"Application of Optimal Control Theory to actuarial gains and losses of a private retirement pension plan"

Octavio Maupomé-Carvantes, Act. & M.Sc.
México

Summary

In a private retirement pension plan, one of the tasks of the actuary is to monitor that gains and losses behave within acceptable limits. The methodology for controlling gains and losses hereby presented takes some ideas proposed by the IASC and the IFAA and merge them into an optimization model. The model assumes that gains and losses [1] are stochastic processes with particular statistical properties, [2] not necessarily converge to zero nor gains are offset with losses, as is usually assumed, and [3] are controllable.

Current pension GAAP recommend the application of amortization of gains and losses as part of the Net Periodic Pension Cost. Based on the Optimal Control Theory, here we suggest the adoption of a Second Corridor such that all gains and losses above an optimal limit u must be recognized completely and immediately in the year, instead of being amortized in linear form. The Second Corridor is a mechanism to control gains and losses according to their magnitude and other relevant characteristics of each plan. This Second Corridor could also avoid the selection of deliberately biased assumptions and represents a way to avoid reductions in equity.

"Aplicación de la Teoría del Control Óptimo a las pérdidas y ganancias actuariales de un plan privado de pensiones"

Octavio Maupomé-Carvantes, Act. & M.Sc.
México

Resumen

En un plan privado de pensiones, una de las tareas del Actuario es monitorear que las ganancias y pérdidas se comporten dentro de límites aceptables. La metodología para controlar las ganancias y pérdidas aquí presentada, toma algunas ideas propuestas por el Comité Internacional de Estándares Contables y la Federación Internacional de Asociaciones Actuariales, mezclándolas en un modelo de optimización. El modelo asume que las ganancias y pérdidas [1] son procesos estocásticos con características estadísticas particulares, [2] que no necesariamente convergen a cero ni que las ganancias eliminan las pérdidas, como se asume usualmente, y [3] que son controlables.

Los principios contables generalmente aceptados actualmente recomiendan la aplicación de una amortización de las ganancias y pérdidas como parte del Costo Neto del Período. Basados en la Teoría del Control Óptimo, aquí sugerimos la adopción de un Segundo Corredor tal que todas las ganancias y pérdidas por encima de un límite óptimo u deben ser reconocidas completa e inmediatamente en el año, en vez de ser amortizadas en forma lineal. El Segundo Corredor es un mecanismo para controlar ganancias y pérdidas de acuerdo a su magnitud y otras características relevantes de cada plan. Este Segundo Corredor puede también evitar la selección de supuestos actuariales sesgados deliberadamente y representa una manera de evitar reducciones al Capital.

Application of Optimal Control Theory to actuarial gains and losses of a private retirement pension plan.

by Octavio Maupomé-Carvantes, Act. & M.Sc.

1. Introduction

In a private employee benefit plan, such as pension plans, the duties of the plan actuary consist, essentially, of [1] assessing the monetary value of liabilities, [2] determining the amortization or financing strategy (and thus the plan's year cost) and [3] monitoring that the plan behaves within expected limits.

Actuarial gains and losses (referred to as G/L from now on), are the measure of the difference between expected values of certain variables of the plan and their actual values. Those G/L are produced by the deviations in demographic and economic assumptions.

In accordance with current pension accounting methods,[1] in each actuarial valuation G/L must be computed so as to consider its amortization as part of the net periodic pension expense. That amortization considers that if G/L do not surpass a given amount, it is not included into the cost. Such amount is calculated by means of a Corridor,[2] which is a range of 10% of the greater value of liabilities (Projected Benefit Obligation, PBO) and assets of the plan. The G/L in excess of the Corridor is amortized in linear form during the average future working lifetime (AFWL) of participants; the portion below the Corridor is to be deferred.

Due to the close relationship of actuarial and accounting professions in this field, in many countries is now common that accounting principles consider and incorporate the opinion of actuaries. The International Accounting Standards Committee (IASC) suggested, in 1997, to recognize in an immediate manner those G/L that surpass the current 10% limit and G/L below the corridor would

[1] e.g., IAS-19 and FAS-87
[2] it should be noted, however, that the use of this "first"Corridor is optional to the plan sponsor at the moment of adoption of FAS-87. Once adopted, it must be used all subsequent years.

be deferred. The International Forum of Actuarial Associations (IFAA) submitted its opinion regarding Exposure Draft E54 document. Among other suggestions, the IFAA proposed the application of a 25% corridor instead of the current 10%, with the intention of amortizing during the corresponding AFWL those G/L below that 25% corridor and to recognize immediately those G/L above it.

