Risk and Savings Contracts

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Cancun, March 2002

1 Time-capitals

- X = *remaining lifetime* of the insured of age x at policy issue.
- A *time-capital* is a series of amounts $c_k(X)$ payable at times $t_k(X)$.
- Notation:

$$Q^{\circ\circ} = \sum_{k=1}^{n} \left(c_k(X),\ t_k(X)\right).$$

- Sums, differences and scalar multiplication are defined in the obvious way.

- Consider the *time-capital* $Q^{\circ\circ}$:

$$Q^{\circ\circ} = \sum_{k=1}^{n} \left(c_k(X),\ t_k(X) \right).$$

- The *present value* of time-capital $Q^{\circ\circ}$ is defined as

$$Q^{\circ} = \sum_{k=1}^{n} c_k(X)\ v^{t_k(X)}.$$

- The *actuarial value* (or *price*) of time-capital $Q^{\circ\circ}$ is defined as

$$Q = E\left[Q^{\circ}\right] = \sum_{k=1}^{n} E\left[c_k(X)\ v^{t_k(X)}\right].$$

- **Examples:**

– t-year pure endowment on a life aged x:

$$ {}_tE_x^{\circ\circ} \doteqdot \left(1_{X>t},\ t\right), $$

– n-year temporary life annuity-due on a life aged x:

$$ {}_n\ddot{a}_x^{\circ\circ} \doteqdot \sum_{k=0}^{n-1} {}_kE_x^{\circ\circ}, $$

– whole life insurance on x, payable at the moment of death:

$$ \overline{A}_x^{\circ\circ} \doteqdot \left(1,\ X\right). $$

2 Restricted time-capitals

- Let $Q^{\circ\circ}$ be a l.c.of deterministic time-capitals, pure endowments, life annuities and life insurances on a life aged x.
- ${}_{s|t}Q^{\circ\circ} \doteqdot$ the restriction of $Q^{\circ\circ}$ to $[s, s+t[$.
- ${}_{\bullet s|t}Q^{\circ\circ} \doteqdot$ the reactualized restriction of $Q^{\circ\circ}$ to $[s, s+t[$.
- Notation: ${}_{s|\infty}Q^{\circ\circ} = {}_{s|}\ Q^{\circ\circ}$

3 Reserves of a time-capital

- The reserve at t when (x) is alive at t:

$$ {}_{\bullet t|}Q_x \doteqdot E\left[{}_{\bullet t|}Q^\circ | X > t\right] $$

- The reserve at t, when (x) died at $s < t$:

$$ {}_{\bullet t|}Q_{x|}(s) \doteqdot E({}_{\bullet t|}Q^\circ | X = s \leq t), \qquad (s \leq t). $$

4 Life insurance contracts

- A life insurance contract on a life (x) is a couple $(B^{\circ\circ}, P^{\circ\circ})$ of time-capitals, depending on X:

$$\begin{aligned} B^{\circ\circ} &: \text{benefits to be paid by the insurer,} \\ P^{\circ\circ} &: \text{premiums to be paid by the insured.} \end{aligned}$$

- $V^{\circ\circ} = B^{\circ\circ} - P^{\circ\circ} =$ the reserve time-capital.

- The equivalence principle:

$$(B^{\circ\circ}, P^{\circ\circ}) \text{ is fair} \Leftrightarrow B = P \Leftrightarrow V = 0.$$

- A <u>general life insurance contract:</u>

Consider $(B^{\circ\circ},\ P^{\circ\circ})$ with

$$B^{\circ\circ} = \ \sum_{j=0}^{n} L_j \ {}_jE_x^{\circ\circ} + \sum_{j=0}^{n-1} D_k \ {}_{k|1}\hat{A}_x^{\circ\circ},$$

$$P^{\circ\circ} = \ \sum_{j=0}^{n} P_j \ {}_jE_x^{\circ\circ}.$$

Assumption: $(B^{\circ\circ}, P^{\circ\circ})$ is fair.

- The *associated deterministic savings contract*

  $(B^{(ads)\circ\circ},\ P^{(ads)\circ\circ})$ of a life insurance contract $(B^{\circ\circ},\ P^{\circ\circ})$ on a life (x) is obtained by replacing the d.f. of X by the d.f. of a status that will exist eternally.

- Remark: The (ads)-contract of a fair life insurance contract is in general not a fair contract.

