REINSURANCE BERMUDAN STYLE

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Abstract

We present a class of new reinsurance covers with features similar to Bermudan options popular in the financial industry. The insurer can buy, say, a cover which will pay the claims emanating during a fixed time interval within one year. For example, the insurer has the right to exercise an option at the end of any month. The reinsurer will then pay the claims of that month and the contract expires.

If the covered claim interval is very short, for instance only a few days, then the cover works similar to a catastrophe treaty except that there is always a payout since at the end of the year it is rational for the option holder to exercise. On the other hand, if the period agreed on is long, say one full year in the extreme case, the contract collapses to a traditional reinsurance cover. Hence, the cover provides for a very flexible tool to optimize existing reinsurance programs.

We derive the optimal strategy the insurer will have to pursue to decide on when to exercise the option. To determine the net–premium of the cover, we use the techniques of dynamic programming and derive explicit recursive solution schemes for a variety of problems and general claim distributions.

1. Introduction

The family of reinsurance covers we present in the following addresses the need for coverage which is in scope between a Catastrophe–XL and a standard one– or multi–year cover. A Catastrophe–XL can be seen as covering the claims emanating from a single event which is highly localized in time, i.e. a catastrophe is in general defined as an event with a short (often less than three days) duration. In contrast, a normal one–year reinsurance cover pays obviously for claims which occur during a whole year. In between these two extremes there was so far no reinsurance coverage available except some specialized tailor–made solutions.

However there might very well be the need for coverage which on the one hand pays out in more situations than a Catastrophe–XL – which by its very nature only rarely is exercised and which is, consequently, very difficult to price – and on the other hand a cover which is 'smaller' and therefore also cheaper than a full–year reinsurance solution.

We propose a flexible suite of reinsurance covers which are of Bermudan Option type in the sense that the client can choose –at a predefined set of dates during the lifetime of the cover - when the cover shall be exercised. For this the client first decides on the duration of the coverage, say one week or one month. During the year she then has the option to decide on the time interval during which the reinsurance is activated. There are several different possibilities when the client is

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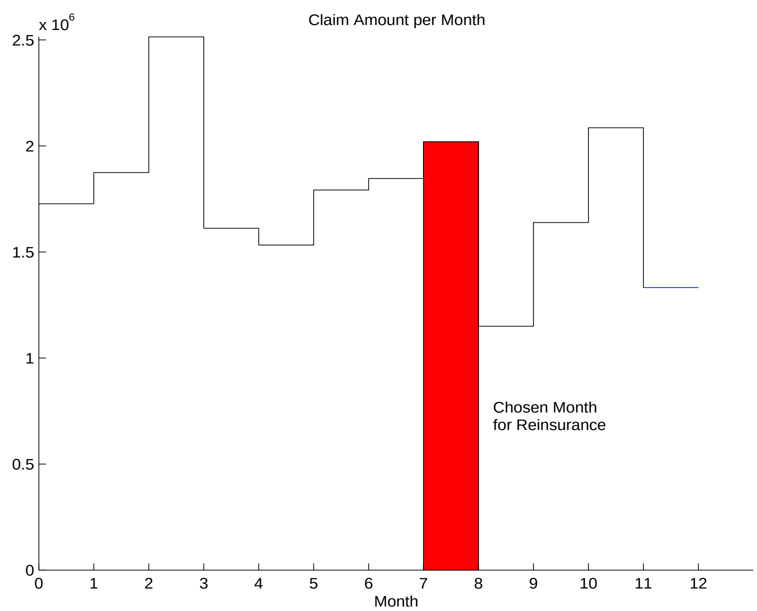
allowed to make her choice. The most expensive cover will be one when the client can wait and decide at the very end of the year. At this time she will know the time interval with the maximal amount of claims and will consequently choose this as the one to be reinsured.

A cheaper cover, and a mathematically more challenging one, is one where the client has to choose when to exercise the option during the year. At each time t she can exercise the cover to pay the claims emanating during the last, say, d days. The figure below shows an example of such a contract, where the choice is at the end of each month whether to exercise or not. In this example, she chooses to invoke the cover at the end of month 8 and the reinsurer then pays the claims which were incurred during month 8. Would it would have been better to exercise at the end of month 1 or 11? To decide whether it was rational to exercise the cover after month 9, we need a definition of the objective of the buyer of the option.



