PREDICTION OF MORTALITIES. A COMPARATIVE DANISH STUDY.*

Peter Fledelius†
Aarhus School of Business &
Codan Forsikring

Jens Perch Nielsen‡
Codan Forsikring

September 3, 2001

Abstract

Prediction of the future mortality of life annuity portfolios is one of the major concerns faced by life insurance companies. Prediction of mortality curves is, however, a complicated challenge and a number of prediction models have been developed by scientists, actuaries, demographers and political planners. This paper investigates the simplest and perhaps most popular prediction model, namely the prediction model using the currently observed mortality as a prediction for future mortality. We call this predictor the 'at risk' estimator. This prediction estimator is compared to the actually observed mortality throughout the period 1900-1996 in Denmark. Our investigations show that the maximal error made by the simple prediction model corresponds to an interest rate of 0.7% on life annuities. Another way of formulating this is to say that old age retirees of age 65 have lived up to two years longer than the 'at risk' prediction model has indicated.

Key Words: Model Validation, Back-test, Longevity; Expected Remaining Life Time; Kernel Hazard Estimation; Cohort

*We thank Kirill Andreev for his help regarding the data used in the investigations.
†Codanhus, 60 Gammel Kongevej, DK-1790 Copenhagen V, Denmark. Email: pfs@codan.dk.
‡Codanhus, 60 Gammel Kongevej, DK-1790 Copenhagen V, Denmark. Email: npj@codan.dk.

1 Introduction

Prediction of the future mortality of life annuity portfolios is one of the major concerns faced by life insurance companies. This longevity is a known problem. Welfare plans, general life planning and actuarial products are strongly depending on reliable mortality estimation and life span prediction. Previous studies have established that life expectancy is increasing (e.g. Macdonald, Cairns, Gwilt and Miller, 1998). Most of these studies investigate the decline in mortality of the general population. In this paper we will investigate the prediction problem and go through back-testing for validation of the model. The single most important rule of prediction is that any prediction model has to be validated. This paper makes a thorough examination of the simplest and most popular prediction model, namely the prediction model using the currently observed mortality as a prediction for future old-age mortality. We call this predictor for the 'at risk' estimator. This prediction estimator is compared to the actually observed old-age mortality throughout the period 1900-1996 in Denmark. The latter comparison corresponds to what mathematical statisticians would describe as a back-test validation of the model. The errors made by the prediction model throughout the last century can be instructive as a rule-of-thumb type of measure of the magnitude of the errors made by the simple 'at risk' prediction model. Our study shows that the maximal error made by the simple prediction model corresponds to an interest rate of 0.7% on life annuities. Another way of formulating this is that old-age retirees have lived up to two years longer than the 'at risk' prediction model has indicated. We consider this result as an illustration of the usefulness of the simple prediction model, since it did relatively well during the turbulent 20'th century. If actuaries, selling life annuities today, reserve, lets say, something around 1% on the interest rate as a safety margin, then we would consider the reserving problem satisfactorily solved. Our point of validation implies that any other prediction model used should be validated in a similar manner as the validation in this paper. It is an established fact from general prediction theory that complication of the model often harms the quality of a prediction. Therefore, we expect it to be difficult, but not impossible, to make a better prediction than the one performed by the simple 'at risk' method. However, we do expect many currently used prediciton models to actually do worse than the simple prediction model considered here. Some more complicated prediction techniques include parametric graduation techniques (see Cramer and Wold, 1935, Buus, 1960 and Benjamin and Soliman, 1995) followed by a projection of these parameters into the future.

Methodologically our starting point is the two-dimensional mortality estimator that was defined theoretically in Nielsen and Linton (1995) and applied to Danish and Spanish mortality data in Felipe, Guillen and Nielsen (2001). This two-dimensional estimator considers the mortality, $\alpha(t, x)$, as a function of chronological time, t and age x. All the statistics of this paper are functions of the two-dimensional curve $\alpha(t, x)$.

In §2 we specify the Danish data for the period from 1900 to 1996. In §3 we consider the prediction principle and the underlying hazard estimator. In §4 we compare the 'at risk' hazard estimator with the experienced hazard estimator. In §5 we compare life expectancy derived from the two different hazard estimators and in §6 we discuss the economic aspects of the two hazards when considering liabilities for old age retirees.

All technicalities and estimation techniques are deferred to the appendix that starts with the general model formulation based on counting processes followed by the definition of the version of the kernel hazard estimator of Nielsen and Linton (1995) which is of particular interest to our study.

2 Data

The present case study is based on Danish population data. We have chosen to work with the subset covering the period 1900-1996 selected from the original data covering the period 1835-1996, see Andreev (1999). The data are discretized annually, i.e. we have observed $O_{t,x}$, the number of occurred deaths in each year at a certain age, and $E_{t,x}$ the number of people under exposure of death in the same period. We have separate reporting on men and women. Here we show an example of the data for Danish women.

Table 1. Danish women. Number of occurred deaths $O_{t,x}$

for Danish women aged 48-53 in the years 1950-1955

	48	49	50	51	52	53
1950	153	135	154	150	163	164
1951	140	113	140	144	183	163
1952	115	127	145	139	122	176
1953	124	131	109	137	171	180
1954	121	148	123	138	153	150
1955	112	102	137	138	141	173

Table 2. Danish women. Number of Danish women $E_{t,x}$aged

48-53 in the years 1950-1955 under exposure for dying.

