"Mathematical Programming and Pension Funds"

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Summary

In the paper we would like to give two applications of mathematical programming (MP) in the field of pension funds. On pension fund we mean the funds in the Hungarian social insurance system. The first application is almost "ready-to-use" and it is entirely connected to this system. Due to its nature the other is only outlined but the idea perhaps may be applied elsewhere in insurance, too.To understand the MP models it is necessary to deal with some details of the Hungarian system. Though we try to give as few details and comments as possible these will be a larger part of the paper. In fact, we are not too much interested in the technical parts of the MP problems. (How to solve them, etc.) It can be briefly said that with the help of these models the (interpretation of the) system's legal framework could be made clearer or the models help to give a clear interpretation at all. The first model deals with certain redistribution among the fund members1 savings (and this makes the problem specific for the Hungarian case). The other one is about the determination of unisex annuities or unisex life tables (which appears in any social insurance system).As far as we know, there is not such regulation where something is defined by the solution of an MP model. It is not an impossible idea since there are many regulations where several formulae are applied.

"Mathematische Programmierung und Pensionskassen"

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Zusammenfassung

Im Vortrag möchten wir zwei Anwendungen der mathematischen Programmierung im Bereich der Pensionskassen darstellen. Unter Pensionskasse werden die Pensionskassen des ungarischen Sozialversicherungssystems verstanden. Die Beschreibung des ersten Modells ist fast komplett und sie knüpft sich an das vorige System. Das zweite wird wegen seiner Natur nur konturiert, aber die Idee ist auch in einem anderen Bereich anzuwenden. Um die MP zu verstehen, ist es nötig, einige Details des ungarischen Systems zu haben. Zwar wird versucht, sich auf die wenigsten Details und Anmerkungen zu beschränken, dennoch bilden diese den größeren Teil des Vortrags. Zugleich wollen wir uns auch nicht mit den technischen Fragen der Modelle beschäftigen. (Wie kann es lösen, usw.) Im großen und ganzen ist zu sagen, dass es mit Hilfe der Modelle möglich ist, die Auslegung der gesetzlichen Rahmen des Systems klarer zu machen, oder so eine klare Auslegung zu geben. Das erste Modell beschäftigt sich mit der Umverteilung unter den Erparnissen der Mitglieder , (was eine spezifische Eigenschaft des ungarischen Systems ist). Das zweite beschäftigt sich mit den unisexen Renten bzw. der Bestimmung der Sterbetafeln (, die in allen Sozialversicherungssysemen zu finden ist) Soweit uns bekannt ist, gibt es keine Regelung durch die Lösung des MP Modells. Zugleich ist es gar nicht unvorstellbar, denn viele Regelungen enthalten Formeln.

Mathematical Programming and Pension Funds

In this paper we would like to deal with two subjects concerning pension funds. By pension fund we mean the funds in the Hungarian pension system. The common points of the two subjects are the problem of redistribution and the application of mathematical programming (MP). The first application is almost “ready-to-use” and it is entirely connected to this system. Due to its nature the other is only outlined but perhaps the idea might be applied elsewhere in insurance, too.

To understand the problems and the MP models given below it is necessary to make known some details of the Hungarian system. Though we try to give as few details and comments as possible, these will be the larger part of the paper.

Recently in Hungary a certain part of the individual’s social insurance contribution of about two million people is paid into a fund. (The other part of it and the employer’s contribution will be paid into the pay-as-you-go pillar.) The funds allocate about 95% of these contributions to the fund members’ personal accounts. The money is invested and at the end of the accumulation period, i.e. at retirement it will be converted into an annuity paid additional to another one from the pay-as-you-go pillar.

Yield redistribution

One must be particularly careful with redistribution in such a case where the personal saving feature of the capital on the personal account is given such a so strong emphasis. The return adjustment reserve is a neat example of the unnecessary and non-transparent redistribution.

There is an expected yield rate for the investment rate of each fund in each year. In fact, this is (a method for calculating) an interval determined by a Board for each year in advance. (This Board is an advisory committee of the President of the Supervisory Authority.) The interval is depending on parameters of the economy and the fund. At the end of the year (or somewhat later) the interval and the fund’s yield rate can be calculated and compared. If the fund’s yield rate is in the interval, then no action is taken. If the fund's yield rate is greater than the upper value then some money will be put into a special reserve (called yield adjustment reserve) from those personal accounts where the yield rate is greater than this upper value. If the fund's yield rate is less than the lower value then some money will be put from this reserve to those personal accounts where the yield rate is less than this lower value.

