, z_{m-1})$, with all the z_l's, except z_j, set equal to 1, giving

$$P_{R_{ij}}(z_j) = \exp\{\lambda_i[P_i(p_j z_j + 1 - p_j) - 1]\}.$$

The random variable R_{ij} therefore has a pgf similar to that of N_i, but is evaluated at $p_j z_j + (1 - p_j)$, the pgf of a Bernoulli random variable with probability p_j. The moments of R_{ij} are obtained by differentiating $P_{R_{ij}}(z_j)$ and setting $z_j = 1$. Its mean and variance equal

$$E(R_{ij}) = \lambda_i p_j P_i'(1),$$

$$\text{Var}(R_{ij}) = \lambda_i p_j^2 P_i''(1) + \lambda_i p_j P_i'(1),$$

and the covariance between R_{ij} and $R_{ij'}$, $j \neq j'$, is equal to

$$Cov(R_{ij}, R_{ij'}) = \lambda_i p_j p_j' P_i''(1).$$

The pgf of U_i is

$$P_{U_i}(z) = \exp\left\{\lambda_i \left[P_i\left(z(1 - \sum_{j=0}^{k-i} p_j) + \sum_{j=0}^{k-i} p_j\right) - 1\right]\right\}, \tag{2}$$

from which we get its mean

$$E(U_i) = \lambda_i \left(1 - \sum_{j=0}^{k-i} p_j\right) P_i'(1),$$

and variance

$$\text{Var}(R_{ij}) = \lambda_i \left(1 - \sum_{j=0}^{k-i} p_j\right)^2 P_i''(1) + \lambda_i \left(1 - \sum_{j=0}^{k-i} p_j\right) P_i'(1).$$

Let us now look at some specific distributions for N_i. If $P_i(z) = z$, N_i follows a Poisson (λ_i) distribution, and we obtain the model analyzed by Jewell (1989). If the compounding distribution is logarithmic with pgf

$$P_i(z) = \frac{\log\left[1 - \beta_i(z-1)\right] - \log(1+\beta_i)}{-\log(1+\beta_i)},$$

N_i follows a negative binomial distribution, and we obtain

$$P_{R_{i0},\ldots,R_{i,m-1}}(z_0, \ldots, z_{m-1}) = \left[1 - \beta_i(p_0 z_0 + \ldots + p_{m-1} z_{m-1} + p_m - 1)\right]^{-\lambda_i/\log(1+\beta_i)},$$

which is the pgf of the negative multinomial distribution (see Johnson and Kotz (1969)). We analyze this model in section 4.

The above analysis is for a single accident period. For the complete exposure period $\{1, 2, \ldots, T\}$, we assume that the random variables $N_1, \ldots, N_T$ are independent of each other, with a compound Poisson distribution $(\lambda_i, P_i(z))$; the total number of claims incurred in the exposure period, $N_\bullet = N_1 + \ldots + N_T$, has pgf

$$\begin{aligned}
P_{N_\bullet}(z) &= E\left(z^{N_1+\ldots+N_T}\right) \\
&= \prod_{i=1}^{T} E\left(z^{N_i}\right), \quad \text{by independence of the } N_i\text{'s}, \\
&= \exp\left\{\sum_{i=1}^{T} \lambda_i \left[P_i(z) - 1\right]\right\} \\
&= \exp\left\{\sum_{i=1}^{T} \lambda_i \left[\frac{\sum_{i=1}^{T} \lambda_i P_i(z)}{\sum_{i=1}^{T} \lambda_i} - 1\right]\right\} \\
&= \exp\left\{\Lambda\left[\bar{P}(z) - 1\right]\right\}, \quad \text{where } \Lambda = \sum_{i=1}^{T} \lambda_i.
\end{aligned}$$

Therefore, $N_\bullet$ has a compound Poisson distribution $(\Lambda, \bar{P}(z))$, where Λ is the sum of the parameters λ_i, and $\bar{P}(z)$ is the weighted average of the compounding distributions $P_i(z)$, with weight λ_i. If the functions $P_i(z)$ are all equal, $P_i(z) = P(z), \forall i$, then $P_{N_\bullet} = \exp\{\Lambda\left[P(z) - 1\right]\}$, i.e. $N_\bullet$ has the same type of compound Poisson distribution as the N_i's.