	48	49	50	51	52	53
1950	27,950	27,240	26,861	26,139	26,066	24,946
1951	28,038	27,634	27,080	26,540	26,322	25,559
1952	28,448	27,915	27,492	26,941	26,403	26,160
1953	28,956	28,331	27,771	27,358	26,800	26,257
1954	29,151	28,829	28,191	27,649	27,208	26,651
1955	29,526	29,027	28,693	28,065	27,506	27,055

In table 1 $O_{1950,49}$ denotes the number of people at age 49 dying in year 1950. The occurrence $O_{1950,49}$ is calculated from the original data by taking the sum of the death counts in two lexis triangles. In this case 135 is the total of 72 deaths of people born in 1900 and 63 deaths of people born in 1901 all aged 49. In table 2 $E_{1950,49}$ denotes the number of people at age 49 under exposure of dying in year 1950. The exposure is calculated as the average of people aged 49 at 1 January 1950 and people aged 49 at 1 January 1951.

The simple estimator of the mortality of a 50 year-old in 1952 can be defined as:

$$\hat{\alpha}(1952, 50) = O_{1952,50}/E_{1952,50} = 145/27,492 = 0.005274262$$

A simple kernel smoothed estimator can be expressed as:

$$\alpha(1952, 50) = \sum \overline{O}_{1951:1953,49:51} / \sum \overline{E}_{1951:1953,49:51} = 1,185/247,062 = 0.004796367$$

Regarding the $\alpha\,(1952, 50)$ calculated above, we used the uniform kernel, i.e. the kernel where all observations within the window applies in the expression for α with the same weight. In the empirical study we use the Epanechnikov kernel function weighting the observations with respect to the distance from the chosen values of year, t and age, x.

The standardised occurrence- and exposure data are used for all further calculations in this paper.

3 The prediction principle and the underlying hazard estimator

In this section we discuss the difference between the mortality of the population and the mortality of a selected cohort of lives. We use our data plane to explain the differences between the population risk and the cohort risk. Assume that we tried to predict the future mortality in 1960.

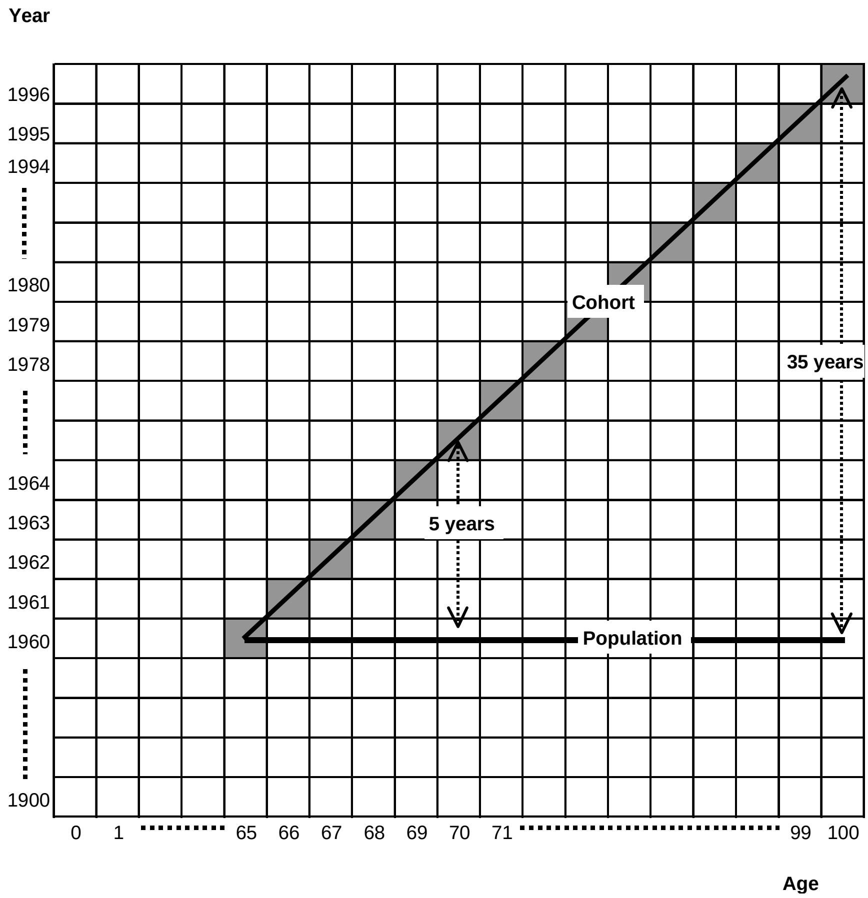

Fig 1. Cohort vs. Population.

The striped area in Fig. 1 describes the populations' mortality in 1960 for the age range 65-100 years. The estimator based on the available data in 1960, $\alpha\,(1960, x)$, is called the 'at risk' estimator of mortality. This is the simplest and most widely used predictor of mortality. The correct future mortality of a 65 year-old person in 1960 is, however, not described by data of a 65 year-old in 1960, a 66 year-old in 1960 and a 67 year-old in 1960 and so on. It is described by a data of 65 year-old in 1960, a 66 year-old in 1961 and a 67 year-old in 1962 and so on. Therefore, data from the striped area is used to predict the mortality corresponding to the solid dark grey area of Fig. 1. As mentioned in the introduction, some prediction methods, (see among many others Daykin (1998, p333)), try to develop methods that take the difference of the known striped area and the unknown dark grey area into account. Daykin's conclusion is based on a rate of decline in mortality and it corresponds more or less to a model assumption of a multiplicative relationship of the hazard with the entering components age and chronological time. It is seen from the study of Felipe et al. (2001) and Fledelius, Guillen, Vogelius and Nielsen (2001) that this mulplicative model can not be supported by historical data. This does not imply that the model assumption of multiplicative hazards is useless for prediction, but it does indicate that we have to be very careful with this type of quite strong model assumptions. A careful validation method should be implemented of the multiplicative hazard assumption before it can be said that its predictive power is better than the predictive power of the simple 'at risk' estimator considered in this paper.

