Finite Time Probability of Ruin

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Summary

Explicit expression is derived for the probability of ruin of an insurance company, whose premium income is represented by arbitrary, increasing real function, the claim amounts are assumed to be dependent, integer valued random variables and their inter-occurrence times are exponentially distributed with different parameters.
Key words: finite time ruin probability, risk reserve process.

1. INTRODUCTION

Let us consider the counting process $N_t = \#\{i : \tau_1 + \ldots \tau_i \le t\}$, where # in the right-hand side denotes the number of elements in the set $\{\cdot\}$, and $\tau_1, \ldots \tau_i, \ldots$ are independent, exponentially distributed random variables (r.v.s) with different means $E\{\tau_i\} = 1/\lambda_i$, $\lambda_i > 0$, $\lambda_i \ne \lambda_j$ for $i \ne j$, i.e., $P(\tau_i > y) = e^{-\lambda_i y}$, for $y > 0$, and $P(\tau_i > y) = 1$, for $y \le 0$, $i = 1, 2, \ldots$

Consider the integer-valued r.v.s $W_1, W_2, \ldots$ independent on N_t, denoting the severities of the successive claims. We will denote the joint distribution of $W_1, W_2, \ldots W_i$ by $P(W_1 = w_1, \ldots W_i = w_i) = P_{w_1, \ldots w_i}$, where $w_1 \ge 1, w_2 \ge 1, \ldots w_i \ge 1$, $i = 1, 2, \ldots$

The aggregate claims amount at time *t* will be

$$S_t = \sum_{i=1}^{N_t} W_i,$$

and the surplus of an insurance company

$$R_t = h(t) - S_t,$$

where $h(t)$ is a function, representing the premium income. We will assume, that $h(t)$ is a nonnegative, increasing, real function, defined on R_+ and such that $\lim_{t\to\infty} h(t) = +\infty$. The classical risk model (see, for example, Bowers, Gerber, Hickman, Jones (1997)) assumes $h(t) = u + ct$ with u the initial surplus and c the premium rate per unit time.

We assume that $h^{-1}(y) = \inf\{z : h(z) \ge y\}$. We will denote $v_i = h^{-1}(i)$, for $i = 0, 1, \ldots$ Clearly, $0 = v_0 \le v_1 \le v_2 \le \ldots$ Let us denote the instance of ruin by T, i.e. $T = \inf\{t : t > 0, R_t \le 0\}$. We will be interested in the probability of non-ruin, i.e. $P(T > x)$ in a finite time interval $[0, x]$, $x > 0$. Here we derive the explicit formula for this probability.

Picard & Lefevre (1997) have considered $P(T > x)$ in the case when the counting process N_t is a Poisson process with intensity λ and r.v.s $W_1, W_2, \ldots$ are integer valued, independent and identically distributed. Under these assumptions they have derived an expression for the probability $P(T > x)$ in terms of generalized Appell polynomials.

Ignatov & Kaishev (2000) have recently obtained two-sided bounds for the probability $P(T > x)$. They have showed that, in the case of compound

Poisson aggregate claims these bounds coincide producing an explicit representation of the survival probability.

2. EXPLICIT EXPRESSION FOR THE PROBABILITY $P(T>x)$

We will assume that the length x of the time interval is fixed. We will use the notation $n=[h(x)]+1$, where $[h(x)]$ is the integer part of $h(x)$. Since $w_1 \ge 1, \ldots, w_n \ge 1$, we have $w_1+\ldots+w_n \ge n$, hence there exists an integer k, $1\le k\le n$ such that $w_1+\ldots+w_{k-1}\le n-1$, and $w_1+\ldots+w_k \ge n$.
We can equivalently require k to be such that $v_{w_1+\ldots+w_{k-1}} \le x < v_{w_1+\ldots+w_k}$. Note that k is a suitable function of $w_1,\ldots,w_n$, i.e. $k=k(w_1,\ldots,w_n)$. Let $I\{\ldots\}$ denotes an indicator of the event {.}.