The expected payout of the cover will depend on the strategy of the buyer. For the pricing we have to assume that the buyer of the cover pursues a strategy which maximizes in the long run her expected utility from the cover. It is interesting to note that this problem is closely related to the so called *Harem problem*:

> A sultan can choose each year among a selection of n ladies one for his harem. Once rejected, a lady cannot be recalled again. It is assumed that the sultan has a preference ordering of the girls. Depending on what he intends to maximize, the sultan has different optimal strategies of how to choose his ladies. For instance if he wants to maximize the probability of obtaining the most beautiful lady, then his best strategy is for large n to let go of the first n/e ($e = 2.7182818\cdot$) lady and then choosing the first one more beautiful than all before.

However, it should be noted that the above would not be an optimal strategy for the insurer, at least not if she considers maximizing the expected payout instead of maximizing the probability of getting the maximal payout. This, and also political correctness, made us refrain from calling it the *Harem Cover* although it would have been a somewhat catchier title than *Reinsurance Bermudan Style*.

There are several generalizations and extensions of the cover possible. The main one is when one can buy not one but m options to exercise during the year. For instance she might buy four weekly covers such that she would be covered for a total of one month during the year. However, the four exercised time–intervals do not have to be consecutive. In the limit a client might buy 52 weekly covers, protecting her portfolio thereby the whole year. Also possible would be to consider the time–continuous case or the case where the reinsurer does not pay claims coming from the immediately preceding time–interval but from one, say, a month in the past.

2. The Basic Model

We assume that the duration of coverage chosen by the client is d days and that the time intervals which can be reinsured are of the form $[(k-1)d, kd)$ for $k = 1, \ldots, n$, where n is (usually) such that the product dn will be approximately one year. We furthermore denote the reinsured claims emanating from time–interval $[(k-1)d, kd)$ by D_k, $k = 1, \ldots, n$ and we assume that they are independent but not necessarily identically distributed. Note that the distributions F_{D_k} depend crucially on the type of reinsurance. For instance, for the same given portfolio, the claim distributions are different for a stop–loss, a quota share or an excess–of–loss reinsurance cover.

The main simplifying assumption we use is that a client can choose the time interval under coverage only at specific times t of the form $t = kd$, $k = 1, \ldots, n$, i.e. that the cover is of Bermudan type in the sense that there is a finite number of predefined possible exercise dates. At each time kd the client can decide to exercise the cover and the reinsurer then pays for the claims emanating from the immediately preceding interval $[(k-1)d, kd)$ or, alternatively, the client can decide to wait. At the end of the year, i.e. at time nd, the client has to choose obviously the last time interval $[(n-1)d, nd)$.

To price the cover, we have to assume that the client pursues an optimal strategy. By an optimal strategy we understand the implementation of a decision rule on when to stop, i.e. on when to invoke the cover such that on the long run the expected payout of the cover is maximized.

Let $\{\mathcal{F}_k\}$, $k = 0, \ldots, n$, be an increasing family of σ–algebras generated by the claim process $\{D_k\}_k$, $k = 1, \ldots, n$ and $\mathcal{F}_0 = \emptyset$. We assume the stopping time T to take on values in $\{d, 2d \ldots, nd\}$ and that the event $\{T \leq kd\} \in \mathcal{F}_k$ for all $k = 1, \ldots, n$.