Note that we need 35 years of data to be able to compare the future mortality, up to 100 years, of a 65 year-old. Therefore 1960 is the last year, where we are able to calculate and validate the prediction.

In the rest of the paper we compare the 'at risk' prediction method with the actual observed future mortality of the 65 years with chronological starting points in 1900-1960. We consider both the mortality itself, the expected life times and present values of annuities while performing these comparisons. All of these three quanties are functionals of the underlying two-dimensional hazard described in the introduction. We use two-dimensional kernel smoothing as introduced by Nielsen and Linton (1995) to get a nice starting point for our investigations, (see Appendix). As a quick introduction to the concept of multidimensional smoothing of hazards consider the mortality estimate based on the raw fractions of the yearly occurences and exposures from Section 2, see Fig. 2. Clearly, the ragginess is rather confusing, while the smoothed two-dimensional hazard is a better starting point for an analysis (see Fig. 3).

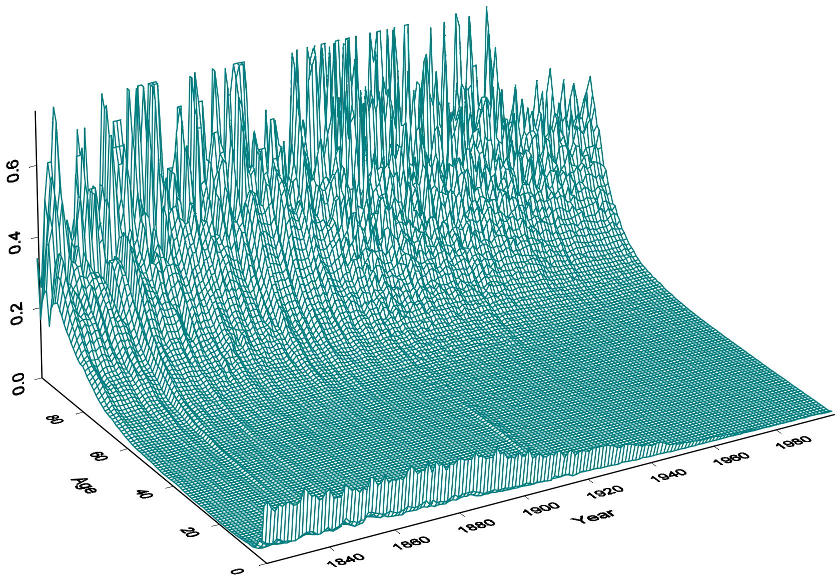

Fig 2. The raw two-dimensional mortality estimator.

Danish women 1835-1996

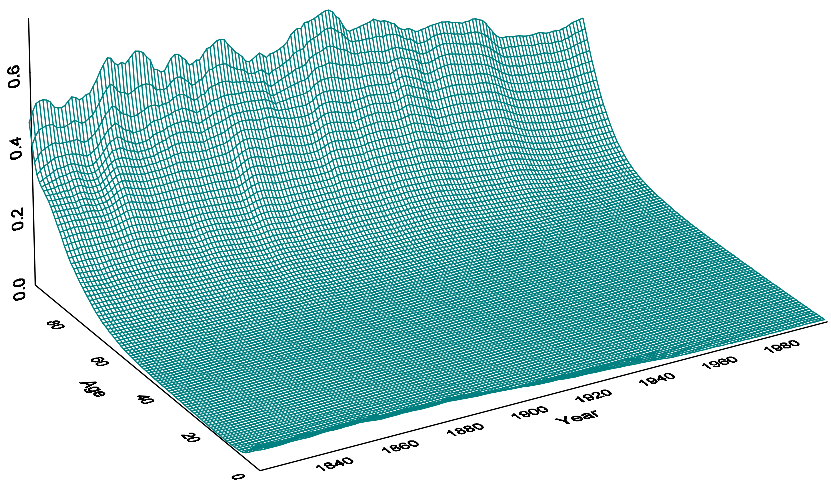

Fig 3. The kernel smoothed two-dimensional mortality estimator.

Danish women 1835-1996. Bandwith 5 years in age direction and 5 years in time direction.

4 Comparing the predicted mortality and the observed mortality

In this section we compare the predicted values based on the 'at risk' estimator with the observed mortalities. We illustrate this through the ratio between the mortality in year t of the population aged 65-100 years and the realized mortality for a cohort of people aged 65 in year t. We graphically display the results for the years 1900-1980. For the years 1970 and 1980 we were not able to observe the cohorts until a 100 years of age. Therefore, the graphs for the 1970 cohort the 1980 cohort depict age ranges from 65-90 and 65-80 respectively.

Regarding women we see that from year 1930 and onwards the 'at risk' prediction of mortality is overestimating the cohorts experienced mortality by 15-40 percent. This is a natural consequence of the decline in mortality rate in this period. Regarding Danish men, again the population's mortality is an overestimation of the experienced mortality. Yet the results for Danish men are not as significant as for Danish women. In Fig. 4 the first graph shows the ratio between the 'at risk' mortality in year 1900 and the experienced mortality of the cohort of women aged 65 in year 1900. The value at age 65 is exactly one, as it is defined as $\frac{\alpha(1900,65)}{\alpha(1900,65)}$. The value at age 66 corresponds to $\frac{\alpha(1900,66)}{\alpha(1901,66)}$, the value at age 100 corresponds to $\frac{\alpha(1900,100)}{\alpha(1935,100)}$. Values larger than one indicate declining levels of mortality with respect to calender time. All the graphs are provided with confidence bands. The confidence bands are calculated by using the bootstrap method, (see Fledelius et al. 2001). The conclusion of Fig. 4 on Danish women is that prediction errors on the mortality are observed up to 40% of the underlying mortality.