The expected yield rate (i.e. the interval) is a certain guarantee for a smooth and/or constant-like yield rate over the years. We will comment the idea later. Now we take it as given and we want only to deal with the realisation. The reserve itself is built from the members' contributions: if the reserve is under a level prescribed by the law then a small part of each contribution must be put into this reserve.

(Even accepting the idea of the smooth yield rate the regulation is not complete. It does not say explicitly what to do in the first case with those personal accounts where the yield rate is less than the lower value and does not deal in the second case with those personal accounts where the yield rate is greater than the upper value. Furthermore it can happen now that the fund's yield rate is in the interval but some of the personal accounts' yield rate are outside.)

Let

> r be the yield rate of the fund;
>
> r_u be the upper value of the fund's yield rate prescribed by the Board;
>
> r_l be the lower value of the fund's yield rate prescribed by the Board;
>
> r_i –s are the yield rates of personal accounts of yield rates less than r_l and E_i-s are their recent balances;
>
> r_j –s are the yield rates of personal accounts of yield rates greater than r_u and G_j-s are their recent balances;
>
> R be the size of the yield adjustment reserve

and let us denote by $d(\ldots r_i \ldots r_j \ldots)$ a measure expressing how much the r_i-s and r_j-s are outside of the interval $[r_l, r_u]$. Let furthermore be the x_i-s and y_j-s the yields given to and taken from the personal accounts and let us denote the modified yield rates by $\underline{r}_i$–s and $\underline{r}_j$–s, resp. For the several yield rate formula the $\underline{r}_i$–s and $\underline{r}_j$–s are simple functions of the x_i-s and y_j-s. (These functions are linear in the Hungarian case.)

The yield redistribution, i.e. the x_i-s and y_j-s should be determined in such a way that

$$\underline{r}_i \leq r_l, \quad (*)$$

since the smooth yield rate principle does not promise more than r_l, and

$$\underline{r}_j \geq r_u, \quad (*)$$

since the smooth yield rate principle wants to guarantee at least r_u. Furthermore, in the yield redistribution process one has to take into account the size of the yield adjustment reserve. This gives the constraints

$$R-\Sigma_i x_i + \Sigma_j y_j \geq 0 \quad (*)$$

and

$$R-\Sigma_i x_i + \Sigma_j y_j \leq 0.05*(\Sigma E_i + \Sigma_i x_i + \Sigma G_j - \Sigma_j y_j$$

+the balance of those personal accounts which do not change in the yield redistribution). (*)

The meaning of the first constraint is obvious. The second one comes from the law. At the yield redistribution the yield adjustment reserve should not go above a limit of 5% of the sum of the balances of the personal accounts.

One has to make the yield redistribution, i.e. to determine the x_i-s and y_j-s in such a way that the new $d(\ldots\underline{r}_i \ldots\underline{r}_j\ldots)$ should be less than $d(\ldots r_i \ldots r_j\ldots)$. To minimise a given $d(\ldots\underline{r}_i \ldots\underline{r}_j\ldots)$ subject to the above constraints (*) is an MP problem.

The only question now is the expression for the $d(\ldots\underline{r}_i \ldots\underline{r}_j\ldots)$ objective of the MP. We are not interested at all in technical problems of solving an MP. What we are interested in is such a choice for $d(\ldots r_i + x_i \ldots r_j - y_j \ldots)$ that the result of optimisation can be considered to be a more or less fair redistribution among the personal accounts since the owners of the personal accounts are the fund's members.

Such a choice is

$$d(\ldots \underline{r}_i \ldots \underline{r}_j \ldots) = \max_{i,j} (\ldots \underline{r}_j - r_u \ldots r_l - \underline{r}_i \ldots)$$

(Let us allow so much technique that with the objective above we have an easy-to-solve LP problem.) This objective minimises the greatest distance from the interval. Individual weight factors and/or exponents associated to the $\underline{r}_j$-r_u-s and r_l-$\underline{r}_i$-s would definitely not be considered fair but to give a greater weight to the r_l-$\underline{r}_i$-s than to the $\underline{r}_j$-r_u-s may seem to be reasonable and not necessarily unfair. (I.e. a difference at the bottom of the interval is less desirable/acceptable than a difference on the top.)

