INFERENCE ABOUT MORTALITY IMPROVEMENTS IN LIFE ANNUITY PORTFOLIOS

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ABSTRACT

Recent mortality trends lead to the use of projected mortality tables when pricing and reserving for life annuities (as well as for other living benefits, such as Long Term Care benefits, whole life sickness benefits, etc.). However mortality patterns continuously evolve along time, so that any projection might reveal weak when used for pricing new annuities and reserving for in-force business. Hence, adjustments must be made in pricing and reserving bases. Monitoring mortality provides data, while an appropriate inferential model should constitute the structure underpinning the adjustment procedure. In this paper, inference about portfolio mortality trends is first focussed. Then, a Bayesian inferential model is proposed, aiming at mortality adjustments based on prior information and statistical evidence. Numerical examples illustrate the inferential mechanism. Finally, some actuarial applications are proposed.

KEYWORDS

Mortality trends, Mortality projections, Longevity risk, Pricing and reserving for annuities, Solvency, Monitoring mortality, Bayesian inference

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1. INTRODUCTION

Recent trends in mortality (see for example Macdonald (ed.), 1997; Macdonald et al, 1998) lead to the use of projected survival models when pricing and reserving for life annuities and other long-term living benefits. Several projection models have been proposed and are actually used in actuarial practice (for example, see Benjamin and Pollard, 1993; Benjamin and Soliman 1993; CMIR, 1990; CMIR, 1999; Renshaw and Haberman, 1996; Sithole, Haberman and Verrall, 2000). However, the future mortality trend is random and hence, whatever kind of model is adopted, systematic deviations from the forecasted mortality may take place. Then, a "model" (or "parameter") risk originates, which is clearly a non-pooling risk. Changes in the mortality pattern refer to both young and old ages. When the random mortality trend at old ages is concerned, the risk is usually referred to as "longevity risk".

The longevity risk has been dealt with by Marocco and Pitacco (1998), where reinsurance arrangements facing this risk are also discussed. Olivieri (2001) considers future mortality trends at young ages and old ages as well, and suggests an assessment of the impact of systematic deviations on term insurance and life annuities portfolios. The longevity risk affecting sickness benefits for the elderly (for example, post-retirement sickness benefits) is analysed by Olivieri and Pitacco (1999). The longevity risk in life annuities portfolio and the relevant solvency requirements are dealt with in Olivieri and Pitacco (2000), where also investment risk is considered. The papers by Ferri and Olivieri (2000) and Olivieri and Pitacco (2001) concern long-term care (LTC) benefits in a moving scenario in which both future mortality and future senescent disability are random. Longevity risk under a risk management perspective is analysed by Riemer-Hommel and Trauth (2000).

The continuous evolution of mortality patterns requires progressive adjustments in pricing and reserving bases. Monitoring mortality provides data, while an appropriate inferential model should constitute the structure underpinning the adjustment procedure. In this paper, inference about portfolio mortality trends is first focussed. Then, a Bayesian inferential model is proposed, aiming at mortality adjustments based on prior information and statistical evidence. Finally, actuarial evaluations following the adjustments in demographical bases are discussed. Some numerical examples illustrate the inferential mechanism and the relevant actuarial consequences.

The paper is organized as follows. In Section 2 the basic aspects of mortality trends are presented and the relevant demographical scenarios are depicted. Section 3 deals with the concept of longevity risk, and describes demographical models allowing for this risk. Section 4 deals with Bayesian inference on future mortality. A particular model is then proposed in Section 5; numerical examples illustrate its implementation. In Section 6 actuarial issues are considered. Finally, in Section 7 some conclusions are presented.

2. MORTALITY TRENDS

Mortality experience over the last decades shows some aspects affecting the shape of curves representing the mortality as a function of the attained age. In particular (see Olivieri, 2001):

(a) an increasing concentration of deaths around the mode (at old ages) of the curve of deaths (the graph of the probability density function of the random lifetime in an age-continuous setting) is evident; so the survival function moves towards a rectangular shape, whence the term “rectangularization” to denote this aspect (see Figure 2.1);
(b) the mode of the curve of deaths (which, owing to the rectangularization, tends to coincide with the maximum age ω) moves towards very old ages; this aspect is called “expansion” of the survival function (see Figure 2.2);
(c) higher levels and a larger dispersion of accidental deaths at young ages (the so-called young mortality hump) have been more recently observed.

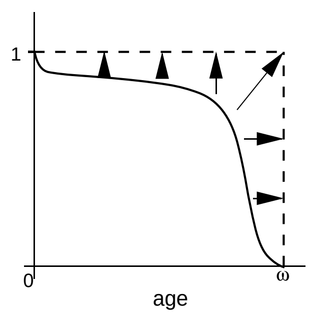

Figure 2.1 – Survival function: rectangularization

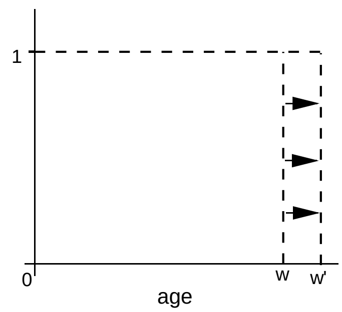

Figure 2.2 – Survival function: expansion

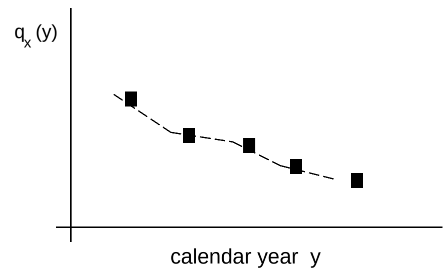

Figure 2.3 – Mortality profile at age x

Consider the mortality rate $q_x(y)$, i.e. the probability for an individual aged x in the calendar year y, of dying between ages x and $x+1$. The mortality profiles (i.e. the behaviour of the mortality rates $q_x(y)$ as functions of the calendar year y) are typically decreasing at adult and old ages (as sketched in Figure 2.3). Rectangularization and expansion phenomena are witnessed by mortality experience in a number of countries. The reader can refer to Macdonald et al (1998) for an interesting international comparison.

Observed mortality trends lead to the construction of projected mortality models for actuarial use, especially when living benefits are concerned. When projecting mortality, the basic idea is to express the mortality itself as a function of the (future) calendar year y. If a single-figure representation of mortality is concerned, a projected mortality model is a real-valued function $\Psi(y)$. For example, the expected lifetime for a newborn, denoted by e_0 in a non-projected context, is represented by a function $e_0(y)$ when the future mortality trend is considered.

In actuarial calculations, the expression of mortality as a function of age is needed. Then, in a projected context mortality at any age x must be considered as a function of the calendar year y. Hence, in a rather general setting, a projected mortality model is a function $\Psi(x, y)$, which may be a real-valued or a vector-valued function. The function is selected and, in particular, its parameters are estimated by applying appropriate statistical procedures to past mortality experience. In concrete terms, a real-valued function Ψ may represent mortality rates, mortality odds, a force of mortality, a survival function, a mortality density function, some transform of the survival function, etc. When a vector-valued function Ψ is concerned, its components typically refer to various causes of death.

A number of projection models are described in the actuarial literature. General presentations are provided, for example, by Benjamin and Pollard (1993); Benjamin and

Soliman (1993); Petauton (1996). Specific models are proposed, in particular, by CMIR (1990); CMIR (1999); Renshaw and Haberman (1996); Sithole, Haberman and Verrall (2000). Projection models can be classified according to several points of view. In particular, we can recognize the following types of models.

- Projection models referring to the mortality rates $q_x(y)$ and mainly based on observations concerning the mortality profiles (see Figure 2.3). The models themselves consist in straight extrapolation procedures of the mortality rates observed in the past. Generally, extrapolation procedures do not ensure the representation of sensible future scenarios. Moreover, inconsistencies may emerge as a result of the extrapolations; for example we may find, for some future calendar year y, $q_{x'}(y) < q_{x''}(y)$ with $x' < x''$, even at old ages.
- Projection models based on mortality laws (such as Gompertz, Makeham, Thiele, Weibull, Heligman-Pollard, etc.). These models are based on the extrapolation of the parameters of the assumed mortality law, and allow us to express the main features of the evolving scenario, such as the rectangularization and the expansion.

In this paper, we only focus on the latter type of models, which also allow for a simple representation of the uncertainty in mortality trends. Moreover, in this framework the definition of an inferential procedure is straightforward.

3. THE LONGEVITY RISK

The future mortality trend is obviously random and hence, whatever kind of projection procedure is adopted, systematic deviations from the forecasted mortality may occur. Depending on the statistical model adopted in analysing past data and forecasting mortality, in some cases the assessment of uncertainty in future mortality trends constitutes an output of the statistical model itself.