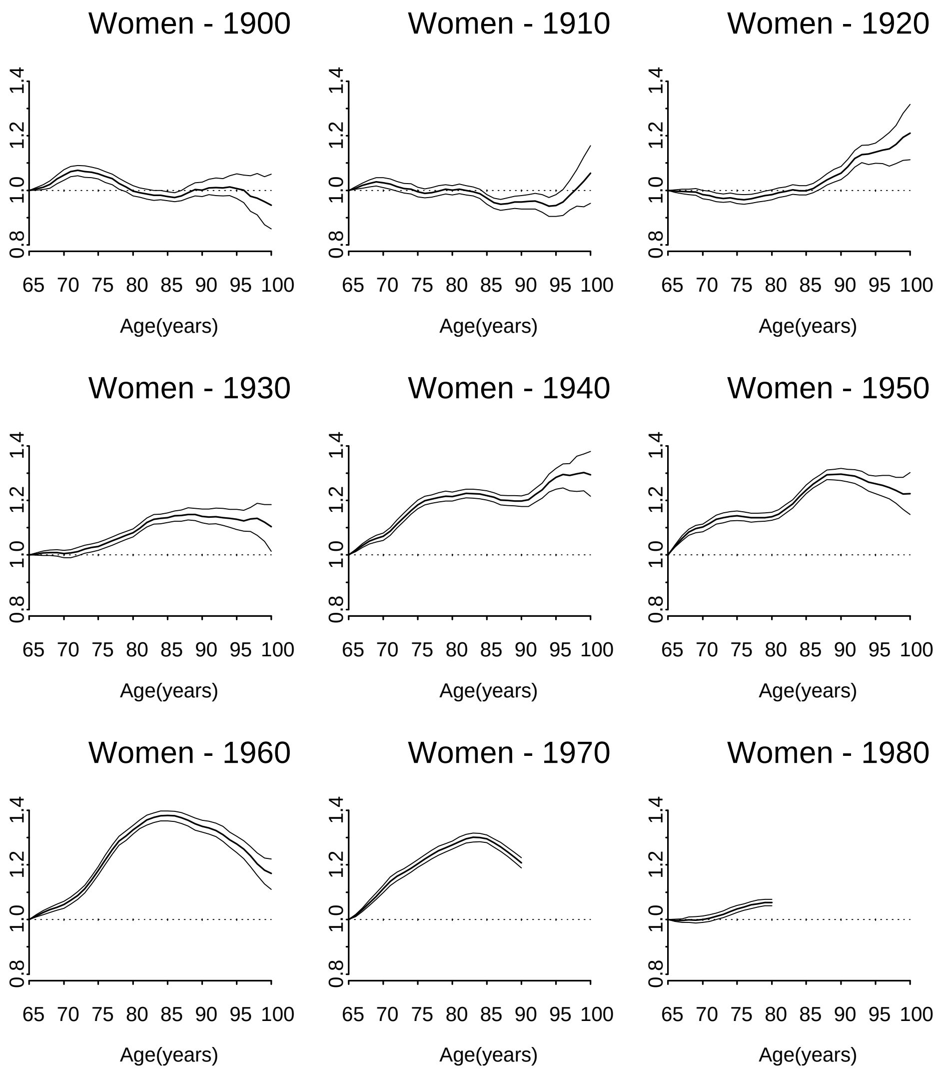

Fig 4. Hazard Ratios. Danish Women 1900-1980

It is also clear from Fig. 4 that there has been a tendency towards a longevity effect during the observed period leading to a systematic overestimation of the mortality by the 'at risk' estimator. However, since the two first graphs from 1900 and 1910 both have values at approximately 1 there is no indication of decreasing mortality through the first twenty years of the previous century. On the 1920-graph we see that from age 85 years until 100 years the ratio increases to a level $15-20\%$ over the baseline. The direct interpretation is that the mortality of 95 year-old in year 1920 was $12-13\%$ higher than the mortality of a 95 year-old in 1950. All the years 1930-1970 result in ratios larger than one. The largest difference is seen in year 1960, where the difference increases to almost 40%. We were not able to follow the 1970 and 1980 cohorts for until age 100. The 1970 graph indicates decreasing mortality, the 1980 graph does not give any precise conclusion. The overall conclusion is that Danish women tend to have a declining mortality and that the 'at risk' method has underestimated the prediction of the future mortality significantly in the period 1940-1960.