Similarly, if we let $U_\bullet = \sum_{i=1}^{T} U_i$ be the total number of IBNR claims arising from accidents in the exposure period, at the end of the observation period, $U_\bullet$ also has a compound Poisson distribution $(\Lambda, \tilde{P}(z))$, where $\tilde{P}(z)$ is a weighted average of the compounding distributions of $P_{U_i}(z)$ defined in (2).

3 A Poisson model

In this section, we look at the case where N_i follows a Poisson (λ_i) distribution. Let us first consider the single accident period i. Let $\theta = (\theta_l)$ be the vector of parameters of the reporting delay distribution W. The probability of n_i accidents occurring in period i, and r_{ij} claims reported j periods later ($j = 0, 1, \ldots, k-i$), with the observation period $\{1, 2, \ldots, k\}$, where $i + j \leq k$, is

$$\begin{aligned} P(n_i, r_{i0}, \ldots, r_{i,k-i}; \lambda_i, \theta) &= P(r_{i0}, \ldots, r_{i,k-i}; \theta, n_i) \times P(n_i; \lambda_i) \\ &= n_i! \left(\prod_{j=0}^{k-i} \frac{p_j(\theta)^{r_{ij}}}{r_{ij}!} \right) \frac{\left(1 - \sum_{j=0}^{k-i} p_j(\theta)\right)^{n_i - r_{i\bullet}}}{(n_i - r_{i\bullet})!} \frac{e^{-\lambda_i} \lambda_i^{n_i}}{n_i!}, \quad (3) \end{aligned}$$

where $r_{i\bullet} = \sum_{j=0}^{k-i} r_{ij}$ is the total number of claims incurred in month i which are reported during the observation interval $(0, k]$. The first terms in (3) represent the probability function of a multinomial distribution. Note that when $k - i \geq m$, all the claims for accident period i have been reported and the likelihood function $L(\theta, \lambda_i)$ can be factorized a product of the likelihood functions of θ and λ_i

$$L(\theta, \lambda_i) = L(\theta) \times L(\lambda_i),$$

facilitating the calculation of the MLE's.

Calculating the marginal probability obtained by summing the joint probability (3) over all possible values of n_i, we obtain

$$\begin{aligned} P[R_{ij} = r_{ij},\ j = 0, ..., k-i; \lambda_i, \theta] &= \sum_{n_i=0}^{\infty} P[N_i = n_i, R_{ij} = r_{ij},\ j = 0, ..., k-i; \lambda_i, \theta] \\ &= \prod_{j=0}^{k-i} \left[\frac{(\lambda_i p_j(\theta))^{r_{ij}}}{r_{ij}!} \right] \exp\left(-\lambda_i \sum_{j=0}^{k-i} p_j(\theta) \right) \\ &= \prod_{j=0}^{k-i} \left[\frac{\exp\left(-\lambda_i p_j(\theta)\right) \left(\lambda_i p_j(\theta)\ \right)^{r_{ij}}}{r_{ij}!} \right]. \end{aligned}$$

This well-known result in probability (see Ross (1985)) shows that $R_{i0}, \ldots, R_{i,k-i}$ are independent Poisson random variables, with parameter $\lambda_i p_j(\theta)$. Also, the random variable U_i, the number of claims incurred in month i which are not reported by month