One can propose to make the x_i –s (y_j -s) proportional to/dependent of the F_i-s (G_j-s). This idea really exists (existed) since in the law one can find its evidence. (It is definitely not fair to take only into account the balances calculating a yield rate. The date of the payment and the earlier balances is of importance, too.)

The recent regulation prescribes a rather complicated procedure for the administration of the transfer between the yield adjustment reserve and the personal accounts. Anyway, if we put this regulation into the above MP framework, then we have a rather strange objective. Let s_1, s_2, …be the $\underline{r}_j$-r_u-s and the r_l-$\underline{r}_i$-s. According to the recent regulation one has

$$d(\ldots \underline{r}_i \ldots \underline{r}_j \ldots) = \Sigma_k\, c_k s_k,$$

where for the weights $c_l < c_k$ if $s_l > s_k$, i.e. the greater the distance from the interval the less the weight of the distance. The recent regulation not only does result in an unnecessarily complicated yield redistribution procedure but it is also hard to be considered as fair. (There is no reason in considering a greater distance more preferable then a lesser one.)

As far as we know, there is no regulation where something is defined by the solution of an MP model. It is not an impossible idea since there are many regulations where several formulae are applied. Of course, one should be careful. In the case of MP one has to take into account the possibility of more optimal solutions, how accurately the solution is calculated, etc. (Though this last point is important in the case of formulae, too.)

The occasional differences among the personal accounts' yield rates are a serious fault in the regulation. The principle of smooth and/or constant like yield-rate over the years is questionable in itself. The real objective is as much capital on the personal accounts as possible (assuming that personal accounts can not be discriminated from each other). Of course, no one can tell how to realise this objective but the return adjustment reserve is inconsistent with this objective. Obviously this is the case if the yield rate of the investments of this reserve is greater or lesser than that of the personal accounts. In the third case (i.e. the yield rate of the personal accounts is the same as the yield rate of the yield adjustment reserve) the potential redistribution (among fund members) are the only consequence of the yield adjustment reserve's existence in the system. A further point may be the unnecessarily complicated procedure of the realisation of the redistribution.

There are two further remarks here. In a fund the (personal accounts and their investments of the) accumulating fund members are strictly separated from the (reserves of the) members getting benefit. They are the members of the same legal entity and nothing more. The only exception is the return adjustment reserve where according to the law the yield-rate of the service reserve must be taken into account, too and the necessary transfers should be made. Such transfers can cause several difficulties in the case if the technical interest rate is greater than r_u.

The system prescribes the forming of a specific demographic reserve, too. The forming of more reserves decreases the benefits and not necessarily makes the system safer. At the same time forming of further reserves will originate needless redistribution.

To be fair we have to remark that some modifications of the law and decrees regulating the pension funds have already been made but further ones are still necessary. As we have already mentioned it is correct to assure that all the personal accounts have the same yield rate. If all personal accounts' yield rate and the yield rate of the fund were equal then one can easily define a fair procedure for the redistribution among personal accounts through the yield adjustment reserve. However, the best thing would be to omit the whole subject of yield adjustment reserve (and some other reserves).

Unisex annuities

At the end of the accumulation period a fund member can have annuities of several types.

(The law contains the types of annuities that can be provided. The choice consists of

- simple life annuities,
- two or more life annuities (i.e. annuity is paid as long as at least one of the individuals is alive),
- guaranteed life annuities (i.e. the annuity is paid in all cases or even after the beneficiary's death at least until a previously agreed-on date).

On the other hand the law is rather liberal. Even in the case if the fund provides annuities the members have the right to receive the benefit from any other fund or he/she is entitled to ask his/her fund to buy the benefit from any insurance company providing such annuities. (There are some sections in the law suggesting that the provider can be changed even later, i.e. in the service period. This possibility definitely could have a selection effect. The possibility of choosing among different types of annuities also can have such effects.)

Furthermore the law states (in several forms at several places) that the benefit provided by a fund should not be dependent on the member's gender. It is still not clear what will happen in that case if the benefit is provided by an insurance company. In the (pension fund) law there is a phrase stating that the insurance company must sell to the fund special products developed for pension fund's members. Presumably the intention was to extend the unisex annuity for insurance companies, too. However, there is a law for regulating the activity of insurance companies. This law does not mention special products and these companies do not apply unisex life tables. We will discuss later further imperfect/controversial points on unisex annuities in the law.