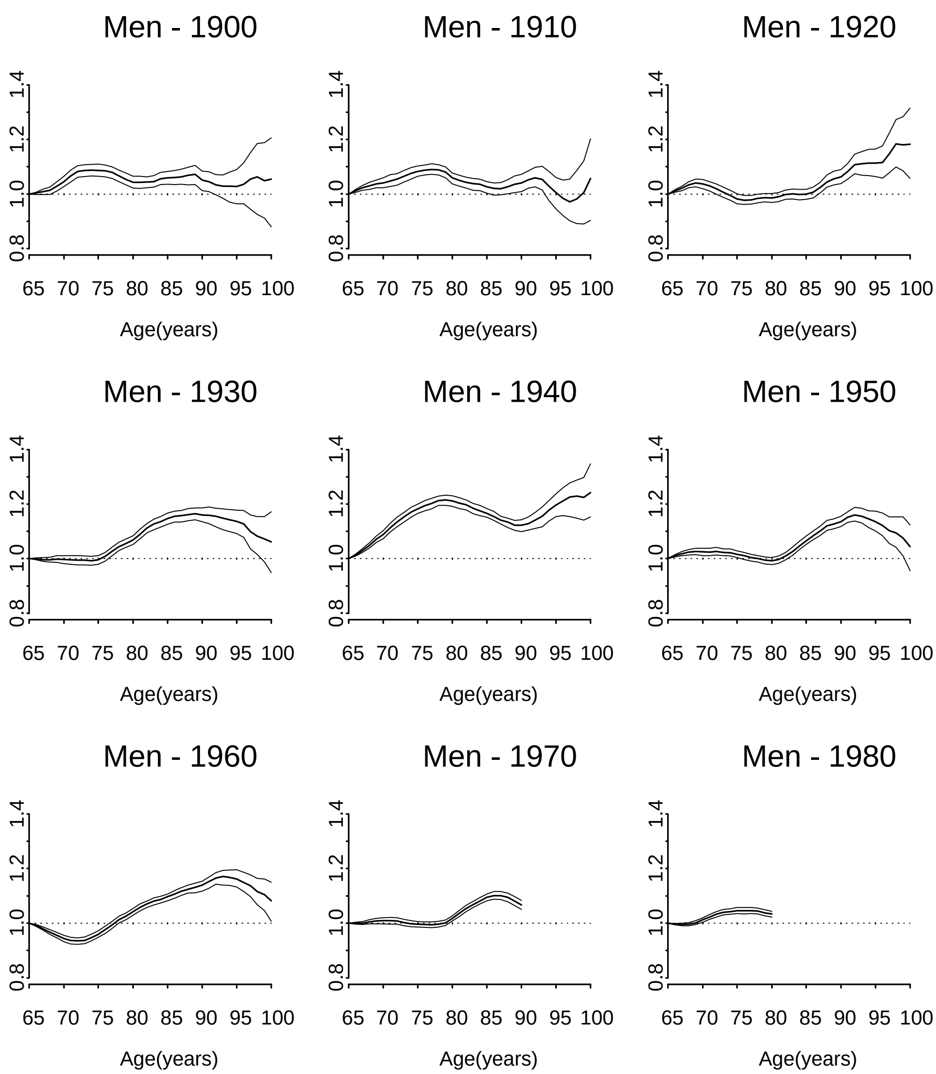

Fig 5. Hazard Ratios. Danish Men 1900-1980

In Fig. 5, we show the same graphs for Danish men. They also show signs of decreasing mortality with respect to calender time. Regarding the 1900 and the 1910 cohort, we note that the level of the ratio is generally greater than one. In 1920 the level is very close to one. The 1930 cohort follows the baseline the first ten years before the ratio slopes upward indicating that the mortality is dropping to lower levels. The 1940 cohort clearly has a lower mortality than the 1940 'at risk' estimator . This is less significant in 1950 and 1960, but the levels are generally larger than one. The graph from the 1960 observed cohort even shows an increase in mortality the first ten years compared to the mortality of the 'at risk' population in 1960. In 1970 and 1980 we note that the ratios follow the baseline, i.e. there are no significant changes in mortality for elderly Danish men in these years. The overall conclusion concerning Danish men is that the 1900 and the 1910 cohorts underwent a slow decrease in mortality. In 1930 and more significantly in 1940, again we experienced declining mortality for Danish men aged 65-100. In 1960 we have a short period of increasing mortality followed by a decrease in mortality. The error of prediction experienced by the 'at risk' estimator was between 0% and 20%.

5 Comparing the predicted life expectancy and the observed life expectancy

In this section we perform the same type of comparison of past prediction errors on life expectancies as we did in the previous section on the mortality curve itself. Life expectancy is a functional of the two-dimensional hazard $\alpha(t, x)$. For a given age and a given chronological time, the remaining lifetime for a cohort is defined as:

$$\overset{\circ}{e}_{t,x} = \int_0^\infty \exp\left\{ - \int_0^s \alpha(t + u, x + u)\, du \right\} ds$$

and life expectancy can therefore be estimated simply by plugging in the estimated smooth two-dimensional hazard as described in the appendix, see also Gerber (1995) or Jordan (1967) for classical references to expected remaining lifetime formulation and estimation. We call the estimator $\overset{\circ}{e}_{t,x}$ the observed expected life time. The problem we face while applying $\overset{\circ}{e}_{t,x}$ in practice is that it can only be estimated retrospectively and it is therefore not suitable for prediction purposes. The time invariant estimator defined as

$$e_{t,x} = \int_0^\infty \exp\left(- \int_0^s \alpha\left(t, x + u\right) du \right) ds$$

can, however, be used for prediction. Newman (1986) considered this (calendar) time invariant estimator that exactly corresponds to our 'at risk' prediction of the life expectancy when our two-dimensional smooth mortality estimator is plugged in. We calculate the experienced remaining life expectancy for selected cohorts and the expected remaining lifetime for the sample under exposure in every year from 1900 until 1996. We restrict our presentation to 65 years old. The calculation of the remaining life expectancy is carried out under the assumption that we have a maximum obtainable age of 100 years. With data from years 1900 until 1996 we are able to make a full comparison of the two estimators regarding the interval 1900 until 1960.