$i+j$, is independent of $R_{i0},\dots,R_{i,k-i}$ and also follows a Poisson distribution with parameter

$$\lambda_i\left(1-\sum_{j=0}^{k-i}p_j(\theta)\right).$$

The number of IBNR claims for the exposure interval $\{1,\dots,T\}$ also follows a Poisson distribution with mean

$$\sum_{i=1}^{T}\lambda_i\left(1-\sum_{j=0}^{k-i}p_j(\theta)\right).$$

The likelihood using all the claims incurred in the exposure interval $\{1,\dots,T\}$, reported by the end of the observation period k is

$$L(\lambda_1,\dots,\lambda_T,\theta)=\prod_{i=1}^{T}\prod_{j=0}^{k-i}\left(\frac{\exp\left[-\lambda_i p_j(\theta)\right](\lambda_i p_j(\theta))^{r_{ij}}}{r_{ij}!}\right).$$

The MLE of λ_i equals

$$\hat{\lambda}_i=\frac{r_{i\bullet}}{\sum\limits_{j=0}^{k-i}p_j(\hat{\theta})},$$

where $\hat{\theta}$, the MLE of θ, is found numerically. If the parameters λ_i are all equal to λ, we obtain the MLE

$$\hat{\lambda}=\frac{r_{\bullet\bullet}}{\sum\limits_{i=1}^{T}\sum\limits_{j=0}^{k-i}p_j(\hat{\theta})},$$

where $r_{\bullet\bullet}=\sum\limits_{i=1}^{T}\sum\limits_{j=0}^{k-i}r_{ij}$ is the total number of claims from the exposure period reported in the observation period.

An identifiability problem arises if a non-parametric distribution is assumed for the reporting lag: λ_i and p_j are identifiable only up to a multiplicative constant. If $\hat{\lambda}_i$ and $\hat{p}_j$, where $0<\sum\hat{p}_j<1$ maximize the likelihood, then $c\hat{\lambda}_i$ and $\hat{p}_j/c$, where c is any constant such that $0<\sum\hat{p}_j/c<1$, give the same maximum value for the likelihood function.

An identifiability problem will also arise whenever the cdf of the reporting lag W is such that $cF(w;\theta)=F(w;h(\theta,c))$ or its survival function is such that $c\bar{F}(w;\theta)=$

$\bar{F}(w; h(\theta, c))$. For example, the Pareto distribution, with pdf $f(w; \alpha, \lambda) = \alpha\lambda^\alpha w^{-\alpha-1}$, $w \geq \lambda$, satisfies this property for the survival function $\bar{F}$; the property is satisfied for the cdf of the power distribution, with pdf $f(w; k, a) = ak^{-a}w^{a-1}, \quad 0 \leq w \leq k$.

Under the assumption of a Poisson process for the incurral of claims, when the reporting lag W follows an exponential distribution with pdf $f(w; \theta) = \theta e^{-\theta w}, \;\; w > 0$, the probability of a claim being reported j periods after its occurrence, follows a modified geometric distribution (see Jewell (1989)), with probability function

$$p_j(\theta) = \begin{cases} 1 - \theta^{-1}(1 - e^{-\theta}), & j = 0 \\ \theta^{-1}e^{-\theta(j-1)}(1 - e^{-\theta})^2, & j \geq 1 \end{cases}$$

To know more about modified discrete distributions, the reader is referred to Panjer and Willmot (1992).

A modified discrete distribution for $p_j(\theta)$ could be used directly, without introducing first a continuous density for W. Under the assumption of a Poisson process, the occurrence dates of the accidents will be uniformly distributed during the exposure period. A claim occurring during an accident period will therefore occur on average in the middle of the period, and will have less than a full period to be reported, for the reporting lag j to equal 0, while for $j \geq 1$ a full period for reporting is possible, regardless of when the claim occurs during the accident period. This motivates the use of a discrete distribution modified at 0 for the reporting lag.