Let now g be a monotonuously growing, positive function, called the reward function. Then the optimal stopping time T^* for the cover is defined as the solution of

$$T^* = \text{argmax}_T E[g(D_T)] \,.$$

The above equation is not yet very intuitive, however using the discrete time structure of our problem we can simplify the calculations. Denote by E_k, $k = 1, \ldots, n$, the expected value of the cover at time $t = kd$ under the assumption of using an optimal strategy with respect to the reward function g, i.e.

$$E_k = E[g(D_{T^*}) \mid T^* \geq kd] \,.$$

It is instructive to argue by going backward in time and calculating the optimal stopping time T^* recursively. Assume first that $t = nd$. That means that the claim history $D_1, \ldots, D_n$ was such that the cover was never exercised. Then obviously the insurer has to invoke the cover and $T^* = nd$. The value E_n of the cover is then $g(D_n)$ and the expected value is $E_n = E[g(D_n)]$. Next assume that $t = kd$, $k = 1, \ldots, n-1$. Then either $T^* = kd$ or $T^* \geq (k+1)d$. Depending on the reward $g(D_k)$ incurred during the time interval immediately preceding kd the client decides either to stop or wait. To be more precise, if $g(D_k) \geq u_k$, then $T^* = kd$ or if $g(D_k) < u_k$, $T^* \geq (k+1)d$ for a yet to be determined values u_k. The unknowns u_k have to be chosen such that

$$P(g(D_k) \geq u_k)E[g(D_k) \mid g(D_k) \geq u_k] + P(g(D_k) < u_k)E_{k+1}$$

is maximized. The equation above can be rewritten as

$$\int_{g^{-1}(u_k)}^{\infty} g(x) f_{D_k}(x) dx + F_{D_k}(g^{-1}(u_k)) E_{k+1} \,.$$

Taking the derivative w.r.t. $g^{-1}(u_k)$ and setting the thus obtained equation to zero we obtain

$$-u_k f_{D_k}(g^{-1}(u_k)) + f_{D_k}(g^{-1}(u_k)) E_{k+1} = 0 \,,$$

hence $u_k = E_{k+1}$. In this way we have the simple recursive scheme

$$\begin{aligned} u_k &= E_{k+1} \,, \\ E_k &= \int_{g^{-1}(u_k)}^{\infty} g(x) f_{D_k}(x) dx + F_{D_k}(g^{-1}(u_k)) E_{k+1} \,. \end{aligned}$$

We can simplify the above equation and obtain finally the recursive formula for E_k

$$\begin{aligned} E_n &= E[g(D_n)] \,, \\ E_k &= E_{k+1} + \int_{g^{-1}(E_{k+1})}^{\infty} g'(x) \bar{F}_{D_k}(x) dx \,,\ k = 1, \ldots, n-1 \,, \end{aligned}$$

where $\bar{F}_{D_k} = 1 - F_{D_k}$. If we take for the reward function the identity, i.e. $g(x) = x$, the optimal strategy becomes intuitively obvious. The cover will be invoked at time $t = kd$ if the claim D_k immediately preceding t is larger than the future expected payoff E_{k+1} of the cover. Alternatively, one has to wait if the immediate gain D_k is less than the future expected gain.

It is not difficult to calculate the distributions of the optimal stopping time T^* and of the payout of the cover. First, denote by f_{T^*} the density function of the optimal stopping time at the start of the cover. We easily see that

$$f_{T^*}(k) = P(T^* = k) = \begin{cases} \prod_{j=1}^{k-1} F_{D_j}(E_{j+1}) \bar{F}_{D_k}(E_{k+1}) \,, & k = 1, \ldots, n-1 \,, \\ \prod_{j=1}^{n-1} F_{D_j}(E_n) \,, & k = n \,. \end{cases}$$

Next, let F_E be the distribution function of the payout of the cover at the begin of the year, i.e.

$$F_E(x) = P(D_{T^*} \leq x) \,.$$

Then

$$\begin{aligned} F_E(x) &= \sum_{j=1}^{n-1} P(T^* = j) P(D_j \le x | D_j > E_{j+1}) + P(T^* = n) P(D_n \le x) \\ &= \sum_{j=1}^{n-1} P(T^* = j) \frac{(F_{D_j}(x) - F_{D_j}(E_{j+1}))_+}{\bar{F}_{D_j}(E_{j+1})} + P(T^* = n) F_{D_n}(x) \,. \end{aligned}$$

Analogously, the density f_E is

$$f_E(x) = \sum_{j=1}^{n-1} P(T^* = k) \frac{f_{D_j}(x)}{\bar{F}_{D_j}(E_{j+1})} \chi_{F_{D_j}(x) > F_{D_j}(E_{j+1})} + P(T^* = n) f_{D_n}(x) \,,$$

where χ stands for the indicator function.