The requirement of the application of the unisex life tables is very clear from the point that the funds are one of the pillars of the mandatory social insurance system. (Here the attribute mandatory is of importance.) But the consequence of the application of the unisex life tables is a redistribution among the fund members which is different from the redistribution caused by the different life spans of the members.

For the sake of simplicity let us always think of a simple life annuity. (If a fund provides annuities then its choice must contain the single life annuity and at least one of the other annuities affecting the beneficiaries.) Furthermore we do not deal with profit sharing or with any indexation of the annuity. (However, this is an oversimplification.) Let us denote by P_x annuity factor, i.e. the capital needed to be present now to pay 1 HUF to a member of x age. The adverb present means a fixed time point, e.g. the time of preparing this paper or the time of providing the first benefit.

The annuity factor P_x of an institution is a function of several parameters. There could not be big differences in the cost parameters at the different institutions because of the competition. Differences in the technical interest rate are bounded by the law. (For the technical interest rate an upper bound is given by the recent regulation which does not allow the funds to promise too much. The lower bound is given because of the competition.) There could not be essential differences also in the demographic parameters since almost all the funds are open funds and their members' demographic distribution are the nation wide distribution, i.e. the same for all funds or at least there is no basis to assume differences now. Of course, there are differences between the male and female members' mortality.

So the difference of the P_x-s of the several institutions can not be essential because of the competition. (Or it can not be essential to escape from the competition but this would be another story.) They should be even equal (i.e. the P_x should be fund independent) since everybody wants to and due to the free selection among providers can get his/her annuity from that fund or insurance company where he /she gets the most.

Let us try to find out what an actuary of a fund can do with the recent prescriptions of the law. We do not want to be lost in the precise translation of the complicated legal sections but the main rules can be summarised in the following way. The actuary has to apply a unisex life table. As the law states, preparing this table he/she has to take into account both the male and the female (members') mortality. Starting from the balance of the personal account the annuity should be calculated applying the equivalence principle. Calculating the reserves the actuary has to take already into account the member's gender. Though it is not explicitly stated, the risk community consists of the members getting the same type of annuity. (In our case where we have only one simple life annuity this means all the fund members getting an annuity.) So, the actuary has to calculate a unique P_x.

Let us assume that the calculation of the annuities happens each year only once. We will use the following notations.

$C_{M,x,j}$ is the sum of the expected present value of those x age males' (females') personal account balances who will be pensioner j years later;
$C_{F,x,j}$ is the same for female members;
$P_{M,x}$ is the annuity factor calculated only on the male members' mortality;
$P_{F,x}$ is the annuity factor calculated only on the female members' mortality.

(We assume that the cost parameters and the technical interest rate does not depend on the member's gender.)

From $C_{M,x,j}$ ($C_{F,x,j}$) the fund has to provide an annuity of present value $C_{M,x,j}/P_x$ ($C_{F,x,j}/P_x$) and because of the statements of the law one has

$$\Sigma_j \Sigma_x C_{M,x.j} + \Sigma_j \Sigma_x C_{F,x,j} = \Sigma_j \Sigma_x (C_{M,x,j} / P_x)*P_{M,x} + \Sigma_j \Sigma_x (C_{F,x,j} / P_x)*P_{F,x}, \qquad (*)$$

(Since the right hand side contains $P_{M,x}/ P_x$ and $P_{F,x}/ P_x$ the time point of the present does not play any role.)

The system (*) has several solutions. E.g it is easy to see that

$$P_x = (\Sigma_j C_{M,x,j}*P_{M,x} + \Sigma_j C_{F,x,j}*P_{F,x}) / (\Sigma_j C_{M,x,j} + \Sigma_j C_{F,x,j}). \qquad (**)$$

is a solution. Or (*) has a (unique) solution of the form

$$1/P_{x+k} = \lambda*1/P_{M,x+k} + (1-\lambda)*1/P_{F,x+k} \quad (k = 0,1,2\ldots) \quad (***)$$

with $0 \leq \lambda \leq 1$. In fact, λ can be given explicitly from (*). The point is that solution (***) is not the expected(?) solution corresponding to the (weighted) arithmetic average of the $P_{M,x+j}$-s and the $P_{F,x+j}$-s (i.e. of the male and female mortality). We said expected referring to the folklore and/or opinions at the time of introducing the Hungarian fund system.