In general terms, thus disregarding the possibility of finding a risk assessment arising from the statistical procedures adopted, we now focus on possibile systematic deviations from the (point) estimation of future mortality, then on the "model" (or "parameter") risk inherent in mortality projections. Obviously, this risk has important consequences on life insurance valuations, especially when living benefits are concerned. Restricting our attention to trends of the mortality pattern at old ages only, we will refer to this risk as "longevity risk".

Let us now define a probabilistic structure allowing for uncertainty in future mortality trends.

Refer to a newborn and denote by $f(t)$ the probability density function (pdf) of his/her random lifetime T. Consistently with a given forecast of the mortality trend, $f(t)$ should express a mortality projection. However, within a projection context, a specific pdf should be used for each generation. Then, for any given calendar year of birth y, let $f(t, y)$ denote the pdf referring to individuals born in year y.

In order to express several possible evolutions of mortality, a family of projected pdf's should be considered for each year of birth y, rather then a single pdf $f(t, y)$. Denote by $H(y)$ an hypothesis about the mortality trend for people born in year y. Hence, the family of pdf's

$$\{f(t, y|H(y)); H(y) \in \mathcal{H}(y)\} \tag{3.1}$$

should be addressed, $\mathcal{H}(y)$ denoting a given set of hypotheses. In particular, if possible evolutions of mortality are expressed via some parameters of the pdf, the family (3.1) can be denoted as follows:

$$\{f(t, y|\theta(y)); \theta(y) \in \Theta(y)\} \tag{3.2}$$

where $\theta(y)$ denotes a vector-valued (in particular, a real-valued) parameter and $\Theta(y)$ denotes the parameter space.
Note that, for any given calendar year y, $f(t, y|H(y))$ and, respectively, $f(t, y|\theta(y))$ represent the lifetime pdf's conditional on a specific hypothesis, expressed by $H(y)$ and $\theta(y)$ respectively, about the mortality trend.
For simplicity, in what follows we address one generation, i.e. a given year of birth. Then, y can be left out. So, the family of projected pdf's can be simply denoted as follows:

$$\{f(t|H); H \in \mathcal{H}\} \tag{3.1$'$}$$

and respectively:

$$\{f(t|\theta); \theta \in \Theta\} \tag{3.2$'$}$$

Let us focus the parameterized case, viz. the family (3.2′). Let us express our opinion about possible future mortality evolutions by assigning a probability distribution on the parameter space Θ. Denote, in the continuous case, by $g(\theta)$ the pdf of the random parameter $\tilde{\theta}$, with $\int_\Theta g(\theta)d\theta = 1$. If a discrete setting is chosen, we have $g(\theta) = \Pr\{\tilde{\theta} = \theta\}$ with $\sum_{\theta \in \Theta} = 1$.
The unconditional pdf of the random lifetime is given by:

$$f(t) = \int_\Theta f(t|\theta)g(\theta)d\theta \tag{3.3}$$

in the continuous case, and in the discrete one by:

$$f(t) = \sum_{\theta \in \Theta} f(t|\theta)g(\theta) \tag{3.4}$$

Note that, conditional on a given mortality scenario, i.e. conditional on $\tilde{\theta} = \theta$, the expected value and the variance of the random lifetime T are respectively given, in the continuous case, by:

$$E(T|\theta) = \int_0^{+\infty} tf(t|\theta)dt \tag{3.5}$$

$$Var(T|\theta) = \int_0^{+\infty} (t - E(T))^2 f(t|\theta)dt \tag{3.6}$$

The unconditional expected value and variance, on the contrary, are respectively given by:

$$E(T) = \int_0^{+\infty} tf(t)dt = \int_0^{+\infty} \int_\Theta tf(t|\theta)g(\theta)d\theta dt \tag{3.7}$$

$$Var(T) = \int_0^{+\infty} (t - E(T))^2 f(t)dt = \int_0^{+\infty} \int_\Theta (t - E(T))^2 f(t|\theta)g(\theta)d\theta dt \tag{3.8}$$

The corresponding expressions in the discrete case are straightforward.
In both the continuous and the discrete case, for the variance the following well known result holds:

$$Var(T) = E[Var(T|\tilde{\theta})] + Var[E(T|\tilde{\theta})] \tag{3.9}$$

the first term on the right-hand side of (3.9) gives account of random fluctuations around the expected values, whereas the second one expresses systematic deviations of observed values from expected ones, hence representing the longevity risk.
Finally, note that in a Bayesian inferential framework, the unconditional pdf $f(t)$ must be meant as the (prior) predictive pdf, whereas the function $g(\theta)$ represents the prior pdf on the parameter space Θ.

4. BAYESIAN INFERENCE

So far we have introduced a probabilistic structure aiming at describing the randomness of the future mortality evolution. Now we turn to inferential issues, and in particular to the construction of a Bayesian inferential model.

4.1 The Bayesian inferential model. Consider, at a given point of time τ, a homogeneous sample of n individuals, all born at time 0 and hence aged τ. Let T_h denote the (total) lifetime for the h-th individual (so that $T_h - \tau$ represents his / her residual lifetime, given that the individual is alive at age τ), $h = 1, 2, \ldots, n$. Assume that, conditional on any hypothesis $\tilde{\theta} = \theta$, the random variables (r.v.) $T_1, T_2, \ldots, T_n$ are independent and identically distributed (i.i.d.). The sampling pdf is then:

$$f_\tau(t|\theta) = \begin{cases} 0 & \text{for } t \leq \tau \\ \frac{f(t|\theta)}{\int_\tau^{+\infty} f(u|\theta)du} & \text{for } t > \tau \end{cases} \tag{4.1}$$

The quantity $\int_t^{+\infty} f(u|\theta)du$ is the probability for a newborn of being alive at age t, given θ. If considered as a function of t, it represents the conditional survival function $S(t|\theta)$:

$$S(t|\theta) = \Pr\{T_h > t|\theta\} = \int_t^{+\infty} f(u|\theta)du \tag{4.2}$$

So we have:

$$f_\tau(t|\theta) = \begin{cases} 0 & \text{for } t \leq \tau \\ \frac{f(t|\theta)}{S(\tau|\theta)} & \text{for } t > \tau \end{cases} \tag{4.3}$$

The multivariate sampling pdf is then given by:

$$f_\tau(t_1, t_2, \ldots, t_n|\theta) = \prod_{h=1}^{n} f_\tau(t_h|\theta) \tag{4.4}$$

Finally, note that

$$f_\tau(t) = \int_\Theta f_\tau(t|\theta)g(\theta)d\theta \tag{4.5}$$

represents the (prior) predictive pdf restricted to the age interval $[\tau, +\infty)$.

Assume now that the observation period is the (limited) interval of time $[\tau, \tau']$. The number of deaths in this interval is a r.v. Let m ($m \leq n$) denote the relevant realization, observed in τ'. With a proper renumbering, let

$$\underline{x} = (x_1, x_2, \ldots, x_m) \tag{4.6}$$

denote the vector of ages at death. Note that the defined observation procedure implies a Type I censored sampling (see, for instance, Namboodiri and Suchindran, 1987).
The following problem will now be addressed. Using the information provided by the pair $(m, \underline{x})$, construct the (posterior) predictive pdf $f_\tau(t|m, \underline{x})$. To this purpose we adopt the following procedure (usual in the Bayesian context), consisting of two steps:
(1) update our opinion about the possible evolution of mortality, and hence about the probability distribution over the parameter space Θ, calculating the posterior pdf

$$g(\theta|m, \underline{x}) \propto g(\theta)L(\theta|m, \underline{x}) \tag{4.7}$$

where $L(\theta|m, \underline{x})$ denotes the likelihood function;
(2) calculate the (posterior) predictive pdf as

$$f_\tau(t|m, \underline{x}) = \int_\Theta f_\tau(t|\theta)g(\theta|m, \underline{x})d\theta \tag{4.8}$$

Step (1) requires the construction of the likelihood function $L(\theta|m, \underline{x})$. To this purpose, denote by $S_\tau(t|\theta)$ the survival function for a generic individual aged τ, whose random lifetime (at birth) is T. Then for any t, $t > \tau$:

$$S_\tau(t|\theta) = Pr\{T > t|T > \tau; \theta\} \tag{4.9}$$

and obviously

$$S_\tau(t|\theta) = \frac{S(t|\theta)}{S(\tau|\theta)} \tag{4.10}$$

For the likelihood we have (see for example Namboodiri and Suchindran, 1987):

$$L(\theta|m, \underline{x}) \propto [\prod_{h=1}^{m} f_\tau(x_h|\theta)][S_\tau(\tau'|\theta)]^{n-m} \tag{4.11}$$

It is straightforward to rewrite the above equations referring to the case in which a discrete parameter space Θ is concerned.