2.1. **Example.** We consider a simple toy example where we assume the yearly claim D to follow a Gamma law $\Gamma(a, b)$ with parameter a and b such that the yearly average claim is ab and its variance ab^2:

$$f_D(x) = \frac{1}{b\Gamma(a)} (x/b)^{a-1} \exp(-x/b) \,.$$

Then the claim amount during a duration of d days such that dn is equal to one year is $\Gamma(a/n, b)$ distributed. For the reward function g we assume $g(x) = x$.

First we consider a cover with $n = 4$, i.e. where the reinsurer pays one season during the year. If the cover starts at 1 January, the client can choose at the four dates T_1 =1 April, T_2 =1 July, T_3 =1 October and T_4 =1 January of the following year to invoke the cover. We assume that $a = 50$ and $b = 20$ hence the mean claim amount during one year is 1000 and the variance is 20'000. For a season the mean claim is 250 and the variance is 5'000. Using the above equations we can easily calculate the values of E_k, $k = 1, \dots, 4$ and we obtain:

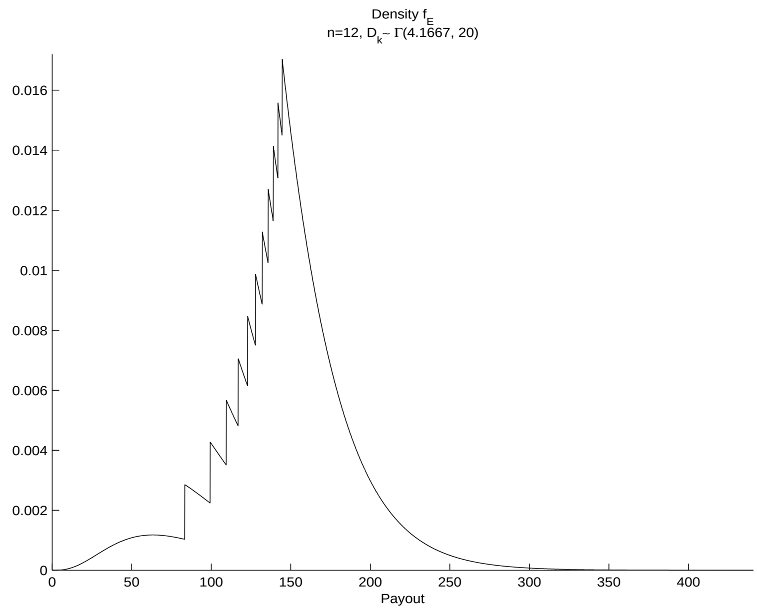
$$\begin{aligned} E_1 &= 307.56 \,, \\ E_2 &= 295.18 \,, \\ E_3 &= 278.02 \,, \\ E_4 &= 250.00 \,. \end{aligned}$$

Since $u_k = E_{k+1}$ in the case where only one time interval is payed by the reinsurer, a client has to invoke the cover at time T_k, $k = 1, 2, 3$ if the claim amount during the preceding three months exceeds E_{k+1}. If the cover was not invoked at the end of the year, the cover obviously has to be exercised at 1 January of the next year, i.e. $u_4 = 0$. The probability that the cover will be chosen at times $T = T_k$ are $f_{T^*}(1) = 0.2428$, $f_{T^*}(2) = 0.2401$, $f_{T^*}(3) = 0.2391$ and $f_{T^*}(4) = 0.2780$.

For $n = 12$, i.e. when the reinsurer pays the claims emanating during one month we obtain the following data