It is hard to believe that either the values in (**) or in (***) are fund independent. (In (***) the λ depends on the C-s.)

But the real problem with the system (*) is that the actuary does not know exactly the $C_{M,x,j}$-s ($C_{F,x,j}$-s). He/she can have only their very uncertain estimate. The uncertainty comes not only from the problem of estimating e.g. future contributions and their present value or the age distribution of the retiring people. The main point is that as a consequence of the results of the above calculations several moves of the members (because of the free choice of provider) may happen. Due to the moves the $C_{M,x,j}$-s and the $C_{F,x,j}$-s and so the P_x-s can change. (Both in the fund left and in the fund chosen and theoretically due to the changes e.g. the fund chosen can already be not so attractive.) In principle it is possible that to escape from the future uncertainties (or at least from some of them) the actuary takes only into account the C-s of the new retirees and each year calculates a new P_x from equation (**). Though such new calculations of the annuity factor are not against the law (it must be approved by the Supervisory Authority) but this can result in too extreme changes in the (starting) annuities. Furthermore the fund independence of P_x are still not solved.

Let us remark that all these problems are not because in (**) the weights of $P_{M,x}$ and $P_{F,x}$ are the sum of the capitals $C_{M,x.j}$-s and $C_{F,x,j}$-s. If the weights were e.g. the (sum of the) number(s) of male and female members then the same problems would arise: the values (**) would not be found independent. (A statistical analysis is going on the (recent) age distribution of the number of male and female members in different funds and on the distribution of capital accumulated by the different cohort. Our arguments are more or less independent of the results of this analysis, though the results can have some consequences on making clearer the statements of the law.)

One possible summary of the above is that the actuary can have a hard time with the recent form of the law.

The problem of unisex annuities does not exist if all the insurance companies may provide annuities to the fund members applying the usual sex dependent life tables. (In this case the funds for male members buy annuities from the insurance companies and all female members get their annuity from a fund or from an insurance company. Then the unisex annuity factor P_x of the fund is simply P_{Fx}. Of course, this solution is against the non-discriminative principle of social insurance. (As we have already mentioned the Hungarian funds are a part of the mandatory social insurance system.)

If there is only one annuity provider (in what follows, one fund) then the case is somewhat better. The difference between the earlier case and the recent one for the fund's actuary is that now the $C_{M,x,j}$-s and $C_{F,x,j}$-s can be considered as given. Or at least they can be much better estimated than in the earlier case since there is only one fund and so the whole population forms the members of the (only) fund. Furthermore the moves resulted earlier may not be considered.

In this case the calculation of a new annuity factor P_x in each year can be done through solving an MP. (We use the same assumptions and simplifications as earlier.) Let us assume that the fund applies the annuity factor $\underline{P}_x$ and let us denote by $L(P_x)$ the present value of the fund's liabilities if the annuity factor is P_x. Now the actuary has the possibility to determine P_x in such a way that an l_p (e.g. l_∞ or l_2) distance of the P_{x+j} and $\underline{P}_{x+j}$-s is minimal and the constraints

$$A + \Sigma_x C_{M,x} + \Sigma_x C_{F,x} \leq L(\underline{P}_x) + \Sigma_x (C_{M,x} / P_x)*P_{M,x} + \Sigma_x (C_{F,x} / P_x)*P_{F,x}$$
$$1/P_{F,x+k} \leq 1/P_{x+k} \leq 1/P_{M,x+k} \quad (k = 0,1,2\ldots)$$

are satisfied where A is the assets of the fund for fulfilling the already existing liabilities $L(\underline{P}_x)$. I.e. the new annuity factor takes into account the new retirees ($C_{M,x}=C_{M,x,0}$, $C_{F,x}=C_{F,x,0}$) but it is not too far from the old one which was the basis of calculating the existing liabilities.

Depending on the regulation of indexation and profit sharing the solution of the MP

minimise an l_p (e.g. l_∞ or l_2) distance of the P_{x+j} and $\underline{P}_{x+j}$-s

subject to

$$A + \Sigma_x C_{M,x} + \Sigma_x C_{F,x} \leq L(P_x) + \Sigma_x (C_{M,x} / P_x)*P_{M,x} + \Sigma_x (C_{F,x} / P_x)*P_{F,x}$$
$$1/P_{F,x+k} \leq 1/P_{x+k} \leq 1/P_{M,x+k} \quad (k = 0,1,2\ldots)$$

can be considered, too. I.e. not only the new annuities are calculated by using the new annuity factor P_x but the existing annuities are also modified.