4.2 Some particular models. Several applications of Bayesian inference can be found within the fields of survival analysis, reliability analysis and actuarial studies as well. However, to our knowledge, up to now no specific analysis has been devoted to mortality trends and, in particular, to longevity risk in life annuities portfolios and pension schemes. Some papers and textbooks follow, dealing with applications of Bayesian inference to problems within the fields mentioned above.
Daboni (1972) deals with inferential problems concerning the mortality of assured lives. A survival function approximating the Makeham law was chosen, aiming at working within the so-called exponential family. Two parameters were considered to be random

and a continuous joint pdf for the parameters was chosen as the natural conjugate of the sampling distribution.
As it is well known, choosing a conjugate prior distribution leads to the property that the posterior distribution belongs to the same family as the prior distribution. However, several Authors do not emphasize this advantage, and actually also use a variety of prior distributions without this property (see, for example, Leonard and Hsu, 1999).
In the textbook by Leonard and Hsu (1999) the case of a Weibull sampling distribution is discussed, in particular using gamma prior distributions for the two random parameters, assumed to be independent.
The textbook by Martz and Waller (1982) deals with various inferential problems. In particular, the case of a Weibull distribution is considered, with one or two random parameters; in the latter case, a prior distribution is chosen, which is continuous with respect to one parameter and discrete with respect to the other one.
Kim and Ibrahim (2000) investigate the properties of the posterior distribution under the uniform improper prior for two commonly used proportional hazard models: the Weibull model and the extreme value model.
Leonard and Hsu (1999) also consider the problem in which the uncertainty is represented by a family of pdf's $\{f(t|H); H \in \mathcal{H}\}$, thus disregarding parameterized representations. Assuming a finite space $\mathcal{H}$, the uncertainty is expressed in terms of a discrete probability distribution over $\mathcal{H}$. The inferential model is embedded in the context of the more general problem of model selection.

5. A BAYESIAN INFERENTIAL MODEL FOR FUTURE MORTALITY

5.1 The model. Assume that the probability distribution of the random lifetime (for a fixed generation of lives) is represented by the Weibull model, hence with pdf given by

$$f(t|\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1} e^{-(\frac{t}{\beta})^{\alpha}}; \qquad \alpha,\ \beta > 0 \tag{5.1}$$

the corresponding survival function is then

$$S(t|\alpha,\beta) = e^{-(\frac{t}{\beta})^{\alpha}} \tag{5.2}$$

It is well known that, whilst the Weibull distribution does not fit well the mortality pattern throughout the whole life span (especially owing to infant and young-adult mortality), it provides a sensible representation of mortality at old ages. Moreover, the choice of the Weibull model is supported by the possibility of easily expressing, in terms of its parameters, the mode (for adult ages) of the distribution of the random lifetime T,

$$Mode(T|\alpha,\beta) = \beta\left(\frac{\alpha-1}{\alpha}\right)^{\frac{1}{\alpha}} \qquad \text{with } \alpha > 1 \tag{5.3}$$

as well as the expected value and the variance,

$$E(T|\alpha,\beta) = \beta\ \Gamma\left(\frac{1}{\alpha}+1\right) \tag{5.4}$$

$$Var(T|\alpha,\beta) = \beta^2 \left[\Gamma\left(\frac{2}{\alpha}+1\right) - \left(\Gamma\left(\frac{1}{\alpha}+1\right)\right)^2\right] \tag{5.5}$$

where Γ denotes the complete gamma function (see, for example, Johnson and Kotz, 1970). This possibility allows us to easily choose projected survival functions reflecting specific future trends of mortality (see also Olivieri and Pitacco, 1999; Ferri and Olivieri, 2000; Olivieri and Pitacco, 2001).
It should be stressed that, anyhow, the overall structure of the inferential model remains unchanged if different laws are used, such as the Gompertz or Makeham law.
Assume now that the two parameters in (5.1) and (5.2) are random, and denote them by $\tilde{\alpha}$ and $\tilde{\beta}$ respectively. Let $\alpha_i,\ i = 1, 2, \ldots, r$ and $\beta_j,\ j = 1, 2, \ldots, s$ indicate their possible values, thus working within a finite parameter space. Finally, let

$$g(\alpha_i, \beta_j) = \Pr\{\tilde{\alpha} = \alpha_i \wedge \tilde{\beta} = \beta_j\} \tag{5.6}$$

trivially with

$$\sum_{i=1}^{r}\sum_{j=1}^{s} g(\alpha_i, \beta_j) = 1 \tag{5.7}$$

The (prior) predictive pdf is hence given by

$$f(t) = \sum_{i=1}^{r}\sum_{j=1}^{s} f(t|\alpha_i, \beta_j) g(\alpha_i, \beta_j) \tag{5.8}$$

Let us turn again to the inference problem introduced in Section 4.1. Since the observation process concern people all aged τ, the pdf and the survival function involved are respectively given by $f_\tau(t|\alpha_i, \beta_j)$ and $S_\tau(t|\alpha_i, \beta_j)$ (see (4.1), (4.9) and (4.10)).
Denoting the observed sample as in Section 4.1, for the likelihood function we have:

$$L(\alpha_i, \beta_j|m, \underline{x}) \propto \left[\prod_{h=1}^{m} f_\tau(x_h|\alpha_i, \beta_j)\right] \left[S_\tau(\tau'|\alpha_i, \beta_j)\right]^{n-m} =$$

$$= \left[\left(\frac{\frac{\alpha_i}{\beta_j}}{e^{-\left(\frac{\tau}{\beta_j}\right)^{\alpha_i}}}\right)^m \left(\prod_{h=1}^{m}\left(\frac{x_h}{\beta_j}\right)^{\alpha_i-1} e^{-\left(\frac{x_h}{\beta_j}\right)^{\alpha_i}}\right)\right] \left[\frac{e^{-\left(\frac{\tau'}{\beta_j}\right)^{\alpha_i}}}{e^{-\left(\frac{\tau}{\beta_j}\right)^{\alpha_i}}}\right]^{n-m} \tag{5.9}$$

For the posterior pdf we have:

$$g(\alpha_i, \beta_j|m, \underline{x}) \propto g(\alpha_i, \beta_j) L(\alpha_i, \beta_j|m, \underline{x}) \tag{5.10}$$

obviously with

$$\sum_{i=1}^{r}\sum_{j=1}^{s} g(\alpha_i, \beta_j|m, \underline{x}) = 1 \tag{5.11}$$

Finally, the (posterior) predictive pdf can be calculated as

$$f_\tau(t|m, \underline{x}) = \sum_{i=1}^{r}\sum_{j=1}^{s} f_\tau(t|\alpha_i, \beta_j) g(\alpha_i, \beta_j|m, \underline{x}) \tag{5.12}$$

5.2 Numerical examples. Let us consider a cohort of $n = 1000$ males aged $\tau = 60$ in current year and choose the observation period [60,65]. The cohort is assumed to be homogeneous; therefore the same pdf can be adopted for the random lifetimes T_h, $h = 1, 2, \ldots, n$; in what follows, we denote the random lifetime of a generic individual in the cohort simply by T. Adopting the Weibull model for representing the pdf of the duration of life, the following parameter space has been considered

$$\begin{aligned}\Theta &= \{(\alpha_1, \alpha_2, \ldots, \alpha_5), (\beta_1, \beta_2, \ldots, \beta_5)\} \\ &= \{(7, 8, 9.15, 10.45, 12), (82, 83.5, 85.2, 87, 89)\}\end{aligned}$$

Hence, 25 scenarios describe the possible future evolution of mortality. Table 5.1–5.3 quote the expected value, variance and mode at adult ages (i.e. the so-called "Lexis point") of the duration of life under each scenario at age 65 (i.e. at the end of the observation period). Note in particular the phenomena of expansion (witnessed by the expected value and mode) and rectangularization (witnessed by the variance) assumed in each scenario.

α, β	82	83.5	85.2	87	89
7	16.097	17.187	18.450	19.816	21.364
8	15.548	16.680	17.991	19.411	21.021
9.15	15.155	16.331	17.695	19.170	20.841
10.45	14.902	16.126	17.542	19.072	20.802
12	14.764	16.036	17.506	19.090	20.877

Table 5.1 – $E(T{-}65|T{>}65;\alpha,\beta)$

α, β	82	83.5	85.2	87	89
7	82.599	90.181	99.035	108.646	119.473
8	69.555	76.119	83.758	92.042	101.422
9.15	58.894	64.518	71.013	77.999	85.857
10.45	50.135	54.895	60.337	66.126	72.563
12	42.406	46.326	50.749	55.389	60.477

Table 5.2 – $Var(T{-}65|T{>}65;\alpha,\beta)$

α, β	82	83.5	85.2	87	89
7	80.214	81.681	83.344	85.105	87.062
8	80.643	82.118	83.790	85.560	87.527
9.15	80.969	82.450	84.129	85.906	87.881
10.45	81.214	82.700	84.384	86.167	88.147
12	81.408	82.897	84.584	86.371	88.357

Table 5.3 – $Mode(T|\alpha,\beta)$ (Lexis point)

The prior pdf chosen for the random parameters $\tilde{\alpha}, \tilde{\beta}$ is quoted in Table 5.4. We have assumed a rather concentrated pdf, where in particular the "central" scenario (i.e. the one corresponding to $\tilde{\alpha} = \alpha_3, \tilde{\beta} = \beta_3$) is the most probable, whilst the probability of the other scenarios is the lower the more they differ from this central mortality hypothesis.