IASC now permits systematic methods of faster recognition, provided that the same basis is applied to both gains and losses and the basis is applied consistently from period to period. Such permitted methods include immediate recognition of all actuarial gains and losses.[3]

On the other hand, in unstable economies [Flores, 1996] the selection of actuarial assumptions must consider additional issues, since deviations, both demographic as well as economic, are inherently larger [Maupomé-Carvantes, 1997] and G/L might have a trend towards larger magnitudes. Among other considerations, this is because G/L not necessarily converge to zero nor gains are offset with losses, as is usually assumed.

Therefore, our proposed methodology intends to answer the question of how to deal with the inevitable difference between expected and actual results [Anderson, 1992] when those differences become extreme and impose a threat to the plan's financial stability.

2. Nature of actuarial gains and losses

According to pension accounting principles, G/L are defined as the sum of

- difference of expected PBO and actual PBO
- difference of actual return and expected return on assets
- difference of expected and actual benefit disbursements
- difference of expected book reserve and actual book reserve
- other differences from what was expected to actual experience.

[3] see IAS 19 (revised in 2000), paragraphs 93 and 95

Confusion should not exist between the actuarial deficit and G/L, since the first is determined as the difference of plan assets and PBO. Furthermore, the actuarial deficit is the liability pending to be financed, and G/L are the sum of differences of expected vs. actual values in certain variables. In fact, G/L become a part of the actuarial deficit.

Since G/L are the monetary value of the "deviations", and in occasions are the result of decisions made by plan administrators, its annual value must be faced by the plan sponsor. Whether gains or losses, it is crucial for the plan sponsor to consider such value into the plan accounting, particularly if the calculated G/L value is large compared to PBO or Plan Assets.

Although it is quite important to revise actuarial assumptions so as to avoid large G/L, there exist unanticipated events that produce them (e.g., a large sudden change in return on assets rate). Therefore, in the dynamics of G/L in a private pension plan, the problem of gains and losses offsetting each other through time must be analyzed to see if their existence implies the need to incorporate a control [Patiño, 1997]. Assuming G/L as stochastic processes, and even if they have a mean equal to zero, it is not reasonable to expect that they will be eliminated by themselves. Thus, they must be controlled.

Assuming that the plan exists indefinitely, the behavior of G/L is a random process that it is assumed stable. G/L are not independent of each other and have the properties of being [1] non correlated, [2] bounded and [3] ergodic. Irrespective of the plan's evolution it is not reasonable to think that G/L will converge to zero and maintain that value.[4] Dufresne [1993] shows that it is not feasible to assume that the sum of G/L, considering or not the interest rate, will tend to zero. This is due to the fact that, if actuarial assumptions are the best estimates of the environment of the plan, i.e., they are not willingly biased, it cannot be expected that G/L will correct themselves. Thus, there exist the need for a control on its dynamics.

As mentioned above, the IASC suggested in 1997 to adopt a criterion for immediate recognition when G/L are larger than the current 10% corridor and to defer G/L when they are below such

[4] the only possibility that it might occur is that the plan would disappear

corridor. The IFAA, on the other hand, suggests the corridor be increased to 25%, and either defer G/L below that limit or recognize immediately G/L above. We believe that these two ideas can be merged into a new one, that would take the best of both. Furthermore, our proposed method is aligned with current IAS-19 permitted methods.

Our proposal is to keep the current 10% Corridor methodology and define a Second Corridor, that is a point from which we must recognize immediately all G/L above. G/L between the first and second corridors will be amortized in linear form during the AFWL.

3. Application of Optimal Control Theory to a private retirement pension plan

In a concise manner, optimal control[5] establishes that for initial conditions (t^0 , x^0), the problem is to find a control function U, such that it maximizes the objective function defined as:

$$\varphi(u) = \int_t^T [\, f_0(\, x(t:u), u(t), t\,)\,]\, dt + g(\, x(T:u), T\,) \qquad [\text{ Eq.1 }]$$

For our problem, i.e., to find a control for the G/L of a pension plan, we need to determine a level from which those G/L must be immediately recognized. Thus, our objective is to calculate a Second Corridor such that we can optimize the recognition of G/L and control them.