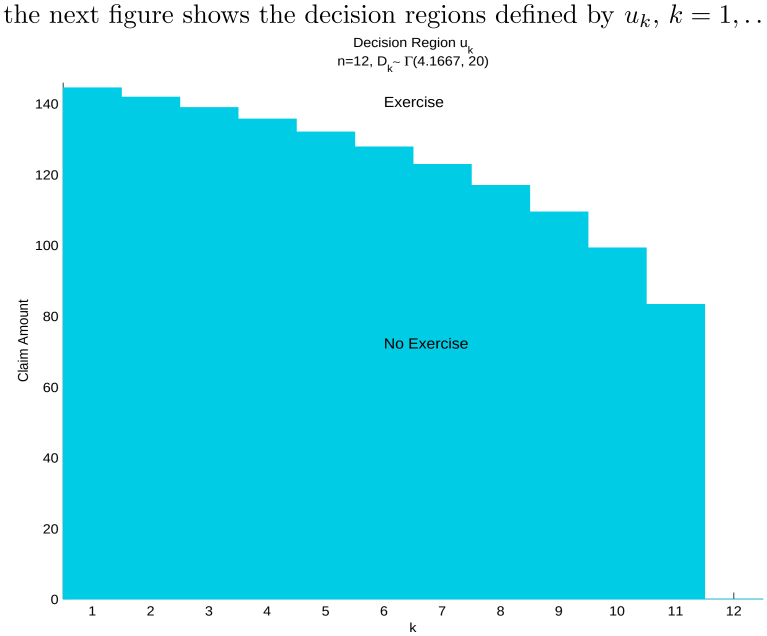
E_1	$= 146.95$	E_7	$= 127.83$	$f_{T^*}(1)$	$= 0.08175$	$f_{T^*}(7)$	$= 0.08051$
E_2	$= 144.53$	E_8	$= 122.90$	$f_{T^*}(2)$	$= 0.08156$	$f_{T^*}(8)$	$= 0.08038$
E_3	$= 141.88$	E_9	$= 116.95$	$f_{T^*}(3)$	$= 0.08135$	$f_{T^*}(9)$	$= 0.08040$
E_4	$= 138.96$	E_{10}	$= 109.46$	$f_{T^*}(4)$	$= 0.08114$	$f_{T^*}(10)$	$= 0.08083$
E_5	$= 135.71$	E_{11}	$= 99.30$	$f_{T^*}(5)$	$= 0.08092$	$f_{T^*}(11)$	$= 0.08281$
E_6	$= 132.04$	E_{12}	$= 83.33$	$f_{T^*}(6)$	$= 0.08071$	$f_{T^*}(12)$	$= 0.10764$

The picture below shows the density of the payout of the cover f_E:



The jagged form of the density stems from the discrete set of possible exercise times. Depending during which month the cover is exercised the payout density differs and the supposition over all months makes it look rather unusual.

Finally the next figure shows the decision regions defined by u_k, $k = 1, \ldots, n$.



3. Multiple Options

As a generalization of the cover described above, we now consider the case when the insurer can buy coverage for m separate time intervals. For instance a client might buy four weekly covers so that during one year she can choose four times to reinsure the preceding week.

To calculate the premium we proceed essentially as before, however we have to add a further recursion with respect to the number m of time intervals. Denote by E_k^j the expected future payout of a cover with j time intervals at time $t = kd$ under the assumption that an optimal strategy is implemented. Obviously E_k^j is only defined for $k \leq n - j + 1$. Were $k > n - j + 1$, then it would not be possible to invoke the cover j times which is in contradiction to the assumption that the insurer pursues an optimal strategy.

Assume first that $t = (n - j + 1)d$. Then the buyer of the cover has to exercise the cover during each time interval $[(k-1)d, kd]$, $k = n - j + 1, \ldots, n$ and

$$E_{n-j+1}^j = \sum_{i=n-j+1}^{n} E[g(D_i)]\,.$$

We proceed as before. First we have to maximize

$$P(g(D_k) \geq u_k^j)\left(E[g(D_k)|g(D_k) \geq u_k^j] + E_{k+1}^{j-1}\right) + P(g(D_k) < u_k^j)E_{k+1}^j\,.$$

Here the cover with j time intervals left will be invoked at time k if the immediate gain $E[g(D_k)|g(D_k) \geq u_k^j]$ together with the expected future gain E_{k+1}^{j-1} with one time interval less is larger than the expected future profit when the cover is not exercised E_{k+1}^j.