4 A negative binomial model

In this section, we asume that the number of claims occuring during exposure period i follows a negative binomial distribution with parameters s and $1 - p = \beta/(1 + \beta)$, denoted $NB(s, p)$, and with probability function

$$P[N_i = n_i] = \binom{s + n_i - 1}{n_i} p^s(1 - p)^{n_i}, \qquad n_i = 0, 1, 2, \ldots.$$

The probability of n_i accidents occurring during exposure period i and r_{ij} claims reported j periods later $(i+j \le k)$, is:

$$P[N_i = n_i, R_{ij} = r_{ij},\ j = 0, \ldots, k-i; s, p, \theta] =$$

$$n_i! \left(\prod_{j=0}^{k-i} \frac{p_j(\theta)^{r_{ij}}}{r_{ij}!} \right) \frac{(1 - \sum\limits_{j=0}^{k-i} p_j(\theta))^{n_i - r_{i\bullet}}}{(n_i - r_{i\bullet})!} \binom{s + n_i - 1}{n_i} p^s (1-p)^{n_i}.$$

Summing over all possible values of n_i, we obtain, after some algebra, the joint probability of the R_{ij}'s

$$P[R_{ij} = r_{ij},\ j = 0, \ldots, k-i; s, p, \theta] =$$

$$\binom{s + r_{i\bullet} - 1}{r_{i0}, \ldots, r_{i,k-i}} \left(\frac{1}{1 + \beta \sum\limits_{j=0}^{k-i} p_j(\theta)} \right)^s \prod_{j=0}^{k-i} \left[\frac{\beta p_j(\theta)}{1 + \beta \sum\limits_{j=0}^{k-i} p_j(\theta)} \right]^{r_{ij}}, \tag{4}$$

where $\beta = (1-p)/p$, by using the fact that $\sum\limits_{n_i = r_{i\bullet}}^{\infty} \binom{s+n_i-1}{n_i - r_{i\bullet}} A^{n_i - r_{i\bullet}} = (1-A)^{-s-r_{i\bullet}}$, with $A = \left(1 - \sum\limits_{j=0}^{k-i} p_j(\theta)\right)(1-p)$.

This is the probability function of the negative multinomial distribution with parameters $\left(s, \frac{\beta p_0(\theta)}{1+\beta \sum\limits_{j=0}^{k-i} p_j(\theta)}, \ldots, \frac{\beta p_{k-i}(\theta)}{1+\beta \sum\limits_{j=0}^{k-i} p_j(\theta)} \right)$ (see Bishop, Fienberg and Holland (1975), section 13.8, or Ratnaparkhi(1985)). Therefore, $R_{i0}, \ldots, R_{i,k-i}$ folow a joint negative multinomial distribution.

From the properties of this distribution, it follows that each component R_{ij}, for $j = 0, \ldots, k-i$, has a marginal negative binomial distribution with parameters $\left(s, \frac{\beta p_j(\theta)}{1+\beta \sum\limits_{j=0}^{k-i} p_j(\theta)} \right)$, but R_{ij} is not independent of $R_{ij'}$, for $j \neq j'$. The mean and variance of R_{ij} are respectively

$$E(R_{ij}) = s\beta p_j(\theta),$$

$$Var(R_{ij}) = s\beta p_j(\theta)[1 + \beta p_j(\theta)],$$

$$\text{while } Cov(R_{ij}, R_{ij'}) = s\beta^2 p_j(\theta) p_{j'}(\theta), \text{ for } j \neq j'.$$

The marginal distribution of the number of IBNR claims for accident period i also follows a negative binomial distribution, with parameters s, and $q_i = \frac{\beta\left[1-\sum_{j=0}^{k-i} p_j(\theta)\right]}{1+\beta\left[1-\sum_{j=0}^{k-i} p_j(\theta)\right]}$. The covariance between U_i and R_{ij} equal to $s\beta^2 p_j(\theta)[1-\sum_{j=0}^{k-i} p_j(\theta)]$, is always positive, as is $Cov(R_{ij}, R_{ij'})$. If we now consider the complete exposure period $\{1, \ldots, T\}$, the marginal distribution of the total number of IBNR claims, $U_\bullet = U_1 + \ldots + U_T$, is a sum of independent negative binomial distributions, all with the first parameter equal to s, but with the second parameter which depends on the exposure month i. The pgf of $U_\bullet$ is