Another possibility is to solve the MP

$$A + \Sigma_x C_{M,x.j} + \Sigma_x C_{F,x,j} - L(P_x) - \Sigma_x (C_{M,x,j} / P_x)*P_{M,x} - \Sigma_x (C_{F,x,j} / P_x)*P_{F,x} = U$$

$$1/P_{F,x+k} \leq 1/P_{x+k} \leq 1/P_{M,x+k} \quad (k = 0,1,2\ldots)$$

an l_p (e.g. l_∞ or l_2) distance of the P_{x+j} and $\underline{P}_{x+j}$-s is not greater than a small value

$$U \geq 0$$

$$\min U$$

(In fact, we have two possibilities here, since one can also have $L(\underline{P}_x)$ instead of $L(P_x)$.)

We have still not mentioned the problem of managing the excess and/or shortage of money. This is not a question in the case of an annuity product of an insurance company though they do not apply unisex life tables. If the mortality parameters are wrong then the stockholders gain or loose. But this is not the case at a fund. On one hand, here the owners are the fund members and the wrong P_x must be compensated by (further) redistribution. Since the funds are elements of the social insurance system not only to supply the shortage is problematical but at least as questionable is the distribution of the excess money. On the other hand, the several redistributions caused are very much against the spirit of the law. This is particularly the case if the redistribution are not transparent.

The U is the excess of the available assets over the (present) values of the guaranteed annuities. To minimise U means that we want to give as much as possible to the pensioners.

The next possibilities are to consider the MP

$$A + \Sigma_x C_{M,x.j} + \Sigma_x C_{F,x,j} - L(P_x) - \Sigma_x (C_{M,x,j} / P_x)*P_{M,x} - \Sigma_x (C_{F,x,j} / P_x)*P_{F,x} = U - V$$

$$1/P_{F,x+k} \leq 1/P_{x+k} \leq 1/P_{M,x+k} \quad (k = 0,1,2\ldots)$$

an l_p (e.g. l_∞ or l_2) distance of the P_{x+j} and $\underline{P}_{x+j}$-s is not greater than a small value

$$U, V \geq 0$$

$$\min (U + V)$$

(As in the last case, we have again two possibilities here.)

The V is the shortage of the available assets below the (present) values of the guaranteed annuities. (U is the same as in the last case.) Obviously, the objective requires that either U or V should be minimal.

To minimise V means that because of the application of the unisex P_x-s we want to redistribute as little as possible, i.e. the application of the unisex P_x-s should be as cheap as possible. Of course, these cases raise the question if who pays the V.

The idea of one fund (which is not necessarily a government institution) does not mean that we would like to suggest to modify the system of funds dealing with the personal accounts in the accumulation period. The recent funds probably do not give readily up the managing of the reserve in the benefit period though they would get rid of a lot of risks (and they would not probably be regulated with a distinctive rigour in profit sharing.) We would also like to emphasise that the application of the above MP models for solving the problem of unisex annuities is only one possible idea. In a given application the constraints and the objectives of the above MP models may undergo radical changes. Only our attitude and objective is sure. To make the system as transparent as possible.

Similar MP model(s) can be used (in a similar sense) to choose a starting $\underline{\boldsymbol{P}}_x$ from the solution of the system (*).

E.g. a "safe" $\underline{\boldsymbol{P}}_x$ can be chosen by solving the MP

$$\Sigma_j \Sigma_x C_{M,x,j} + \Sigma_j \Sigma_x C_{F,x,j} = \Sigma_j \Sigma_x (C_{M,x,j} / \underline{\boldsymbol{P}}_x)*P_{M,x} + \Sigma_j \Sigma_x (C_{F,x,j} / \underline{\boldsymbol{P}}_x)*P_{F,x}$$

$$1/P_{F,x+k} \leq 1/P_{x+k} \leq 1/P_{M,x+k} \quad (k = 0,1,2\ldots)$$

$$\text{maximise the } l_p \text{ (e.g. } l_\infty \text{ or } l_2\text{) norm of } \underline{\boldsymbol{P}}_{x+j}$$