Such Second Corridor can be calculated as a function of each plan's particular characteristics, based on the actuarial assumptions, economic environment and, obviously, the amount of the G/L.

3.1 Suggested model

$$c(u) = \frac{u}{I + S\,u^3} \qquad [\text{ Eq. 2 }]$$

where

C is the optimal corridor
u is the control variable, that represents the optimal recognition level
I is the amortization calculated according to the first corridor
S are the G/L in excess of I

The plan needs to maintain a minimum level of amortization of G/L and a recognition level that optimizes its efforts to keep P/G within an optimal level. Applying the optimal control theory to the model above, we want to find:

$$max \frac{u}{I + S u^3} \qquad [\text{ Eq. } 3]$$

If we differentiate with respect to U and equate to zero, we obtain

$$u'(I + S u^3) - (I + S u^3)'(u) = 0 \qquad [\text{ Eq. } 4]$$

and then we have

$$I - 2 S u^3 = 0 \qquad [\text{ Eq. } 5]$$

and finally we get

$$\boxed{u = [\frac{I}{2 S}]^{\frac{1}{3}} \qquad [\text{ Eq. } 6]}$$

If the value of AFWL implies long periods for G/L to be amortized, according to the first corridor method, thus we need an additional recognition so as to control G/L. On the other hand, for small G/L and/or short AFWL, the use of the control is unnecessary since the current first corridor methodology will produce enough amortization amounts.

3.1 **Application of the model**

The application of the first and second corridors is as follows:

- If G/L are below the 10% corridor, there is no amortization nor immediate recognition,
- If G/L are between the first and second corridors, G/L are linearly amortized over the AFWL,

---

[5] see Appendix B for a more detailed description

- If G/L are above the second corridor, there are both an amortization and an immediate recognition: the amortization is based on the G/L between both corridors; the immediate recognition is the G/L in excess of the second corridor.

Then:

$$R = abs(G/L) - u * max(PBO, PA) \qquad [\text{ Eq. 7 }]$$

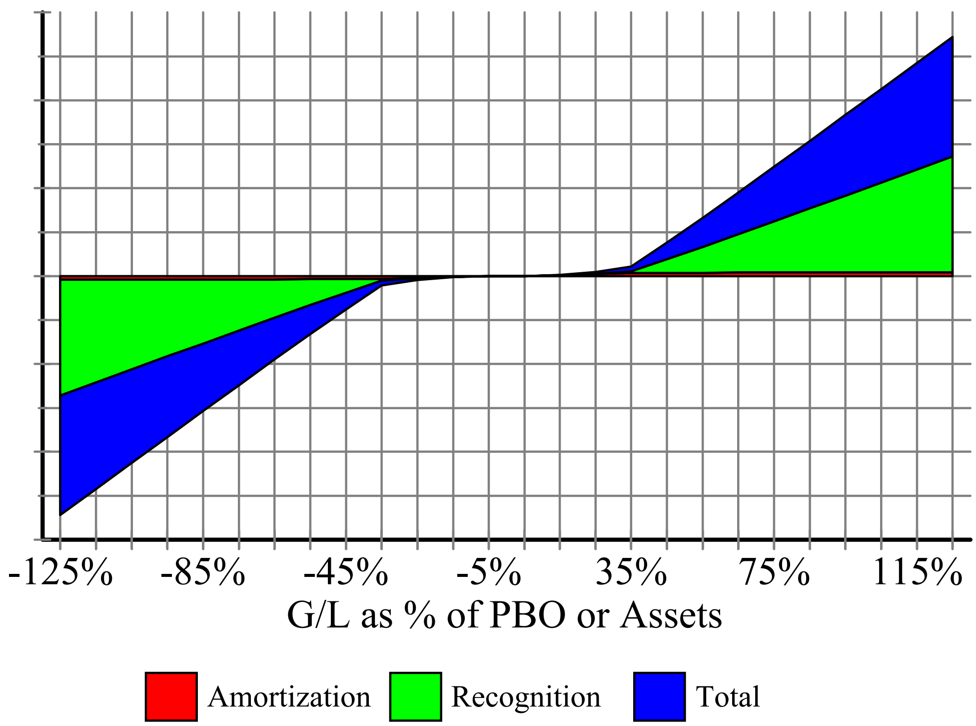
where:
abs is absolute value
G/L are the gains and losses
u is the control variable
PBO is the amount of plan liabilities (Projected Benefit Obligation)
PA is the amount of plan assets
R is the optimal recognition, according to the methodology of the second corridor.