- A <u>general life insurance contract</u>:

Consider $(B^{\circ\circ},\ P^{\circ\circ})$ with

$$B^{\circ\circ} = \ \textstyle\sum_{j=0}^{n} L_j \ {}_jE_x^{\circ\circ} + \sum_{j=0}^{n-1} D_k \ {}_{k|1}\hat{A}_x^{\circ\circ},$$

$$P^{\circ\circ} = \ \textstyle\sum_{j=0}^{n} P_j \ {}_jE_x^{\circ\circ}.$$

The (ads)- contract is given by $(B^{(ads)\circ\circ},\ P^{(ads)\circ\circ})$ with

$$B^{(ads)\circ\circ} = \ \textstyle\sum_{j=0}^{n}(L_j,\ j).$$

$$P^{(ads)\circ\circ} = \ \textstyle\sum_{j=0}^{n}(P_j,\ j)$$

5 Savings contracts (on a life (x))

- A life insurance contract $(B^{\circ\circ},\ P^{\circ\circ})$ is a *savings contract* if

$$\begin{aligned} {}_{\bullet k|}V_x &= {}_{\bullet k|}V^{ads} \\ &= \sum_{j=0}^{k-1}(P_j - L_j)\ u^{k-j}, \qquad (k = 0, 1, ..., n). \end{aligned}$$

- Survival probabilities need not be known to determine the reserves.

- <u>THEOREM</u>: A life insurance contract on a life (x) is a savings contract if and only if

$$\sum_{j=0}^{n} P_j \ v^j = \sum_{j=0}^{n} L_j \ v^j$$

and for $k = 0, 1, ..., n$,

$$\bullet_{k|1} B_x = L_k + \left(\sum_{j=0}^{k} (P_j - L_j) \ u^{k+1-j} \right) \ {}_1A_{x+k}.$$

6 Risk contracts on a life (x)

- A risk contract on a life (x) is a life insurance contract with only payments-at-death (no survival benefits) and such that

$$_{\bullet k|}V_x = 0, \qquad (k = 0, 1, ..., n).$$

- THEOREM: A life insurance contract on a life (x) with no survival benefits is a risk contract if and only if

$$_{\bullet k|1}B_x = P_k.$$

7 Combination of life insurance contracts

- $(B^{(1)\circ\circ}, P^{(1)\circ\circ}) + (B^{(2)\circ\circ}, P^{(2)\circ\circ})$
$$\doteqdot \left(B^{(1)\circ\circ} + B^{(2)\circ\circ}, P^{(1)\circ\circ} + P^{(2)\circ\circ}\right)$$
- $_{\bullet t|}V_x^{(1)+(2)} = {}_{\bullet t|}V_x^{(1)} + {}_{\bullet t|}V_x^{(2)},$
- <u>THEOREM</u>:
Any life insurance contract $(B^{\circ\circ}, P^{\circ\circ})$ is the unique combination of a savings contract $(B^{(s)\circ\circ}, P^{(s)\circ\circ})$ and a risk contract $(B^{(r)\circ\circ}, P^{(r)\circ\circ})$.

- <u>Term Insurance</u> : $\left({}_{n}\hat{A}_x^{\circ\circ},\ \pi\ \ddot{a}^{\circ\circ}_{x:\overline{n}|}\right)$.

$$
\begin{aligned}
B^{(s)\circ\circ} &= \sum_{k=0}^{n-1} {}_{\bullet k+1|}V_x\ v^{\frac{1}{2}}\ {}_{k|1}\hat{A}_x^{\circ\circ}; \\
P^{(s)\circ\circ} &= \sum_{k=0}^{n-1} \left({}_{\bullet k+1|}V_x\ v - \ {}_{\bullet k|}V_x\right)\ {}_{k}E_x^{\circ\circ}; \\
{}_{\bullet k|}V_x &= \pi\ \ddot{s}_{\overline{k}|} - \sum_{j=0}^{k-1} q_{x+j}\ \left(1 - \ {}_{\bullet j+1|}V_x\ v^{\frac{1}{2}}\right)\ u^{k-j-\frac{1}{2}}.
\end{aligned}
$$

- Pure Endowment : $\left({}_{n}E_{x}^{\circ\circ},\ \pi\ \ddot{a}^{\circ\circ}_{x:\overline{n}|}\right)$.

$$
\begin{aligned}
B^{(s)\circ\circ} &= {}_{n}E_{x}^{\circ\circ} + \sum_{k=0}^{n-1} {}_{\bullet k+1|}V_x \; v^{\frac{1}{2}} \; {}_{k|1}\hat{A}_{x}^{\circ\circ}; \\
P^{(s)\circ\circ} &= \sum_{k=0}^{n-1} \left({}_{\bullet k+1|}V_x \; v - {}_{\bullet k|}V_x \right) \; {}_{k}E_{x}^{\circ\circ}; \\
{}_{\bullet k|}V_x &= \pi \; \ddot{s}_{\overline{k}|} + \sum_{j=0}^{k-1} q_{x+j} \; {}_{\bullet j+1|}V_x \; u^{k-j-1}.
\end{aligned}
$$