α, β	82	83.5	85.2	87	89	
7	0.0025	0.0075	0.03	0.0075	0.0025	0.05
8	0.0075	0.0225	0.09	0.0225	0.0075	0.15
9.15	0.03	0.09	0.36	0.09	0.03	0.6
10.45	0.0075	0.0225	0.09	0.0225	0.0075	0.15
12	0.0025	0.0075	0.03	0.0075	0.0025	0.05
	0.05	0.15	0.6	0.15	0.05	1

Table 5.4 – Case I: prior pdf $g(\alpha,\beta)$

Observations have been obtained through stochastic simulation, assigning for each simulation a given mortality scenario, chosen among the available 25 ones. In order to reduce the effects of random fluctuations, a number of deaths equal to the one expected under the assumed actual scenario has been generated in each simulation. Results obtained in terms of posterior pdf on the parameter space are quoted in Tables 5.5–5.9, each experience belonging to a different scenario, as specified in the relevant table.

α, β	82	83.5	85.2	87	89	
7	0.19441	0.30156	0.15176	0.00121	0	0.64894
8	0.22923	0.07390	0.00552	0.00001	0	0.30866
9.15	0.04085	0.00153	0.00001	0	0	0.04238
10.45	0.00002	0	0	0	0	0.00002
12	0	0	0	0	0	0
	0.46450	0.37699	0.15729	0.00121	0	1

Table 5.5 – Case Ia: posterior distribution $g(\alpha,\beta|m,\underline{x})$*; actual scenario* (α_1,β_1)

α, β	82	83.5	85.2	87	89	
7	0.00107	0.02087	0.17260	0.02461	0.00120	0.22035
8	0.02316	0.13262	0.23821	0.00612	0.00004	0.40015
9.15	0.16863	0.16569	0.03734	0.00010	0	0.37177
10.45	0.00701	0.00070	0.00001	0	0	0.00772
12	0.00001	0	0	0	0	0.00001
	0.19987	0.31989	0.44816	0.03084	0.00125	1

Table 5.6 – Case Ib: posterior distribution $g(\alpha,\beta|m,\underline{x})$*; actual scenario* (α_2,β_2)

α, β	82	83.5	85.2	87	89	
7	0	0.00001	0.00085	0.00218	0.00243	0.00547
8	0.00001	0.00102	0.04422	0.03034	0.00728	0.08287
9.15	0.00294	0.07640	0.64172	0.07426	0.00248	0.79780
10.45	0.01139	0.04688	0.04853	0.00064	0	0.10744
12	0.00457	0.00172	0.00013	0	0	0.00642
	0.01891	0.12602	0.73545	0.10742	0.01219	1

Table 5.7 – Case Ic: posterior distribution $g(\alpha,\beta|m,\underline{x})$*; actual scenario* (α_3,β_3)

α, β	82	83.5	85.2	87	89	
7	0	0	0	0	0.00005	0.00006
8	0	0	0.00011	0.00113	0.00525	0.00649
9.15	0	0.00030	0.05227	0.13679	0.13383	0.32320
10.45	0.00010	0.00889	0.28427	0.12977	0.01975	0.44276
12	0.00342	0.04437	0.17017	0.00938	0.00016	0.22750
	0.00351	0.05356	0.50681	0.27707	0.15904	1

Table 5.8 – Case Id: posterior distribution $g(\alpha,\beta|m,\underline{x})$*; actual scenario* (α_4,β_4)

α, β	82	83.5	85.2	87	89	
7	0	0	0	0	0	0
8	0	0	0	0	0.00002	0.00002
9.15	0	0	0.00003	0.00098	0.01491	0.01592
10.45	0	0.00001	0.00506	0.04105	0.14152	0.18764
12	0.00001	0.00253	0.23389	0.34680	0.21320	0.79642
	0.00001	0.00254	0.23898	0.38883	0.36964	1

Table 5.9 – Case Ie: posterior distribution $g(\alpha,\beta|m,\underline{x})$*; actual scenario* (α_5,β_5)

Note that in each case the inferential mechanism increases the probability of the actual scenario, reducing to zero the probability of very different mortality hypotheses. The mechanism is more satisfying, in terms of the relative weight of the posterior probability of the actual scenario with respect to that of the others, the higher is the prior probability of the actual scenario itself.
In order to realize the effect of the information obtained in terms of some synthetic quantities, in Table 5.10 the prior and posterior expected value, the variance and its components of the residual duration of life for an individual aged 65 are quoted, together with the Lexis point.

	prior	posterior case Ia	posterior case Ib	posterior case Ic	posterior case Id	posterior case Ie
$E(T-65\|T>65)$	17.792	16.677	17.092	17.675	18.417	19.352
$E(Var(T-65\|T>65;\tilde{\alpha},\tilde{\beta}))$	71.921	82.952	77.779	70.998	65.567	58.739
$Var(E(T-65\|T>65;\tilde{\alpha},\tilde{\beta}))$	1.466	1.030	1.605	0.815	1.807	1.696
$Var(T-65\|T>65)$	73.387	83.982	79.384	71.813	67.374	60.435
$Mode(T)$	84.072	81.298	82.429	83.940	84.941	86.368

Table 5.10 – Case I: prior and posterior valuation of the duration of life

From the posterior expected value and mode of the lifetime it is evident the effect of expansion implied by each experienced scenario. From the posterior variance and its component $E(Var(T-65|T>65;\tilde{\alpha},\tilde{\beta}))$ (due to random fluctuations) it emerges the effect of rectangularization. However, the quantity $Var(E(T-65|T>65;\tilde{\alpha},\tilde{\beta}))$, witnessing the longevity risk, does not always reduce as a result of the new information; this is due to the possibly higher posterior dispersion of the pdf $g(\alpha,\beta|m,\underline{x})$, mainly experienced when the prior probability of the actual scenario is low.
In order to understand the influence of the prior pdf $g(\alpha,\beta)$ on the inferential results, calculations have been performed also using a prior uniform pdf (see Table 5.11). We point out that a prior uniform pdf on the parameter space originates from a serious lack of information about mortality evolution. The posterior pdf's are quoted in Tables 5.12–5.16 and confirm what shown by case I. However, a greater dispersion than case I may arise, as suggested for example by the values of $Var(E(T-65|T>65;\tilde{\alpha},\tilde{\beta}))$ (see Table 5.17, where the expected value, variance and mode of the lifetime are reported).

α, β	82	83.5	85.2	87	89	
7	0.04	0.04	0.04	0.04	0.04	0.2
8	0.04	0.04	0.04	0.04	0.04	0.2
9.15	0.04	0.04	0.04	0.04	0.04	0.2
10.45	0.04	0.04	0.04	0.04	0.04	0.2
12	0.04	0.04	0.04	0.04	0.04	0.2
	0.2	0.2	0.2	0.2	0.2	1

Table 5.11 – Case II: prior pdf $g(\alpha,\beta)$

α, β	82	83.5	85.2	87	89	
7	0.49067	0.25371	0.03192	0.00101	0.00001	0.77732
8	0.19285	0.02073	0.00039	0	0	0.21397
9.15	0.00859	0.00011	0	0	0	0.00870
10.45	0.00002	0	0	0	0	0.00002
12	0	0	0	0	0	0
	0.69213	0.27454	0.03231	0.00102	0.00001	1

Table 5.12 – Case IIa: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_1,β_1)

α, β	82	83.5	85.2	87	89	
7	0.01289	0.08390	0.17346	0.09893	0.01451	0.38368
8	0.09309	0.17771	0.07980	0.00821	0.00017	0.35897
9.15	0.16947	0.05551	0.00313	0.00003	0	0.22814
10.45	0.02816	0.00094	0	0	0	0.02911
12	0.00011	0	0	0	0	0.00011
	0.30371	0.31806	0.25639	0.10717	0.01468	1

Table 5.13 – Case IIb: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_2,β_2)