Taking the derivative w.r.t. $g^{-1}(u_k^j)$, setting the thus obtained equation to zero and solving for u_t^j we obtain

$$u_k^j = -E_{k+1}^{j-1} + E_{k+1}^j$$

and

$$\begin{aligned} E_k^j =& \bar{F}_{D_k}(g^{-1}(-E_{k+1}^{j-1} + E_{k+1}^j))E_{k+1}^{j-1} \\ &+ \int\limits_{g^{-1}(-E_{k+1}^{j-1}+E_{k+1}^j)}^{\infty} g(x)f_{D_k}(x)dx + F_{D_k}(g^{-1}(-E_{k+1}^{j-1} + E_{k+1}^j))E_{k+1}^j\,. \end{aligned}$$

Again we can simplify the above equation, yielding the recursion

$$\begin{aligned} E_n^1 &= E[g(D_n)]\,, \\ E_k^1 &= E_{k+1}^1 + \int\limits_{g^{-1}(E_{k+1}^1)}^{\infty} g'(x)\bar{F}_{D_k}(x)dx\,,\ k < n\,, \\ E_k^j &= E_{k+1}^j + \int\limits_{g^{-1}(-E_{k+1}^{j-1}+E_{k+1}^j)}^{\infty} g'(x)\bar{F}_{D_k}(x)dx\,. \end{aligned}$$

It is relatively easy to calculate the densities $f_{T_j^*}$, $j = 1, \ldots, m$ of the m optimal stopping times $1 \leq T_1^* < T_2^* < \cdots < T_m^* \leq n$.

$$
\begin{aligned}
f_{T_1^*}(k) &= \prod_{j=1}^{k-1} F_{D_j}(u_j^m)\bar{F}_{D_k}(u_k^m)\,,\ k = 1,\dots,n-m\,,\\
f_{T_1^*}(n-m+1) &= \prod_{j=1}^{n-m} F_{D_j}(u_j^m)\,,\\
f_{T_l^*}(k) &= \sum_{i=1}^{k-1} f_{T_{l-1}^*}(i) \prod_{j=i+1}^{k-1} F_{D_j}(u_j^{m+1-l})\bar{F}_{D_k}(u_k^{m+1-k})\,,\ l = 2,\dots,m\,,\\
&\quad k = l,\dots,n-m+l\,.
\end{aligned}
$$

The above system of equations can again be solved recursively.

In principle we can determine the distribution of the payout of the cover as in the case when only one time–interval can be chosen during the year. However, the calculations get much more involved and tedious. We would have to calculate

$$P(E \le x) = P(D_{T_1^*} + \dots + D_{T_m^*} \le x)\,,$$

which necessitates an m–fold convolution.

3.1. **Example.** We use the same assumptions and data as in the previous example. For $n = 4$ we obtain

$$
\begin{array}{lllllllllll}
E_1^1 &=& 307.56 & E_2^1 &=& 295.18 & E_3^1 &=& 278.02 & E_4^1 &=& 250.00\\
E_1^2 &=& 572.17 & E_2^2 &=& 543.23 & E_3^2 &=& 500.00\\
E_1^3 &=& 803.39 & E_2^3 &=& 750.00\\
E_1^4 &=& 1000.00
\end{array}
$$

Hence a cover for, say, three three month periods during the year would cost 803.39.

In order to pursue the optimal strategy one has to use the u_k^j which are given below for a reward function $g(x) = x$.

$$
\begin{array}{lllllllllll}
u_1^1 &=& 295.18 & u_2^1 &=& 278.02 & u_3^1 &=& 250.00 & u_4^1 &=& 0.00\\
u_1^2 &=& 248.05 & u_2^2 &=& 221.98 & u_3^2 &=& 0.00\\
u_1^3 &=& 206.77 & u_2^3 &=& 0.00\\
u_1^4 &=& 0.00
\end{array}
$$