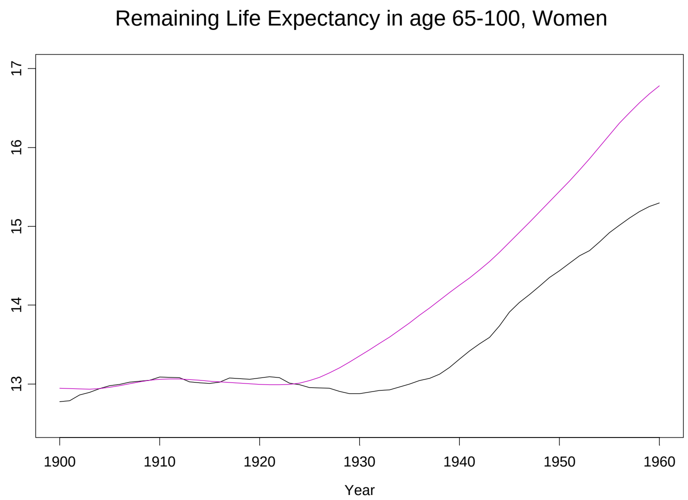

Fig 6. Comparison between life expectancy of a 65 year-old and the realized life span of a 65 year-old. Women.

Regarding the 'at risk' estimator for Danish women, see Fig. 6, we note that it remains at a level around 13 years from 1900 until 1933. From 1935 until 1960 the graphs slope upwards showing an increase in life expectancy. In the late 1950's the estimator reaches 15 years. We experience an increase from 13 to 15 years over 25 years. We see a constant slope from 1940 until 1960. There is no sign of convergency. The realized life expectancy of Danish women also starts at a level around 13 years. In 1924 the graph starts sloping upward. From 1925 until 1956 the graph has an increasing slope. From 1957 to 1960 we experience an increase in life expectancy, but we do not see an increasing slope in this interval. The observed life expectancy for Danish women aged 65 in 1960 was approximately 16.8 years.

When comparing the two graphs we see that from 1900 until 1923 there was no signifcant difference between the two estimators. From 1924 until 1960 we see a significant difference between the two estimators. The 'at risk' life expectancy predictor underestimates the life span up to $9 - 10\%$.

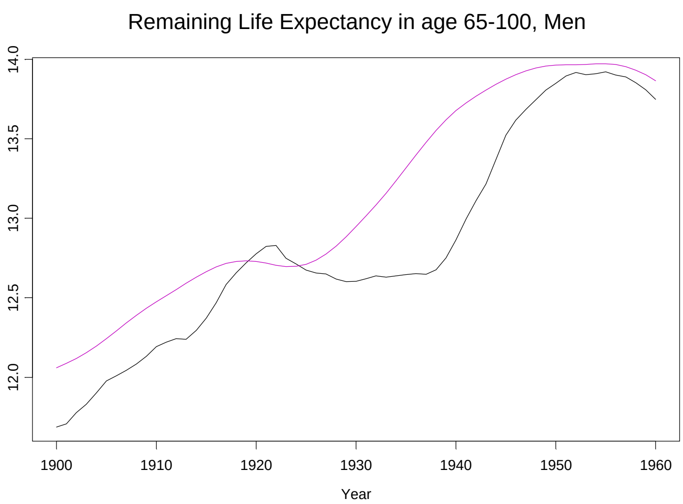

Fig 7. Comparison between life expectancy of a 65 year-old and the realized life span of a 65 year-old. Men.

Regarding the 'at risk' estimator for Danish men in Fig. 7, we see an increase in life expectancy from 11.6 years to 12.8 years in the interval 1900 until 1922. From 1923 until 1938 life expectancy starts a slow decline to a level of 12.7 years. From 1938 until 1952 again we see an increase from 12.7 years to 13.9 years. From 1953 until 1960 the graphs decline to a level of 13.7. The overall impression is a general increase in life expectancy. The observed life expectancy for Danish men aged 65 was 12.1 years in 1900. We see an increase in life expectancy until 1955, where the level almost reaches 14 years. From 1956 until 1960 we experience a minor decrease in life expectancy. A comparison of the two graphs makes us conclude that the observed life expectancy estimator generally has larger values than the 'at risk' estimator. In 1938 we experience a difference of 7%. The overall impression is, that the observed life expectancy estimator has values $2-3\%$ larger than the 'at risk' estimator. Before calculating the estimators, the two-dimensional hazard estimator $\alpha(t,x)$ was smoothed by kernel smoothing. Yet it is easy to see that the 'at risk' estimator is ragged compared to the observed life expectancy estimator. The reason for this is that the 'at risk' estimator primarily is based on data from a single year. If this year was an 'outlier' year, i.e. a year with one or more epidemics, we would expect a high mortality that single year. When calculating the life expextancy for the population at risk this year, we therefore experience a high mortality causing a low life expectancy. Therefore the 'at risk' estimator will be more volatile than the cohort estimator. When using the 'at risk' estimator as a predictor of future mortality, we need to take these aspects into consideration. In the next section, we consider different interest rate scenarios and discuss how the longevity tendencies can be taken into account when calculating reserves in life insurance.

6 Financial aspects of prediction errors

In this section we will discuss the financial aspects of prediction errors due to changes in mortality. When mortality changes the value of a life annuity changes. We discuss the prediction errors due to the longevity effect under different basic interest rate scenarios. Clearly, an insurance company selling life annuities is forced to reserve for the uncertainty in future mortality. Furthermore we conclude that a reasonable safety margin is around 1% on the technical interest rate corresponding to a safety margin of two more lived years for each old-age retiree. Originally, we used three levels of interest rates, r, namely 3%, 5% and 10%, but we have restricted the presentation to the analysis based on the 5% scenario, since our conclusions were surprisingly robust to the choice of basic interest rate.

Basically the technique is to add discounting factors to the survival probabilities.