α, β	82	83.5	85.2	87	89	
7	0	0.00006	0.00202	0.02078	0.06939	0.09225
8	0.00009	0.00323	0.03505	0.09618	0.06922	0.20378
9.15	0.00700	0.06055	0.12716	0.05886	0.00589	0.25946
10.45	0.10836	0.14861	0.03846	0.00202	0.00002	0.29748
12	0.13033	0.01639	0.00031	0	0	0.14703
	0.24579	0.22885	0.20300	0.17784	0.14452	1

Table 5.14 – Case IIc: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_3,β_3)

α, β	82	83.5	85.2	87	89	
7	0	0	0	0.00001	0.00064	0.00066
8	0	0	0.00004	0.00151	0.02113	0.02268
9.15	0	0.00010	0.00438	0.04586	0.13461	0.18496
10.45	0.00039	0.01192	0.09531	0.17404	0.07945	0.36110
12	0.04123	0.17852	0.17117	0.03773	0.00196	0.43060
	0.04162	0.19054	0.27090	0.25915	0.23779	1

Table 5.15 – Case IId: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_4,β_4)

α, β	82	83.5	85.2	87	89	
7	0	0	0	0	0	0
8	0	0	0	0	0.00002	0.00002
9.15	0	0	0	0.00007	0.00309	0.00316
10.45	0	0	0.00035	0.01134	0.11726	0.12895
12	0.00003	0.00210	0.04845	0.28735	0.52996	0.86788
	0.00003	0.00210	0.04880	0.29876	0.65032	1

Table 5.16 – Case IIe: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_5,β_5)

	prior	posterior case IIa	posterior case IIb	posterior case IIc	posterior case IId	posterior case IIe
$E(T-65|T>65)$	17.979	16.351	17.097	17.454	18.323	20.159
$E(Var(T-65|T>65;\tilde{\alpha},\tilde{\beta}))$	73.621	82.218	81.003	70.609	62.427	60.075
$Var(E(T-65|T>65;\tilde{\alpha},\tilde{\beta}))$	4.189	0.502	2.502	4.752	3.362	1.045
$Var(T-65|T>65)$	77.810	82.720	83.505	75.361	65.789	61.120
$Mode(T)$	83.923	80.783	82.039	82.804	84.586	87.416

Table 5.17 – Case II: prior and posterior valuation of the duration of life

It is known that results coming from a Bayesian inferential procedure are affected by the specific sample we are dealing with; in the case of mortality investigations, this means that random fluctuations may significantly affect our opinion about future trends. In order to investigate this aspect, five samples have been generated under the same mortality scenario (scenario $\tilde{\alpha} = \alpha_3, \tilde{\beta} = \beta_3$), adopting the prior pdf in Table 5.4. The posterior pdf's originated are quoted in Tables 5.18–5.22 and valuations concerning the lifetime are in Table 5.23. Actually, the inferential mechanism is affected by random fluctuations (note, for example, the posterior values of $E(Var(T-65|T>65;\tilde{\alpha},\tilde{\beta})))$; however, the mechanism itself is able to catch the mortality trend (the magnitude of $g(\alpha_3,\beta_3|m,\underline{x})$ is roughly the same for any sample). Observations generated by scenarios with a lower prior probability confirm such conclusions.

α, β	82	83.5	85.2	87	89	
7	0	0.00001	0.00176	0.00437	0.00471	0.01086
8	0.00002	0.00156	0.06503	0.04319	0.01006	0.11984
9.15	0.00315	0.07882	0.63961	0.07186	0.00233	0.79578
10.45	0.00786	0.03126	0.03138	0.00040	0	0.07090
12	0.00188	0.00069	0.00005	0	0	0.00262
	0.01291	0.11233	0.73784	0.11982	0.01711	1

Table 5.18 – Case IIIa: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_3,β_3); sample no. 1

α, β	82	83.5	85.2	87	89	
7	0	0.00001	0.00158	0.00394	0.00426	0.00980
8	0.00001	0.00147	0.06154	0.04098	0.00957	0.11357
9.15	0.00314	0.07883	0.64137	0.07222	0.00235	0.79792
10.45	0.00837	0.03335	0.03355	0.00043	0	0.07570
12	0.00216	0.00079	0.00006	0	0	0.00301
	0.01368	0.11446	0.73811	0.11757	0.01618	1

Table 5.19 – Case IIIb: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_3,β_3); sample no. 2

α, β	82	83.5	85.2	87	89	
7	0	0.00001	0.00130	0.00327	0.00356	0.00814
8	0.00001	0.00131	0.05549	0.03728	0.00878	0.10288
9.15	0.00309	0.07833	0.64330	0.07302	0.00239	0.80013
10.45	0.00928	0.03734	0.03788	0.00049	0	0.08499
12	0.00276	0.00102	0.00008	0	0	0.00386
	0.01515	0.11801	0.73805	0.11406	0.01473	1

Table 5.20 – Case IIIc: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_3,β_3); sample no. 3

α, β	82	83.5	85.2	87	89	
7	0	0.00001	0.00113	0.00287	0.00315	0.00716
8	0.00001	0.00121	0.05157	0.03492	0.00828	0.09598
9.15	0.00304	0.07769	0.64365	0.07361	0.00243	0.80042
10.45	0.00992	0.04025	0.04116	0.00054	0	0.09187
12	0.00326	0.00121	0.00009	0	0	0.00456
	0.01622	0.12038	0.73760	0.11193	0.01386	1

Table 5.21 – Case IIId: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_3,β_3); sample no. 4

α, β	82	83.5	85.2	87	89	
7	0	0	0.00067	0.00174	0.00196	0.00438
8	0.00001	0.00089	0.03889	0.02693	0.00652	0.07323
9.15	0.00287	0.07517	0.63749	0.07439	0.00250	0.79241
10.45	0.01273	0.05287	0.05520	0.00073	0	0.12153
12	0.00600	0.00228	0.00017	0	0	0.00845
	0.02160	0.13120	0.73243	0.10379	0.01097	1

Table 5.22 – Case IIIe: posterior distribution $g(\alpha,\beta|m,\underline{x})$; actual scenario (α_3,β_3); sample no. 5

	prior	posterior case IIIa	posterior case IIIb	posterior case IIIc	posterior case IIId	posterior case IIIe
$E(T-65|T>65)$	17.792	17.764	17.750	17.725	17.710	17.645
$E(Var(T-65|T>65;\tilde{\alpha},\tilde{\beta}))$	71.921	72.416	72.191	71.801	71.541	70.549
$Var(E(T-65|T>65;\tilde{\alpha},\tilde{\beta}))$	1.466	0.829	0.824	0.816	0.814	0.823
$Var(T-65|T>65)$	73.388	73.245	73.015	72.618	72.355	71.371
$Mode(T)$	84.072	84.009	83.999	83.981	83.969	83.913

Table 5.23 – Case III: prior and posterior valuation of the duration of life

As far as the length of the observation period is concerned, the period [60,70] has been considered in alternative to [60,65]. The results obtained are similar to those discussed above. However, when dealing with actuarial applications, such as reserving or capital allocation (see Section 6), an interval of five years seems on the one hand long enough

for catching the mortality trend and on the other hand more suitable then a ten year period for updating the reserving basis or the solvency margin.
Finally, further investigations have concerned observations generated by a scenario with a prior probability equal to zero. In this case, the inferential model assigns the higher posterior probability to the scenario, among those with a positive prior probability, more similar to the actual one in terms of the phenomena of expansion and rectangularization. For the sake of brevity, relevant numerical results are omitted.

5.3 Possible generalizations. It should be stressed that the proposed inferential model regards the mortality in one generation only. Hence, a straightforward actuarial application of monitoring mortality according to the described procedure consists in updating the demographical bases to be used for evaluating some generation-related quantities after the observation period elapsed. Typically, the portfolio reserve and the required solvency margin will be concerned (see Section 6).
The extension of mortality adjustments to more generations requires (a) some particular assumptions, or (b) a much more complicated inferential model.

(a) Assume that, for all couples of generations in a given set, the mortality of one generation is linked to the mortality of the other one in a given way. For example (referring to notation (3.2)), assume that

$$\theta(y) = \phi_y[\theta(y-1)] \tag{5.13}$$

for all y, where ϕ_y is an assigned (vector-valued or, in particular, real-valued function); a further simplification could result in:

$$\theta(y) = \phi[\theta(y-1)] \tag{5.13$'$}$$

Referring to the Weibull model, the following link

$$\alpha(y) = \gamma\ \alpha(y-1) \tag{5.14a}$$

$$\beta(y) = \delta\ \beta(y-1) \tag{5.14b}$$

with $\gamma, \delta > 0$, can express hypotheses of rectangularization and expansion both increasing as the calendar year of birth increases.
The assumption of a fixed link allows us to extend the inferential results concerning a generation to other generations (at least to generations closed to the monitored one).