It is instructive to consider in detail a cover with $n = 12$ and $m = 3$ and

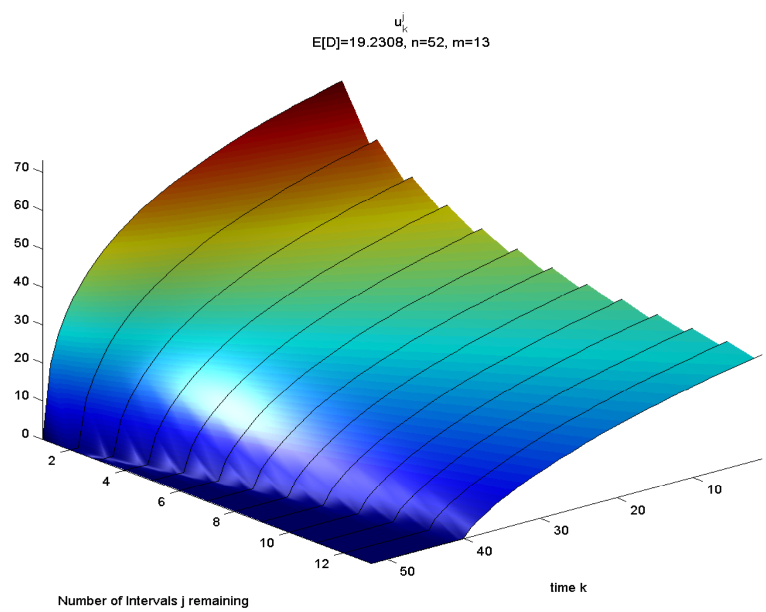
u_k^j	1	2	3
1			107.72
2		119.18	104.58
3	138.96	115.96	101.07
4	135.71	112.36	97.09
5	132.04	108.25	92.51
6	127.83	103.48	87.09
7	122.90	97.78	80.45
8	116.95	90.73	71.82
9	109.46	81.40	59.14
10	99.30	67.37	0.00
11	83.33	0.00	
12	0.00		

Consider as an example now the sequence of claims

$$\begin{array}{llllllllllll} D_1 & = & 79.77 & D_2 & = & 34.95 & D_3 & = & 141.23 & D_4 & = & 47.04 \\ D_5 & = & 125.29 & D_6 & = & 39.05 & D_7 & = & 152.66 & D_8 & = & 37.34 \\ D_9 & = & 99.63 & D_{10} & = & 50.92 & D_{11} & = & 49.31 & D_{12} & = & 64.09 \end{array}$$

The first month the insurer has to pass since $79.77 < 107.72$ ($D_1 < u_1^3$), as she has at the end of February. However at the end of March she has to invoke the cover for the first time since $141.23 > 101.07$ ($D_3 > u_3^3$). The fourth month she again has to pass since $39.05 < 112.36$ ($D_4 < u_4^2$). She has to exercise the cover for the second time for May since $125.29 > 108.25$ ($D_5 > u_5^2$). Again she passes June but she exercises for the last time for July since $152.66 > 122.9$ ($D_7 > u_7^1$).

We calculated the values E_k^j for a realistic case where a client buys coverage for three months during a full year, subdivided into 13 weekly covers, i.e. $n = 52$, $m = 13$. As always we assume the yearly claim to follow a Gamma law with mean 1'000 and variance 20'000. The figures below show the surfaces defined by E_k^j and u_k^j.



Finally the figure below shows the density of the stopping times $T_1^*, \ldots, T_m^*$.

3.2. **Reinstatements.** One disadvantage of the proposed schemes is that if the cover is exercised before end of treaty period, the company is left with no more reinsurance. However, one can give the company the right to reinstate the cover after exercising it. In particular, we assume that a company will buy a reinstatement which renews the treaty until the end of the year, i.e. if for instance the cover was exercised in June then the reinstatement would buy a cover from July until the end of December.

The price for a reinstatement can quite easily be determined. Assume that the cover gave the right for choosing one time–interval. If the cover was exercised for interval k, i.e. the reinsurer pays for claims occurring during $[(k-1)d, kd]$ then then value of the cover from $[kd, nd]$ is by definition E_{k+1}. Hence to offer reinstatement it suffices to communicate the sequence $\{E_k\}_{k=1,\ldots,n}$. The same can obviously be done for the more complicated cover where a number of time–interval can be chosen during the year, however the principle stays the same.

This procedure has the additional advantage that the buyer of the cover is informed about the optimal strategy which is defined by E_k, $k = 1, \ldots, n$.

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