THEOREM. *The probability of ruin*

$$P(T>x)=\sum_{\substack{w_1\ge1\\ \ldots\\ w_n\ge1}} P_{w_1,\ldots,w_n}\Big(I\{k=1\}e^{-\lambda_1 x}+I\{k>1\}\times$$

$$\times\lambda_1\lambda_2\ldots\lambda_k\sum_{j=1}^{k-1} C_k(\lambda_1,\ldots,\lambda_k,z_1,\ldots,z_{k-1},j)\left(\frac{e^{-\lambda_j x}}{\lambda_j}-\frac{e^{(\lambda_k-\lambda_j)z_{k-1}}e^{-\lambda_k x}}{\lambda_k}\right)$$

provided that $v_{n-1}\le x< v_n$, *where* $z_l = v_{w_1+\ldots+w_l}$, $l=1,2,\ldots$, *and*

$C_k(\lambda_1,\ldots,\lambda_k,z_1,\ldots,z_{k-1},j)=\prod_{m=j}^{k}\frac{1}{\lambda_m-\lambda_j}$ *for* $1\le j\le k-2$, *and*

$C_k(\lambda_1,\ldots,\lambda_k,z_1,\ldots,z_{k-1},k-1)$

$$=\frac{1}{\lambda_{k-1}-\lambda_k}\sum_{j=1}^{k-2} C_{k-1}(\lambda_1,\ldots,\lambda_{k-1},z_1,\ldots,z_{k-2},j)e^{(\lambda_{k-1}-\lambda_{k-1-j})z_{k-2}}.$$

Proof. Applying the formula for the total probability we can express the probability of non-ruin as follows (see Ignatov & Kaishev (2000) p.49)

$$P(T>x)=\sum_{\substack{w_1\ge1\\ \ldots\\ w_n\ge1}} P_{w_1,\ldots,w_n}P(T>x/W_1=w_1,\ldots,W_n=w_n). \qquad (1)$$

But the probability

$$P(T > x / W_1 = w_1, \ldots, W_n = w_n)$$

$$= P\left(\bigcap_{l=1}^{k-1} (\tau_1 + \ldots + \tau_l \geq v_{w_1+\ldots+w_l}) \bigcap (\tau_1 + \ldots + \tau_k \geq x)\right)$$

for $k \geq 2$, and

$$P(T > x / W_1 = w_1, \ldots, W_n = w_n) = \int_x^{+\infty} \lambda_1 e^{-\lambda_1 y_1} dy_1 = e^{-\lambda_1 x} \qquad (2)$$

for $k = 1$.
Since we will assume further on, that $w_1, \ldots, w_n$ are fixed, for simplicity, we will use the notation $z_l = v_{w_1+\ldots+w_l}$, $l = 1, 2, \ldots$. Then for $k \geq 2$

$$P\left(\bigcap_{l=1}^{k-1} (\tau_1 + \ldots + \tau_l \geq v_{w_1+\ldots+w_l}) \bigcap (\tau_1 + \ldots + \tau_k \geq x)\right)$$

$$= \quad P\left(\bigcap_{l=1}^{k-1} (\tau_1 + \ldots + \tau_l \geq z_l) \bigcap (\tau_1 + \ldots + \tau_k \geq x)\right)$$

$$= \lambda_1 \ldots \lambda_k \int_x^{+\infty} \int_{z_{k-1}}^{y_k} \int_{z_{k-2}}^{y_{k-1}} \ldots \int_{z_1}^{y_2} \exp((\lambda_2 - \lambda_1) y_1 + (\lambda_3 - \lambda_2) y_2 + \ldots$$
$$+ (\lambda_k - \lambda_{k-1}) y_{k-1} - \lambda_k y_k) dy_1 dy_2 \ldots dy_k. \qquad (3)$$

Let us denote the right hand of the previous equality by I. To evaluate I we will introduce the notations

$$I(\lambda_i, \lambda_j, z_{i-1}, y_i) = \left(e^{(\lambda_i - \lambda_j) y_i} - e^{(\lambda_i - \lambda_j) z_{i-1}}\right),$$
$$I_{l+1} = \int_{z_l}^{y_{l+1}} \int_{z_{l-1}}^{y_l} \ldots \int_{z_1}^{y_2} \exp((\lambda_2 - \lambda_1) y_1 + (\lambda_3 - \lambda_2) y_2 + \ldots + (\lambda_{l+1} - \lambda_l) y_l) dy_1 dy_2 \ldots dy_l.$$