(b) A random link between generations can be assumed to represent uncertainty in rectangularization and expansion processes. In this case, the randomness of the link itself should be included in the inferential model, which must be applied to a set of generations jointly. Of course, Bayesian modelling in this framework is much more complicated.

6. ACTUARIAL APPLICATIONS

6.1 Some preliminary ideas. As mentioned in Section 5.3, a straight actuarial application of the inferential results, concerning a given generation, consists in updating

the bases to be used for evaluating some generation-related quantities, such as the portfolio reserve and the required solvency margin. We now focus on the calculation of these quantities, thus disregarding the updating of premiums, which should be based on the adjustment of mortality of annuitants in following generations.
As far as updating the reserving basis is concerned, some points should be stressed. Current valuation techniques mostly lie on the adoption of prudential bases, which means, in the case of annuities, survival probabilities higher than the realistic ones. On the contrary, fair value accounting principles require valuations in realistic terms. So, in both cases a realistic, and hence updated, demographical hypothesis is needed. In order to evaluate the portfolio reserve V, Bayes inference results can suggest the choice of the most probable mortality law, according to the posterior pdf $g(\alpha, \beta | m, \underline{x})$ or, in alternative, the posterior (unconditional) expected value of liabilities.
The required solvency margin, M, is the minimum shareholders' capital needed to meet the liabilities with an assigned probability, i.e. to face risks related to the portfolio. The quantity SR,

$$SR = V + M \tag{6.1}$$

is the so-called (minimum) solvency reserve. Taking into account the effect of risks leads to the assessment, possibly via stochastic simulation, of the solvency reserve SR (see for example Olivieri and Pitacco, 2000); then, for any given portfolio reserve V, the required solvency margin is given by $M = \max(SR - V, 0)$.
When life annuity portfolios are concerned, an important component of the demographical risk is given by the longevity risk (which is a model risk, see Section 3), arising from the uncertainty in future mortality. So, in order to assess the demographical risk (and the consequent solvency requirements) taking into account its longevity component, several mortality scenarios should be considered, and the relevant probability distribution used as the underlying stochastic model. In general terms, the predictive pdf provides the tool to be used to express uncertainty in future mortality. In the Bayesian framework, the posterior predictive pdf expresses the uncertainty adjusted by the experienced mortality.

6.2 Applications. We consider a homogeneous cohort of time-continuous straight life annuities, with benefits paid at the instantaneous rate $b = 1$. We denote by Y_t the random present value at age t of future benefits for a given annuitant. Disregarding financial risks (hence focussing only demographical aspects), let δ be the (constant) expected instantaneous investment yield. We have

$$Y_t = \int_0^{T-t} e^{-\delta s} \, ds = \bar{a}_{\overline{T-t}|} \tag{6.2}$$

where $\bar{a}_{\overline{s}|} = \frac{1-e^{-\delta s}}{\delta}$ denotes the present value of a continuous annuity certain paid up to time s. (The generalization of (6.2) to the case of variable benefits or variable investment yields is straightforward.)
Referring to notation (3.2′), the conditional expected value, variance and distribution function are

$$E(Y_t|\theta) = \int_0^{\infty} e^{-\delta s} \, f_t(s|\theta) \, ds \tag{6.3}$$

$$Var(Y_t|\theta) = \int_0^\infty e^{-2\delta s} \; f_t(s|\theta) \; ds - \big(E(Y_t|\theta)\big)^2 \tag{6.4}$$

$$F_{Y_t}(y|\theta) = \Pr\{Y_t \leq y|\theta\} = \int_0^{t^*} \; f_t(s|\theta) \; ds \tag{6.5}$$

where t^*, $t^* = \ln(1 - \delta \; y)^{-1/\delta}$, is the realization of the residual lifetime such that $\bar{a}_{t^*\rceil} = y$. The unconditional quantities can be easily obtained. In the case of a continuous pdf $g(\theta)$ we have

$$E(Y_t) = \int_\Theta E(Y_t|\theta) \; g(\theta) \; d\theta \tag{6.6}$$

$$\begin{aligned} Var(Y_t) &= E(Var(Y_t|\tilde{\theta})) + Var(E(Y_t|\tilde{\theta})) \\ &= \int_\Theta Var(Y_t|\theta) \; g(\theta) \; d\theta + \Big(\int_\Theta \big(E(Y_t|\theta)\big)^2 \; g(\theta) \; d\theta - \big(E(Y_t)\big)^2 \Big) \end{aligned} \tag{6.7}$$

$$F_{Y_t}(y) = \int_\Theta F_{Y_t}(y|\theta) \; g(\theta) \; d\theta \tag{6.8}$$

Expressions for the case of a discrete pdf $g(\theta)$ are straightforward.
The present value of future benefits at the portfolio level, $\hat{Y}_t$, is simply given by the sum of the individual items Y_t. If the portfolio is homogeneous and n annuitants are present at time t, whose lifetimes are independent under any mortality hypothesis, the following results hold

$$E(\hat{Y}_t|\theta) = n \; E(Y_t|\theta) \tag{6.9}$$

$$E(\hat{Y}_t) = n \; E(Y_t) \tag{6.10}$$

$$Var(\hat{Y}_t|\theta) = n \; Var(Y_t|\theta) \tag{6.11}$$

$$Var(\hat{Y}_t) = n \; E(Var(Y_t|\tilde{\theta})) + n^2 \; Var(E(Y_t|\tilde{\theta})) \tag{6.12}$$

Note in particular that the second term of $Var(\hat{Y}_t)$, which witnesses longevity risk, is proportional to n^2; this reflects the fact that longevity risk has a systematic character.
As far as the distribution function of $\hat{Y}_t$ is concerned, given the parameter θ it can be obtained as the convolution of the (i.i.d.) distribution function of the r.v. Y_t; so

$$F_{\hat{Y}_t}(y|\theta) = \Pr\{\hat{Y}_t \leq y|\theta\} = \big[F_{Y_t}(y|\theta)\big]^{n*} \tag{6.13}$$

(where we have used the symbol commonly denoting the convolution operation). Then we have

$$F_{\hat{Y}_t}(y) = \Pr\{\hat{Y}_t \leq y\} = \int_\Theta F_{\hat{Y}_t}(y|\theta) \; g(\theta) \; d\theta \tag{6.14}$$

Similar results hold when the posterior pdf $g(\theta|m, \underline{x})$ is considered.
Turning to reserving aspects, the definition of reserve involves the expected value of future benefits, such expected value being based on a given hypothesis (either conservative or realistic) of the future scenario. As has been mentioned above, the Bayesian inferential model suggests to take the expected value based on the most probable scenario or alternatively the predictive expected value. Given that the Bayesian inferential mechanism is affected by random fluctuations, in our opinion the latter choice is more

appropriate, so that sudden changes of the reserving basis as a result of the new information are avoided. Moreover, in the long run the predictive expected value tends to the expected value of the most probable scenario (to this regard, some examples are discussed at the end of this Section). In our view, the portfolio reserve at time t should be defined as

$$V_t = E(\hat{Y}_t) \tag{6.15}$$

In Table 6.1 and 6.2 the conditional expected value and variance of the present value of future benefits for a given annuitant are quoted. The same hypotheses of Section 5 have been used; the force of interest is $\delta = \ln(1.03) = 0.02956$. Similarly to the valuation of the duration of life, the expected value reflects the hypothesis of expansion, whilst the variance that of rectangularization assumed in the relevant scenario. Tables 6.3–6.5 quote the unconditional values at the individual and portfolio level for the three types of observations discussed in Section 5 (case I, II and III). Note in particular the dramatic increase of the variance of the present value of future payments when a cohort of policies is considered, owing to the longevity risk. Moreover, reflect on the required change of the reserve at age 65 after new information have been obtained and on what would be required if the expected value of the most probable scenario were chosen as reserving basis instead of (6.15).
We point out that in order to obtain a conservative valuation of the reserve, a safety loading can be added to (6.15) and therefore to the results quoted in Table 6.3–6.5.

