27th International Congress of Actuaries

ICA CANCUN 2002

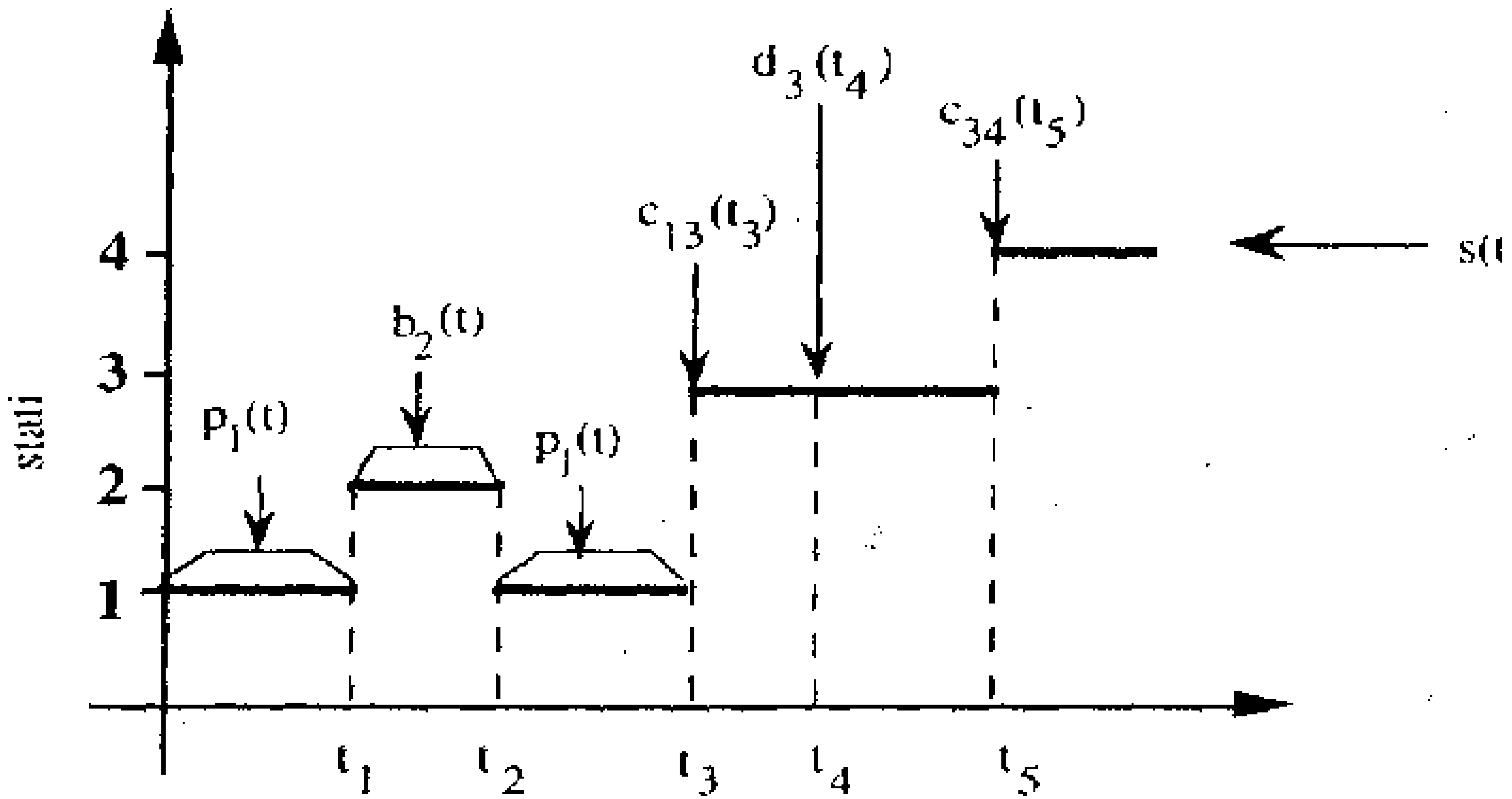
# GENERAL ACTUARIAL MODELS IN A SEMI-MARKOV ENVIRONMENT