$$P_{U_\bullet}(z) = \prod_{i=1}^{T} \left[\frac{1-q_i z}{1-q_i}\right]^s.$$

We have seen in section 2 that this random variable can be represented as a compound Poisson distribution with pgf of the form $\exp\{\Lambda[\tilde{P}(z)-1]\}$, where

$$\Lambda = -s \sum_{i=1}^{T} \ln(1-q_i)$$

$$\text{and } \tilde{P}(z) = \frac{\sum_{i=1}^{T} \ln(1-q_i z)}{\sum_{i=1}^{T} \ln(1-q_i)}.$$

The conditional distribution of U_i given $(R_{i0}, \ldots, R_{i,k-i})$ is the same as that of U_i given $R_{i\bullet}$ and follows a negative binomial distribution with parameters $\left(s + r_{i\bullet}, \frac{\beta\left[1-\sum_{j=0}^{k-i} p_j(\theta)\right]}{1+\beta}\right)$ (see Sibuya et al. (1964)). The regression of U_i on $R_{i\bullet}$ is thus linear in $r_{i\bullet}$,

$$E[U_i \mid R_{i\bullet} = r_{i\bullet}] = (s + r_{i\bullet}) \times \frac{\beta\left[1-\sum_{j=0}^{k-i} p_j(\theta)\right]}{1+\beta\sum_{j=0}^{k-i} p_j(\theta)}.$$

From equation (4), we see that an identifiability problem exists again when a non-parametric distribution is assumed for p_j: since the terms β and p_j always appear through the expression βp_j, they can only be estimated up to a multiplicative constant; and as in the Poisson case, when the cdf or the survival function of the random variable W satisfies one of the conditions stated in section 3, there will also be an identifiability problem for the parameters.

ACKNOWLEDGEMENTS

The author gratefully acknowledges the financial support of the Natural Sciences and Engineering Research Council of Canada, and of the Fonds pour la Contribution à la Recherche du Québec.

5 References

Bishop, Y.M., Fienberg, S.E. and Holland, P.W. (1975). Discrete Multivariate Analysis: Theory and Practice, The MIT Press, Cambridge, Massachusetts.

Hesselager, O. (1995). Modelling of Discretized Claim Numbers in Loss Reserving, Astin Bulletin, **25**, 119-135.

Hesselager, O. and Witting, T. (1988). A credibility model with random fluctuations in delay probabilities for the prediction of IBNR claims, Astin Bulletin, **18**, 79-90.

Jewell, W.S. (1989). Predicting IBNYR events and delays I. Continuous Time, Astin Bulletin, **19**, 25-55.

Jewell, W.S. (1989). Predicting IBNYR events and delays II. Discrete Time, Astin Bulletin, **20**, 93-111.

Johnson, N.L. and Kotz, S. (1969). Distributions in Statistics: Discrete Distributions. Houghton Mifflin Company, Boston.

Neuhaus, W. (1992). IBNR Models with Random Delay Distributions, Scandinavian Actuarial Journal, 97-107.

Panjer, H.H. and Willmot, G.E. (1992). Insurance Risk Models, Society of Actuaries, Schaumburg, IL.

Ratnaparkhi, M.V. (1985). Negative multinomial distribution, in Encyclopedia of Statistical Sciences, edited by S. Kotz and N.L. Johnson, Vol. 5 , 662-665.

Ross, S.M. (1985). Introduction to probability models. Academic Press, Orlando.

Sibuya, M., Yoshimura, I. and Shimizu, R. (1964). Negative multinomial distribution, Annals of the Institute of Statistical Mathematics, **16**, 409-426.