α, β	82	83.5	85.2	87	89
7	12.060	12.681	13.377	14.104	14.895
8	11.819	12.481	13.224	13.997	14.839
9.15	11.658	12.362	13.150	13.969	14.858
10.45	11.572	12.316	13.147	14.009	14.941
12	11.553	12.336	13.208	14.109	15.079

Table 6.1 – $E(Y_{65}|\alpha,\beta)$

α, β	82	83.5	85.2	87	89
7	31.831	33.039	34.242	35.317	36.263
8	28.190	29.278	30.336	31.252	32.024
9.15	24.859	25.795	26.671	27.388	27.939
10.45	21.833	22.595	23.268	23.768	24.085
12	18.915	19.480	19.931	20.205	20.289

Table 6.2 – $Var(Y_{65}|\alpha,\beta)$

	prior	posterior case Ia	posterior case Ib	posterior case Ic	posterior case Id	posterior case Ie
$E(Y_{65})$	13.190	12.416	12.713	13.132	13.616	14.220
$V_{65}=E(\hat{Y}_{65})$	13190.110	12416.068	12713.379	13131.950	13616.320	14220.036
$E(Var(Y_{65}\|\tilde{\alpha},\tilde{\beta}))$	26.701	31.240	29.135	26.585	23.986	20.990
$Var(E(Y_{65}\|\tilde{\alpha},\tilde{\beta}))$	0.454	0.299	0.453	0.238	0.494	0.497
$Var(Y_{65})$	27.155	31.538	29.588	26.822	24.481	21.488
$Var(\hat{Y}_{65})$	480304.577	329871.128	481900.405	264239.086	518299.286	518420.382

Table 6.3 – Case I: prior and posterior valuation of the reserve

Let us finally address solvency aspects. Generally speaking, a portfolio is solvent if assets keep higher than liabilities with a given probability and within a given time

	prior	posterior case IIa	posterior case IIb	posterior case IIc	posterior case IId	posterior case IIe
$E(Y_{65})$	13.270	12.221	12.684	12.975	13.575	14.674
$V_{65}=E(\hat{Y}_{65})$	13269.716	12221.042	12684.090	12975.356	13575.141	14673.668
$E(Var(Y_{65}\|\tilde{\alpha},\tilde{\beta}))$	26.752	31.402	30.019	26.066	22.902	20.755
$Var(E(Y_{65}\|\tilde{\alpha},\tilde{\beta}))$	1.291	0.148	0.694	1.296	0.965	0.309
$Var(Y_{65})$	28.043	31.550	30.712	27.363	23.866	21.064
$Var(\hat{Y}_{65})$	1317648.604	179816.455	723873.018	1322227.581	987431.622	329568.694

Table 6.4 – Case II: prior and posterior valuation of the reserve

	prior	posterior case IIIa	posterior case IIIb	posterior case IIIc	posterior case IIId	posterior case IIIe
$E(Y_{65})$	13.190	13.174	13.167	13.156	13.148	13.118
$V_{65}=E(\hat{Y}_{65})$	13190.110	13174.118	13167.252	13155.846	13148.426	13117.640
$E(Var(Y_{65}\|\tilde{\alpha},\tilde{\beta}))$	26.701	26.948	26.892	26.793	26.725	26.468
$Var(E(Y_{65}\|\tilde{\alpha},\tilde{\beta}))$	0.454	0.236	0.236	0.235	0.235	0.241
$Var(Y_{65})$	27.155	27.185	27.128	27.028	26.961	26.709
$Var(\hat{Y}_{65})$	480304.577	263410.335	262631.392	262031.145	262209.874	267622.322

Table 6.5 – Case III: prior and posterior valuation of the reserve

horizon. Different solvency requirements derive from such a definition, depending on the way liabilities are described (either in terms of the portfolio reserve or of their random present value), on the accepted probability level and on the time span. Considering the random present value of liabilities, an infinite time span is implicitly assumed (see Olivieri and Pitacco, 2000). Given an accepted ruin probability ε, the following definitions of solvency reserve required at time t originates

$$SR_t^{(\theta)} = \inf\left\{y : \Pr\{\hat{Y}_t > y|\theta\} \leq 1-\varepsilon\right\} \tag{6.16}$$

$$SR_t = \inf\left\{y : \Pr\{\hat{Y}_t > y\} \leq 1-\varepsilon\right\} \tag{6.17}$$

The difference between (6.16) and (6.17) consists in the types of demographical risks considered: $SR_t^{(\theta)}$ accounts only for the risk of random fluctuations, whilst SR_t also for systematic risks, in particular the longevity risk. It is evident that if we believe that the solvency reserve has to be tailored to all the risks affecting the portfolio, definition (6.17) must be adopted.

Turning to numerical evaluations, we mention that the solvency reserve according to (6.16) or (6.17) can be easily calculated from the distribution function of $\hat{Y}_t$. Table 6.6 shows the required solvency reserve according to definition (6.16) and for different levels of the ruin probability (for the sake of brevity, only five scenarios have been considered). Note that as a result of a strong assumed rectangularization, lower values of the solvency reserve per unit of the portfolio reserve are required. In Tables 6.7–6.9 definition (6.17) has been adopted. Note in particular the dramatic increase of magnitude of the solvency margin with respect to Table 6.6, given that now longevity risk is considered.

Two final examples follow (see Tables 6.10 and 6.11), in which the mortality has been monitored on a five year basis, assuming two different scenarios. The prior pdf is that of Table 5.4. In order to stress longevity risk, a size of $n = 1000$ annuitants has been considered at the beginning of any observation period (in this way, the influence

ε	α_1,β_1	α_2,β_2	α_3,β_3	α_4,β_4	α_5,β_5
0.5	99.911%	99.958%	99.990%	99.996%	100.010%
0.8	101.128%	101.124%	101.054%	101.006%	100.845%
0.9	101.782%	101.672%	101.580%	101.470%	101.263%
0.95	102.380%	102.118%	102.095%	101.764%	101.613%
0.99	103.399%	102.954%	102.931%	102.548%	102.129%
$V_{65}=E(\hat{Y}_{65}\|\alpha,\beta)$	12060.105	12481.497	13149.624	14008.583	15078.668

Table 6.6 – Solvency reserve, $\frac{SR_{65}^{(\alpha,\beta)}}{V_{65}}$

ε	prior	posterior case Ia	posterior case Ib	posterior case Ic	posterior case Id	posterior case Ie
0.5	99.839%	99.760%	100.019%	100.096%	97.993%	99.558%
0.8	103.500%	103.742%	104.924%	101.712%	104.040%	105.509%
0.9	106.627%	107.278%	106.088%	105.196%	108.801%	106.378%
0.95	109.654%	108.546%	107.233%	106.921%	109.797%	106.909%
0.99	113.954%	110.220%	111.616%	111.865%	111.067%	107.774%
$V_{65}=E(\hat{Y}_{65})$	13190.110	12416.068	12713.379	13131.950	13616.320	14220.036

Table 6.7 – Case I: solvency reserve, $\frac{SR_{65}}{V_{65}}$

ε	prior	posterior case IIa	posterior case IIb	posterior case IIc	posterior case IId	posterior case IIe
0.5	99.574%	99.104%	99.042%	100.007%	99.412%	101.814%
0.8	109.040%	103.175%	105.960%	108.549%	108.420%	103.156%
0.9	112.490%	104.680%	109.914%	113.838%	109.923%	103.702%
0.95	113.451%	105.846%	111.753%	115.167%	110.698%	104.106%
0.99	114.812%	110.464%	116.714%	116.747%	111.754%	104.696%
$V_{65}=E(\hat{Y}_{65})$	13269.716	12221.042	12684.090	12975.356	13575.141	14673.668

Table 6.8 – Case II: solvency reserve, $\frac{SR_{65}}{V_{65}}$

ε	prior	posterior case IIIa	posterior case IIIb	posterior case IIIc	posterior case IIId	posterior case IIIe
0.5	99.839%	100.036%	100.030%	100.014%	99.988%	99.915%
0.8	103.500%	105.720%	105.642%	105.455%	105.258%	103.600%
0.9	106.627%	107.535%	107.447%	107.309%	107.211%	106.967%
0.95	109.654%	111.819%	111.667%	111.265%	110.686%	108.516%
0.99	113.954%	114.247%	114.231%	114.221%	114.194%	114.134%
$V_{65}=E(\hat{Y}_{65})$	13190.110	13174.118	13167.252	13155.846	13148.426	13117.640

Table 6.9 – Case III: solvency reserve, $\frac{SR_{65}}{V_{65}}$

of random fluctuations has been kept roughly constant through time). Note that in both cases in the long run the Bayesian inferential model assigns probability 1 to the assumed actual scenario, the rapidity of the convergence depending on the initial prior probability of such scenario. Results can be easily interpreted.

	age τ'=65 prior	post.	age τ'=70 prior	post.	age τ'=75 prior	post.	age τ'=80 prior	post.
$g(\alpha_3,\beta_3)$	0.36000	0.63961	0.63961	0.79552	0.79552	0.89912	0.89912	0.95334
$E(Y_{\tau'})$	13.190	13.174	10.934	10.953	8.833	8.812	6.901	6.861
$E(Var(Y_{\tau'}\|\tilde{\alpha},\tilde{\beta}))$	26.701	26.948	25.127	25.196	21.597	21.512	16.897	16.709
$Var(E(Y_{\tau'}\|\tilde{\alpha},\tilde{\beta}))$	0.454	0.236	0.289	0.169	0.213	0.071	0.089	0.015
$V_{\tau'}=E(\hat{Y}_{\tau'})$	13190.110	13174.118	10934.388	10953.076	8833.127	8811.828	6901.041	6860.571
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.5	99.839%	99.875%	99.759%	99.610%	99.353%	99.561%	99.307%	99.798%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.8	103.500%	101.581%	102.114%	101.319%	101.498%	101.274%	101.249%	101.557%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.9	106.627%	105.483%	107.009%	103.575%	105.566%	102.682%	103.196%	102.660%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.95	109.654%	106.913%	109.064%	108.488%	111.878%	105.000%	108.208%	104.124%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.99	113.954%	112.474%	116.741%	112.346%	118.039%	114.308%	119.422%	109.350%