*Jacques Janssen*, CESIAF, BELGIUM

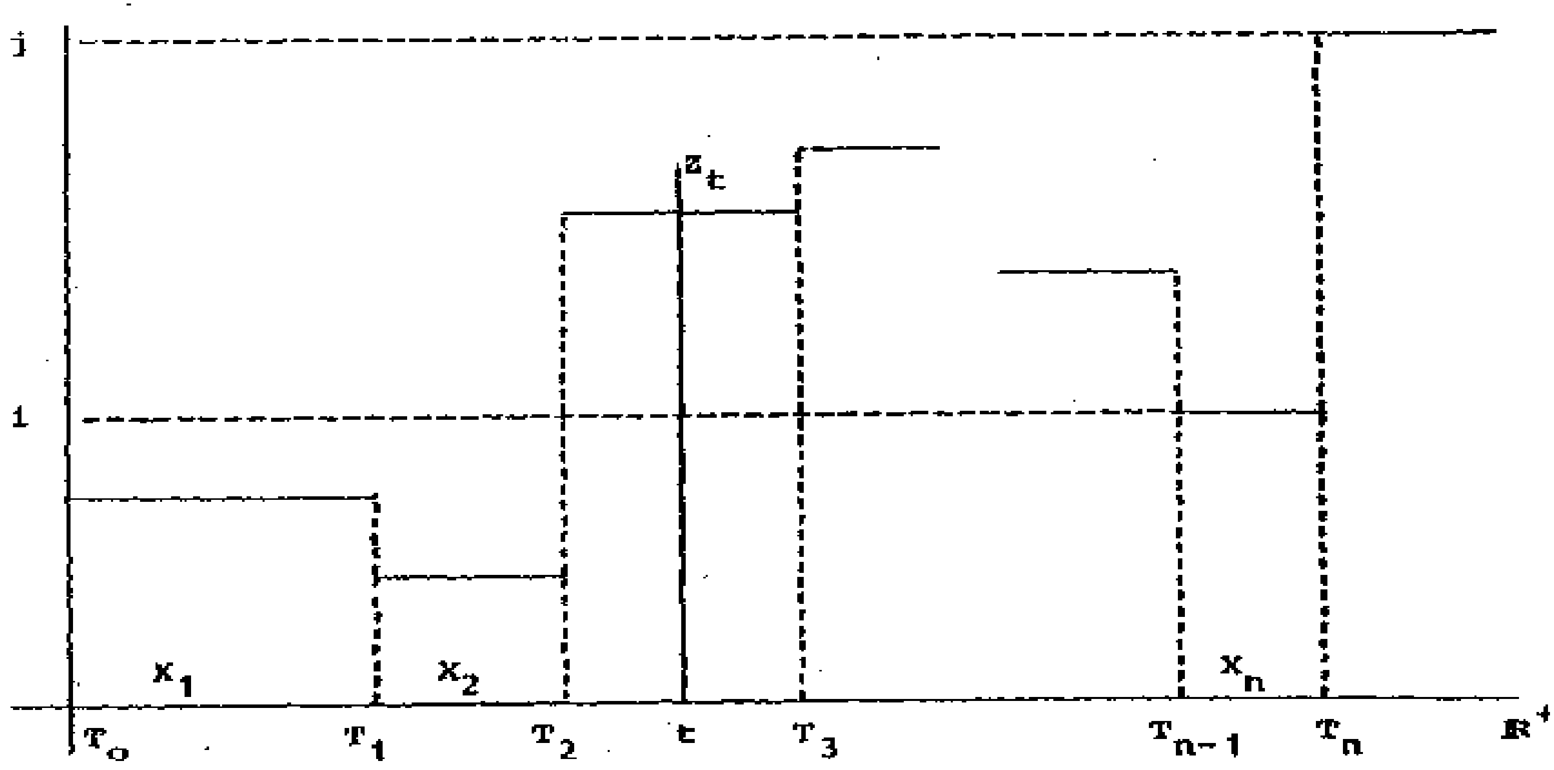
*Raimondo Manca*°, “La Sapienza”, ITALY

° Work supported by a MURST and a CNR grant.