In the previous section we did not have any discounting. That corresponds to a zero interest scenario, i.e. a payment of one unit today is worth the same as a payment of one unit in e.g. five years. When introducing an interest level for discounting, we will define the present value of the future payments as:

$$\overset{\circ}{PV}_{t,x,r} = \int_0^\infty \exp(-\delta_r s) \exp\left(-\int_0^s \alpha(t+u, x+u)\, du\right) ds, \text{ where } \delta_r = \log(1+r)$$

As in the previous chapter we will also introduce the present value based on the time invariant mortality estimator. In this case we define:

$$PV_{t,x,r} = \int_0^\infty \exp(-\delta_r s) \exp\left(-\int_0^s \alpha(t, x+u)\, du\right) ds, \text{ where } \delta_r = \log(1+r)$$

It is seen that $\overset{\circ}{PV}_{t,x,0}$ equals to $\overset{\circ}{e}_{t,x}$ as the term $\delta_r = \log(1+r) = 0$ and the following term $\exp(-\delta_r s) = 1$. With a positive basic interest level, $r>0$, we will see that $\overset{\circ}{PV}_{t,x,r} < \overset{\circ}{e}_{t,x}$. In the zero interest scenario in the last section we saw that the mortality greatly influenced the expected present value of a life annuity. In scenarios with interest we will see, again, that mortality has great influence on present values of future payments.

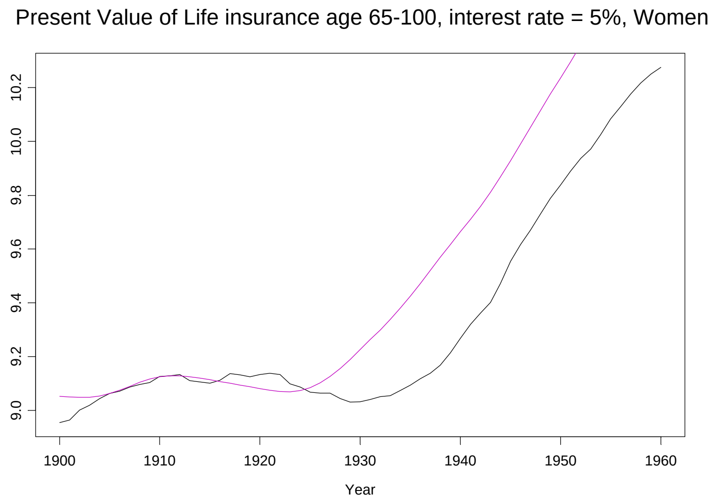

Fig 8. Comparison between expected present value of a life annuity and realized present value of a life annuity for 65 years old Danish women. Basic interest rate equals 5%

Regarding Danish Women in Fig. 8, the shape of the graphs looks similar to the graph in the previous section. As expected the general level of the present values is now lower due to the introduced interest rate, $r = 5\%$. The general level is now $9-11$ years. In the zero interest scenario it was $13-17$ years. Note that the difference between $\overset{\circ}{PV}_{t,x,r}$ and $PV_{t,x,r}$ is reduced to approximately 7% in year 1960. It was $9-10\%$ in the scenario without discounting.

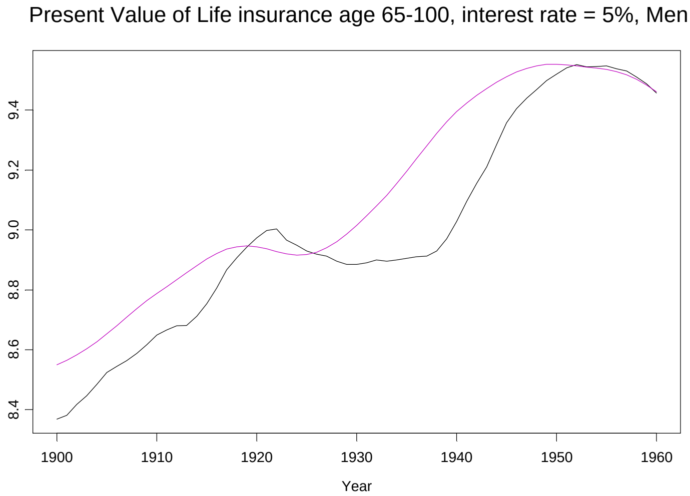

Fig 9. Comparison between expected present value of a life annuity and realized present value of a life annuity for 65 years old Danish men. Basic interest rate equals 5%

Regarding the Danish men in Fig. 9, again we note that the general shape of the curves is the same as before and that the general level is lowered to 8.5-9.5. In the zero interest scenario the level was 12-14. The largest difference between the $\overset{\circ}{PV}_{t,x,r}$ and $PV_{t,x,r}$ in year 1938 is now approximately 5%, in the zero interest scenario it was 7%.

We would now like to describe these differences as an excess discount on the $\overset{\circ}{PV}_{t,x,r}$. We are able to introduce additional discounting on $\overset{\circ}{PV}_{t,x,r}$ simply by this definition:

$$\overset{\circ}{PV}_{t,x,r,\varepsilon} = \int_0^\infty \exp(-\delta_{r,\varepsilon} s) \exp\left\{-\int_0^s \alpha(t+u,x+u)\,du\right\} ds, \text{ where } \delta_{r,\varepsilon} = \log(1+r+\varepsilon)$$

The general formula is left unchanged, but the definition of $\delta_{r,\varepsilon}$ is new. Our aim is to find the level of excess discount ε, resulting in $\mathring{PV}_{t,x,r,\varepsilon} = PV_{t,x,r}$. By different choices of ε we are able to find a level of the excess discount, suitable for valuation the mortality risk, but measured in a standard financial term. We have tried with ε taking values in the value set (0%, 0.1%, 0.3%, 0.5%, 0.7% and 0.9%).

Fig 10a+b. Women. Valuation af the difference between expected present value of life annuity and realized present value of life annuity.