Then

$$I_2 == \frac{1}{\lambda_2 - \lambda_1}\left(e^{(\lambda_2 - \lambda_1) y_2} - e^{(\lambda_2 - \lambda_1) z_1}\right) = c_2(\lambda_1, \lambda_2, z_1, 1) I(\lambda_2, \lambda_1, z_1, y_2),$$

where $c_2(\lambda_1, \lambda_2, z_1, 1) = 1/\lambda_2 - \lambda_1)$. Successively evaluating integrals in (3) we have

$$I_3=\int_{z_2}^{y_3} c_2(\lambda_1,\lambda_2,z_1,1)\,I(\lambda_2,\lambda_1,z_1,y_2)e^{(\lambda_3-\lambda_2)y_2}dy_2=$$

$$=c_2(\lambda_1,\lambda_2,z_1,1)\int_{z_2}^{y_3}\left(e^{(\lambda_2-\lambda_1)y_2}-e^{(\lambda_2-\lambda_1)z_1}\right)e^{(\lambda_3-\lambda_2)y_2}dy_2=$$

$$=c_2(\lambda_1,\lambda_2,z_1,1)\left(\int_{z_2}^{y_3}e^{(\lambda_3-\lambda_1)y_2}dy_2-e^{(\lambda_2-\lambda_1)z_1}\int_{z_2}^{y_3}e^{(\lambda_3-\lambda_2)y_2}dy_2\right)=$$

$$=c_2(\lambda_1,\lambda_2,z_1,1)\left(\frac{1}{\lambda_3-\lambda_1}I(\lambda_3,\lambda_1,z_2,y_3)-e^{(\lambda_2-\lambda_1)z_1}\frac{1}{\lambda_3-\lambda_2}I(\lambda_3,\lambda_2,z_2,y_3)\right)$$

$$=c_3(\lambda_1,\lambda_2,\lambda_3,z_1,z_2,1)\,I(\lambda_3,\lambda_1,z_2,y_3)+c_3(\lambda_1,\lambda_2,\lambda_3,z_1,z_2,2)\,I(\lambda_3,\lambda_2,z_2,y_3)$$

where

$$c_3(\lambda_1,\lambda_2,\lambda_3,z_1,z_2,1)=c_2(\lambda_1,\lambda_2,z_1,1)\frac{1}{\lambda_3-\lambda_1},$$

$$c_3(\lambda_1,\lambda_2,\lambda_3,z_1,z_2,2)=-c_2(\lambda_1,\lambda_2,z_1,1)\frac{1}{\lambda_3-\lambda_2}e^{(\lambda_2-\lambda_1)z_1}.$$

$$I_k=\int_{z_{k-1}}^{y_k}\int_{z_{k-2}}^{y_{k-1}}\ldots\int_{z_1}^{y_2}\exp\left((\lambda_2-\lambda_1)y_1+(\lambda_3-\lambda_2)y_2+\ldots+(\lambda_k-\lambda_{k-1})y_{k-1}\right)dy_1\ldots dy_{k-1}=$$

$$=\sum_{j=1}^{k-1}c_k(\lambda_1,\lambda_2,\ldots,\lambda_k,z_1,\ldots,z_{k-1},j)\,I(\lambda_k,\lambda_j,z_{k-1},y_k).$$

Coefficients $c_k(\lambda_1,\lambda_2,\ldots,\lambda_k,z_1,\ldots,z_{k-1},j)$ are evaluated as it is indicated in the theorem. Now integral I may be expressed as

$$I=\lambda_1\ldots\lambda_k\int_x^{+\infty}\sum_{j=1}^{k-1}c_k(\lambda_1,\lambda_2,\ldots,\lambda_k,z_1,\ldots,z_{k-1},j)\,I(\lambda_k,\lambda_j,z_{k-1},y_k)e^{-\lambda_k y_k}dy_k. \quad (4)$$

Since

$$\int_x^{+\infty}I(\lambda_k,\lambda_j,z_{k-1},y_k)e^{-\lambda_k y_k}dy_k=\int_x^{+\infty}\left(e^{(\lambda_k-\lambda_j)y_k}-e^{(\lambda_k-\lambda_j)z_{k-1}}\right)e^{-\lambda_k y_k}dy_k$$

$$= \frac{1}{\lambda_j} e^{-\lambda_j x} - \frac{1}{\lambda_k} e^{(\lambda_k - \lambda_j) z_{k-1}} e^{-\lambda_k x},$$

combining (1-4) we obtain our main result.

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