It is important to mention that the recognition as well as the amortization have the same sign (both are positive or both are negative). For different levels of G/L in terms of PBO or plan assets, the first and second corridors as well as the amortization and recognition are shown in Table 1 below.

Table 1. Application of Second Corridor to gains and losses of a theoretical pension plan.

G/L AS % OF PBO / ASSETS	G/L	AMORTIZATION FIRST CORRIDOR	AMORTIZATION [A]	U	SECOND CORRIDOR	RECOGNITION [R]	A + R
5%	750	0	0	0.00%	0	0	0
15%	2,250	75	75	25.83%	3,875	0	75
25%	3,750	225	225	31.72%	4,758	0	225
35%	5,250	375	356	33.76%	5,063	187	543
45%	6,750	525	372	34.81%	5,221	1,529	1,901
55%	8,250	675	382	35.45%	5,318	2,932	3,314
65%	9,750	825	388	35.89%	5,383	4,367	4,755
75%	11,250	975	393	36.20%	5,430	5,820	6,213
85%	12,750	1,125	397	36.44%	5,466	7,284	7,681
95%	14,250	1,275	399	36.63%	5,494	8,756	9,155
105%	15,750	1,425	402	36.78%	5,516	10,234	10,635
115%	17,250	1,575	403	36.90%	5,535	11,715	12,119
125%	18,750	1,725	405	37.00%	5,550	13,200	13,605
AFWL = 10; PBO = $15,000; PA = $3,800							

As can be observed in Table 1 as well as in Graph 1 (below) the application of the Second Corridor does not affect the current provision of the 10% corridor. This is due to the fact that the second corridor is only applied when G/L have a significant amount and so we need to control them. For small G/L, the value of variable u is such that it results unnecessary any immediate recognition.



Graph 1

The u value can be easily computed for each plan, using formulae above. However, if simplicity is wanted, a fixed Second Corridor value can be assumed. For that purpose we computed the Second Corridor for different AFWL values (from 1 to 50 years) and assuming that G/L ranges from 5% to 1105% of PBO or Assets. For each AFWL value, the maximum of u is shown (see Table 2 below). The average of all u values is 38.04%; disregarding extreme values, we arrive to a fixed value of 31%.

Table 2. Computation of *u* maximum values for different average future working lifetime periods.

AFWL	*U* (%)	AFWL	*U* (%)
1	379.72	26	27.06
2	78.89	27	26.71
3	62.71	28	26.37
4	54.81	29	26.06
5	49.81	30	25.75
6	46.25	31	25.46
7	43.52	32	25.19
8	41.35	33	24.92
9	39.55	34	24.67
10	38.03	35	24.42
11	36.72	36	24.19
12	35.57	37	23.96
13	34.55	38	23.74
14	33.65	39	23.53
15	32.82	40	23.33
16	32.08	41	23.14
17	31.40	42	22.95
18	30.77	43	22.76
19	30.19	44	22.59
20	29.65	45	22.41
21	29.15	46	22.25
22	28.68	47	22.08
23	28.24	48	21.93
24	27.82	49	21.77
25	27.43	50	21.62

Whereas the first Corridor can be 0% or 10% when applying FAS87 for the first time, we suggest that, at the time of first year of adoption of this Second Corridor approach, the plan sponsor may decide to use a variable *u* value computed by the plan's Actuary each year, or a fixed value of *u* equal to 31% all years.

Using the Second Corridor method would imply larger costs for the plan sponsor, since part of the losses would be now recognized in addition to an amortization. However, such additional expense is the result of an inappropriate selection of actuarial assumptions and/or that the plan is being affected by unexpected events; in either case the plan sponsor must face the costs. However, in the case of gains this method would reduce the net periodic pension cost.