Table 6.10a – Monitoring on a five year basis; actual scenario (α_3,β_3)

	age τ'=85 prior	post.	age τ'=90 prior	post.	age τ'=95 prior	post.
$g(\alpha_3,\beta_3)$	0.95334	0.99137	0.99137	0.99999	0.99999	1
$E(Y_{\tau'})$	5.221	5.203	3.865	3.863	2.819	2.819
$E(Var(Y_{\tau'}\|\tilde{\alpha},\tilde{\beta}))$	12.027	11.947	7.937	7.926	4.920	4.920
$Var(E(Y_{\tau'}\|\tilde{\alpha},\tilde{\beta}))$	0.020	0.004	0.004	0.000	0.000	0.000
$V_{\tau'}=E(\hat{Y}_{\tau'})$	5220.853	5203.472	3865.394	3862.795	2818.770	2818761
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.5	99.658%	99.933%	99.855%	99.912%	99.888%	99.888%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.8	101.636%	101.777%	101.920%	101.940%	102.078%	102.078%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.9	102.839%	102.840%	103.155%	103.167%	103.439%	103.439%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.95	104.548%	103.777%	104.037%	103.789%	104.186%	104.186%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.99	113.864%	105.723%	106.296%	105.671%	106.099%	106.100%

Table 6.10b – Monitoring on a five year basis; actual scenario (α_3,β_3)

7. CONCLUSIONS

The longevity risk, originating from uncertainty in future mortality, has a tremendous impact on the global riskiness of a life annuity portfolio. Uncertainty in future mortality, and hence the magnitude of longevity risk, can be reduced monitoring mortality and implementing inferential procedures. Bayesian theory provides a sound logical framework for the construction of appropriate procedures. In this paper a particular Bayes model has been proposed and some actuarial applications have been presented and discussed. Results seem to be encouraging, from both a theoretical and a practical point of view. Further research work should be devoted to some related problems, such as

	age τ'=65 prior	post.	age τ'=70 prior	post.	age τ'=75 prior	post.	age τ'=80 prior	post.
$g(\alpha_1,\beta_4)$	0.00750	0.02877	0.02877	0.05581	0.05581	0.15783	0.15783	0.68983
$E(Y_{\tau'})$	13.190	12.929	10.774	10.972	8.966	9.307	7.508	8.470
$E(Var(Y_{\tau'}\|\tilde{\alpha},\tilde{\beta}))$	26.701	28.405	26.187	26.987	23.221	24.552	19.964	24.301
$Var(E(Y_{\tau'}\|\tilde{\alpha},\tilde{\beta}))$	0.454	0.352	0.454	0.319	0.417	0.322	0.414	0.180
$V_{\tau'}=E(\hat{Y}_{\tau'})$	13190.110	12928.535	10774.048	10971.798	8965.791	9306.740	7507.832	8469.696
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.5	99.839%	101.211%	101.266%	100.568%	100.896%	98.496%	98.476%	101.783%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.8	103.500%	103.226%	104.429%	103.132%	104.393%	106.089%	108.077%	104.014%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.9	106.627%	104.560%	106.708%	105.688%	108.812%	111.084%	115.447%	104.880%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.95	109.654%	107.808%	110.497%	110.044%	114.146%	112.593%	117.215%	105.618%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.99	113.954%	110.851%	115.105%	113.112%	118.121%	114.644%	119.494%	107.235%

Table 6.11a – Monitoring on a five year basis; actual scenario (α_1,β_4)

	age prior	τ'=85 post.	age prior	τ'=90 post.	age prior	τ'=95 post.
$g(\alpha_1,\beta_4)$	0.68983	0.86547	0.86547	0.99525	0.99525	1
$E(Y_{\tau'})$	6.941	7.122	5.802	5.904	4.785	4.789
$E(Var(Y_{\tau'}\|\tilde{\alpha},\tilde{\beta}))$	19.523	20.351	15.743	16.209	12.090	12.105
$Var(E(Y_{\tau'}\|\tilde{\alpha},\tilde{\beta}))$	0.211	0.060	0.072	0.003	0.003	0.000
$V_{\tau'}=E(\hat{Y}_{\tau'})$	6.941.429	7122.216	5802.389	5904.276	4785.255	4789.089
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.5	102.706%	100.914%	101.387%	100.000%	100.122%	100.044%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.8	105.203%	102.903%	103.464%	101.903%	102.108%	102.026%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.9	106.146%	103.679%	104.285%	102.688%	102.926%	102.844%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.95	106.972%	104.503%	105.228%	103.555%	103.874%	103.791%
$\frac{SR_{\tau'}}{V_{\tau'}}$; ε=0.99	108.642%	105.925%	106.894%	105.100%	105.599%	105.515%

Table 6.11b – Monitoring on a five year basis; actual scenario (α_1,β_4)

inference on the mortality of a set of generations, in particular to gain a deeper insight into the problem of premium adjustment according to mortality evolution.

REFERENCES

Benjamin B and J H Pollard (1993), *The analysis of mortality and other actuarial statistics*, The Institute of Actuaries, Oxford

Benjamin J and A S Soliman (1993), *Mortality on the move*, Actuarial Education Service, Oxford

CMIR (Continuous Mortality Investigation Reports) n. 10 (1990), Institute of Actuaries and Faculty of Actuaries

CMIR (Continuous Mortality Investigation Reports) n. 17 (1999), Institute of Actuaries and Faculty of Actuaries

Daboni L (1972), Un procedimento inferenziale bayesiano sulla mortalità di rischi normali e tarati, in *Studi in onore di F Giaccardi Giraud*, Torino: 69–86

Ferri S and A Olivieri (2000), Technical bases for LTC covers including mortality and disability projections, *Proceedings of the XXXI International ASTIN Colloquium*, Porto Cervo, Italy: 295–314

Johnson NL and S Kotz (1970), *Continuous univariate distributions – I*, John Wiley & Sons, New York

Kim S W and J G Ibrahim (2000), On bayesian inference for proportional hazards models using noninformative priors, *Lifetime Data Analysis*, **6**: 331–341

Leonard T and J S J Hsu (1999), *Bayesian Methods*, Cambridge University Press

Macdonald A S (ed.) (1997), *The second actuarial study of mortality in Europe*, Groupe Consultatif des Associations d'Actuaires des Pays des Communautes Europeennes, Oxford

Macdonald A S et al (1998), An international comparison of recent trends in population mortality, *British Actuarial Journal*, **4**: 3–141

Marocco P and E Pitacco (1998), Longevity risk and life annuity reinsurance, *Transactions of the 26th International Congress of Actuaries*, Birmingham, **6**: 453–479

Martz H F and R A Waller (1982), *Bayesian reliability analysis*, J Wiley & Sons

Namboodiri K and C M Suchindran (1987), *Life table techniques and their applications*, Academic Press

Olivieri A (2001), Uncertainty in mortality projections: an actuarial perspective, to appear on *Insurance: Mathematics & Economics*

Olivieri A and E Pitacco (1999), Funding sickness benefits for the elderly, *Proceedings of the XXX International ASTIN Colloquium*, Tokyo: 135–155

Olivieri A and E Pitacco (2000), Solvency requirements for life annuities, *Proceedings of the AFIR 2000 Colloquium*, Tromsø, Norway: 547–571

Olivieri A and E Pitacco (2001), Facing LTC risks, *Proceedings of the XXXII International ASTIN Colloquium*, Washington

Petauton P (1996), *Theorie et pratique de l'assurance vie*, Dunod, Paris

Renshaw A E and S Haberman (1996), The modelling of recent mortality trends in United Kingdom male assured lives, *British Actuarial Journal*, **2** (II): 449–477

Riemer-Hommel P and T Trauth (2000), Challenges and solutions for the management of longevity risk, in: M Frenkel, U Hommel and M Rudolf (eds), *Risk management. Challenge and opportunity*, Springer: 85–100

Sithole T Z, S Haberman and R J Verrall (2000), An investigation into parametric models for mortality projections, with applications to immediate annuitants and life office pensioners' data, *Insurance: Mathematics and Economics*, **27** (13): 285–312