Additional discounting (0.1%, 0.3%, 0.5%, 0.7%, 0.9%) on realized present value of life annuity.

Regarding Danish women in Fig. 10a the zero interest scenario, we see that the changes in mortality can be expressed as an extra discounting on future payments. We note that from 1935 until 1960 the suitable level of extra discount is approximately $0.7\% - 0.9\%$. The straightforward interpretation of this result is: If we had sold life annuities to 65 years old Danish women, priced on the most updated mortality data, we would have underestimated the liabilitites. The experienced annual payments are supposed to be discounted by $0.7\% - 0.9\%$ to equal the expected present value of the life annuity.

Regarding the 5% in interest scenario in Fig 10b we note that the excess discount is now reduced to $0.6\% - 0.7\%$. So, even in a more realistic scenario, we establish that changes in mortality for elderly people have great impact on the life annuity reserving.

Fig 11a+b. Men. Valuation of the difference between expected present value of life annuity and realized present value of life annuity.

Additional discounting (0.1%, 0.3%, 0.5%, 0.7%, 0.9%) on realized present value of life annuity.

Regarding Danish men in Fig. 11a the zero interest scenario, we see that the largest difference between $\overset{\circ}{PV}_{t,x,r}$ and $PV_{t,x,r}$ is in year 1938-1939. To match the changes in mortality we have to introduce an excess discount, ε equal 0.8%. Furthermore we note that the problem was not a large problem at the end of our observation period 1950-1960. In the 5% in Fig 11b scenario we determine ε at almost 0.7%.

7 Concluding remarks

This paper investigates the simplest and most popular prediction model, the 'at risk' estimator. This prediction estimator is compared to the actually observed mortality throughout the period 1900-1996 in Denmark. Our investigations show that the maximal error made by the simple mortality prediction model corresponds to an interest rate of 0.7% on life annuities. In other words must the insurance company generate a 0.7% better financial result to neutralize the longevity changes. Though the trends for the two genders are quite different, we do obtain the same results when regarding a safety margin on reserve calculations on life annuities.

The results make us conclude that the traditional 'at risk' estimator of future mortality is not that bad and that it can serve as a basis for reserving purposes.

We also note that any other prediction method should be validated in a similar manner. One could for example envision a semiparametric multiplicative model with a non-parametric age effect and a parametric calender time effect. The parametric model on the calender time effect could for example imply an exponential decrease in mortality. However, such a semiparametric model is not necessarily better than the simple prediction model presented in this paper. One way of validating the quality of the semiparametric approach would be to go through the same type of validation calculations for the semiparametric prediction model as we have given in the present paper for the simple 'at risk' estimator of future mortality. A direct comparison of the prediction behaviour of the two prediction approaches would then serve as a useful evaluation of which one to use in praxis.

8 APPENDIX

In this appendix we define our two-dimensional hazard estimator and our basic model. We observe n individuals $i = 1, .., n$. Let $N_i^{(n)}$ count observed failures for the i'th individual in the time interval $[0, 1]$. We assume that $\mathbf{N}^{(n)} = (N_1^{(n)}, .., N_n^{(n)})$ is an n-dimensional counting process with respect to an increasing, right continuous, complete filtration $\mathcal{F}_t^{(n)}$, $t \in [0, 1]$, i.e. one that obeys **les conditions habituelles**, see Andersen et al. (1992, p60). We model the random intensity process $\lambda^{(n)} = (\lambda_1^{(n)}, .., \lambda_n^{(n)})$ of $\mathbf{N}^{(n)}$ as depending on marker values,

$$\lambda_i^{(n)}(t) = \alpha\{X_i^{(n)}(t), t\}Y_i^{(n)}(t),$$

but do not restrict the functional form of $\alpha(\bullet)$. Here, Y_i is a predictable process taking values in $\{0, 1\}$, indicating (by the value 1) when the i'th individual is under risk, t is chronological time and $X_i^{(n)}(t)$ is age of the i'th individual and the chronological time t.

The estimator suggested by Nielsen and Linton (1995) is

$$\hat{\alpha}(x, t) = \frac{\sum_{i=1}^{n} \int_{1900}^{1996} K_{b_1}(t-s) K_{b_2}\{x - X_i(s)\} dN_i(s)}{\sum_{i=1}^{n} \int_{1900}^{1996} K_{b_1}(t-s) K_{b_2}\{x - X_i(s)\} Y_i(s) ds} = \frac{\overline{O}_{x,t}}{\overline{E}_{x,t}}. \qquad \text{(n1)}$$

where

$$\overline{O}_{x,t} = \sum_{i=1} \int_{1900}^{1996} K_{b_1}(t-s)\, K_{b_2}\{x - X_i(s)\}\, dN_i(s)$$

and

$$\overline{E}_{x,t} = \sum_{i=1} \int_{1900}^{1996} K_{b_1}(t-s)\, K_{b_2}\{x - X_i(s)\}\, Y_i(s) ds$$

are respectively the smoothed occurence and the smoothed exposure. Nielsen (1998) pointed out that this estimator could be interpreted as a local constant marker dependent hazard estimator. Let for a moment the kernel $K(x)$ equal $I(|x| < 1)$,then $\overline{O}_{t,x}$ corresponds to the observed number of failures with deaths in the chronological time interval $[t-b, t+b]$ and age in the interval $[x-b, x+b]$ and $\overline{E}_{t,x}$ corresponds to the exposure time observed in the area in the chronological time interval $[t-b, t+b]$ and age in the interval $[x-b, x+b]$.The local constant kernel estimator hereby corresponds to the well known occurrence exposure ratio.

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