4. Conclusions

As is known to pension actuaries, the existence of G/L as part of the plan is inherent in the nature of actuarial cost methods, the demographic environment that affects the group of participants and the economic conditions that affect both the plan and its sponsor. Additionally, when the economy of the host country does not allow to forecast the future behavior of the financial assumptions with an acceptable degree of accuracy, G/L can grow as large as to be considerable in regard to liabilities and/or assets.

If we assume that it is not reasonable to think that gains and losses will offset each other with the passage of time, but to the contrary, they may diverge, it is then necessary to apply a control to keep them within appropriate levels. In that sense, the optimal control theory allow us to establish a simple manner to define a Second Corridor. Either case by case, based on the particulars of the plan, or using a fixed value for u, such Second Corridor can easily be computed.

Also, our Second Corridor approach does not interfere with the application of the first one as established in current GAAP and it also merges the ideas of the IASC and the IFAA. We believe that the adoption of this Second Corridor will benefit plan sponsors despite that they might face larger annual costs in certain years, because the plan's figures will reflect more precisely its status and then disclosed amounts will be more accurate and reliable.

Adopting this methodology of a Second Corridor has two advantages: (1) it avoids the selection of biased actuarial assumptions, since an important deviation translates into an immediate amount to be recognized, and (2) it avoids the reduction in equity, since the relation of unfunded Accrued Benefit Obligation and Accrued Pension Expense changes and so does the relation of Additional Liability and Intangible Asset.

As a final remark, our proposal of the Second Corridor would help the accounting and actuarial professions to reach a consensual approach on one area where their expertise converge.

References

Anderson, Arthur W.(1992); *Pension mathematics for actuaries*; Actex Publications; USA

Asociación Mexicana de Actuarios Consultores (1994); *Principios de Práctica Actuarial, Guía Actuarial Número 1, Valuación Actuarial de Obligaciones Laborales (Boletín D3),* AMAC, Mexico

Astrom, Carl J. (1970); *Introduction to control theory*; Academic Press; USA

Dufresne, Daniel (1994); *Mathematics des caisses de retraite*; Editions Supremum; Canada

Financial Accounting Standards Board (1985); *Employer's Accounting for Pensions (FAS87)*; FASB; USA

Flores-Quiroz, Marcela (1996); "The selection of actuarial assumptions in an unstable economy: Mexico, a case history"; *International Association of Consulting Actuaries (Transactions of the 15th. Conference)*; Scotland

Instituto Mexicano de Contadores Públicos (1997); *Boletín D3, Obligaciones Laborales*, IMCP; Mexico

International Forum of Actuarial Associations (1997); "Comments on International Accounting Standards Committee Exposure Draft E54"; FIAA; England

Maupomé-Carvantes, Octavio (1997); *Pension accounting in Mexico*; Society of Actuaries; USA

Mesterton-Gibbons, Michael (1989); *A concrete approach to mathematical modelling*; Addison-Wesley; USA

Patiño García, Maurilio (1998); "La dinámica de los fondos de retiro: un problema de control estocástico"; *thesis*; Universidad Anáhuac; Mexico

Appendix A. Study case

Using real figures of a real company, whose name is withheld for confidentiality reasons, hereby we present the results obtained should we have used the Second Corridor methodology. The Company had the following results on its Plan as at 12/31/96.

PBO	987,164	Net Periodic Pension Expense FY 1997	
Plan Assets	0	Service Cost	110,929
Funded Status	(987,164)	Interest Cost	85,965
Transition Liability	179,202	Return on Assets	0
Prior Service	0	Amortization of:	
Gains and Losses	624,746	Transition Liability	22,844
(Accrued) Prepaid Pension Expense	(183,216)	Prior Service	0
ABO	490,300	Gains/Losses	44,019
Plan Assets	0	Total	263,757
Unfunded ABO	(490,300)		
Additional Liability	307,084	Expected benefit payments	64,000
Intangible Asset	179,202		
Reduction in Equity	127,882		
AFWL	11.95		

Applying the proposed methodology we have to calculate the optimal corridor, the immediate recognition and the new G/L amortization. Thus:

I = [G/L - 10% max (PBP,PA)] / AFWL = [624,746 - 98,716.4] / 11.95 = 44,019
S = G/L - I = 624,746 - 44,019 = 580,727
u = 33.59%   Second corridor = 33.59% of 987,164 = 331,588
Recognition = G/L - Second Corridor = 624,746 - 331,588 = 293,158

The G/L Amortization is modified accordingly: A = [331,588 - 98,716] / 11.95 = 19,487

Then we have:

PBO	987,164	Net Periodic Pension Expense FY 1997	
Plan Assets	0	Service Cost	110,929
Funded Status	(987,164)	Interest Cost	85,965
Transition Liability	179,202	Return on Assets	0
Prior Service	0	Amortization of:	
Gains and Losses	331,588	Transition Liability	22,844
(Accrued) Prepaid Pension Expense	(476,374)	Prior Service	0
ABO	490,300	Gains/Losses	19,487
Plan Assets	0	Total	239,225
Unfunded ABO	(490,300)		
Additional Liability	13,926	Expected benefit payments	64,000
Intangible Asset	13,926		
Reduction in Equity	0		
AFWL	11.95		

As can be observed, using the Second Corridor methodology:

- we eliminated the Reduction in Equity,
- we reduced the 1997 Net Periodic Pension Expense (from $263,757 to $239,225),
- we recognized during FY 1996 an amount equivalent to $293,158, and
- G/L were reduced (controlled) from $624,746 to $331,588

Appendix B. Optimal Control Theory (summary)

Consider a system the state of which can be described by an n-dimensional vector

$$x = (x_1 , \dots , x_n)$$

[B - 1]

For any time *t*, let

$$u(t) = (u_1(t) , \dots , u_m(t))$$

[B - 2]

be an m-dimensional vector, called the *controls* at time *t*. Once a control function $u : t \rightarrow u\,(t)$ is chosen, the state of the system evolves through time according to the following system of differential equations:

$$\begin{gathered} d\,x_1 / d\,t = f_1(x_1 ,\dots, x_n ; u_1(t) ,\dots, u_m(t) ; t), \\ d\,x_2 / d\,t = f_2(x_1 ,\dots, x_n ; u_1(t) ,\dots, u_m(t) ; t), \\ \dots \\ d\,x_n / d\,t = f_n(x_1 ,\dots, x_n ; u_1(t) ,\dots, u_m(t) ; t) \end{gathered}$$

[B - 3]

Given an initial condition

$$x\,(t^0) = (x_1(t^0) ,\dots, x_n(t^0)) = (x_1^0 ,\dots, x_n^0) = x^0$$

[B - 4]

and a control function $u : t \rightarrow u(t)$, the trajectory of the system is completely determined by [B-3].

Because of the constraints imposed upon the system, we often require that for each time t, the value of the j-th control, namely $u_j(t)$, must belong to some set Ω such that

$$u_j(t) \in \Omega \quad \forall\, j = 1,\ldots,m\,;\, t \geq t^0$$

[B - 5]

The set of admissible controls depend on the problem under consideration. Suppose that a control function $u: t \rightarrow u(t)$ is chosen. Let

$$x(t:u) = (x_1(t:u),\ldots,x_n(t:u))$$

[B - 6]

be the solution of [B-3] under the initial solution [B-4], namely,

$$x(t^0 : u) = x^0 = (x_1^0, \ldots, x_n^0)$$

[B - 7]

Furthermore, suppose that the control function u transfers the system from the initial condition (t_0, x_0) to a point of some terminal set Φ at time T. That T is the first time $x(t:u)$ hits the set Φ. We suppose that once the trajectory $x(t:u)$ hits Φ the problem ends. Depending on the choice of the control function, the system, although beginning at the same initial condition (t_0, x_o), will hit the terminal set Φ along different paths at different terminal times T.

The payoff obtained from the system depends on the control function u, the time path of the system under u, namely $x(t:u)$, and the final point $(T, t(T:u))$. More precisely, the payoff under the control u is

$$\varphi(u) = \int_t^T [\ f_0(\ x(t:u), u(t), t\ )\ ]\ dt + g(\ x(T:u), T\ )$$

[B - 8]

In [B-8] the integral represents the payoff along the path, and *g(x (T:u) , T)* represents the terminal payoff. When the time horizon is infinite, there is no terminal payoff and [B-8] becomes:

$$\varphi(u) = \int_t^\infty f_0(\ x(t:u), u(t), t\ )\ dt$$

[B - 9]

The optimal control problem is to find a control function *u* that maximizes the objective function [B-8].