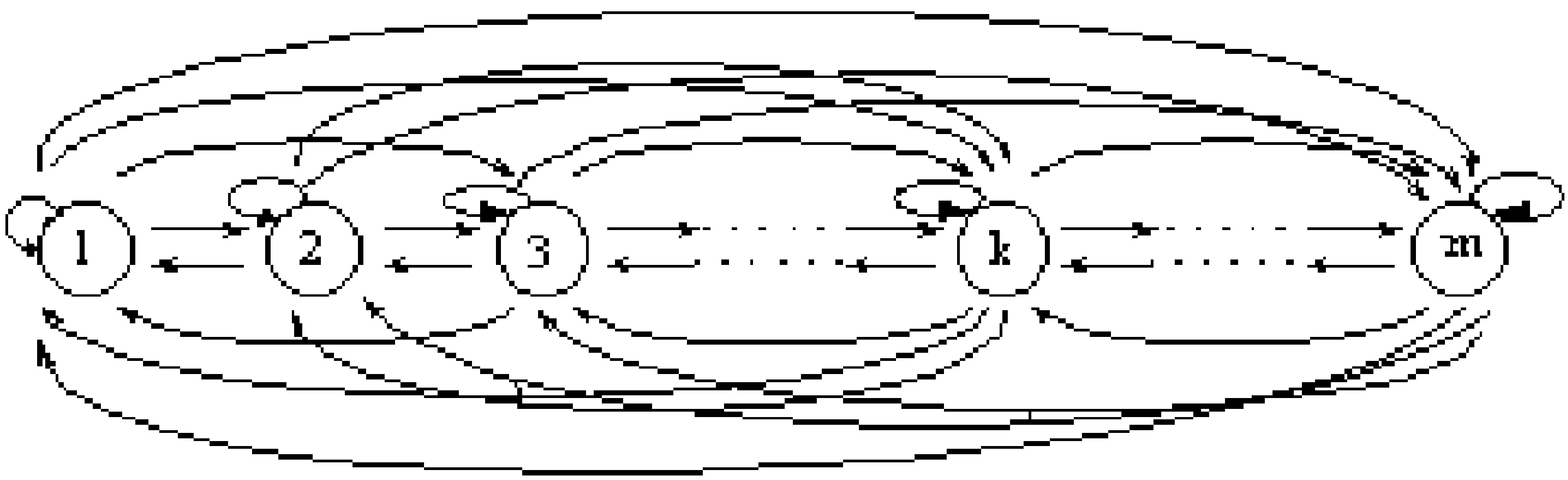
# Trajectory of a generic insurance operation



# Trajectory of a SMP



# A general actuarial model




- arc = possible transition
- weights = couple *(p,r), p* probability transition, *r* reward

# The Discrete Time non-Homogeneous SMP

- $(X_n, T_n)$ non-homogeneous markovian renewal process
- $Q_{ij}(\mathrm{s},t) = \mathrm{P}[\, X_{n+1} = j,\, T_{n+1} \le t \mid X_n = i,\, T_n = \mathrm{s}]$
- $p_{ij}(\mathrm{s}) = \lim_{t \to \infty} Q_{ij}(\mathrm{s},t)$; $i, j \in E$, $\mathrm{s}, t \in \mathrm{N}$
- $S_i(\mathrm{s},t) = \mathrm{P}[\, T_{n+1} \le t \mid X_n = i,\, T_n = \mathrm{s}]$. $S_i(\mathrm{s},t) = Q_{ij}(\mathrm{s},t)$.

# The Discrete Time non-Homogeneous SMP

$$b_{ij}(\mathrm{s},t) = \mathrm{P}[\, X_{n+1} = j,\ T_{n+1} = t \mid X_n = i,\ T_n = \mathrm{s}]$$

$$G_{ij}(\mathrm{s},t) = \mathrm{P}[\, T_{n+1} \leq t \mid X_n = i,\ X_{n+1} = j,\ T_n = \mathrm{s}].$$

$$p_{ij}(\mathrm{s},t) = \mathrm{P}[\, Z_t = j \mid \mathrm{Z_s} = i\,].$$

$$p_{ij}(s,t) = \delta_{ij}(1 - S_i(s,t)) + \sum_{j \in E} \sum_{\vartheta = 1}^{t} p_{\beta j}(\vartheta,t)\, b_{i\beta}(s,\vartheta)$$

# Reward: fixed interest rate and rewards.

$$V_i(s,t) = (1 - S_i(s,t))\psi_i \cdot a_{t-s]r} + \sum_{\beta \in E} \sum_{\vartheta = s}^{t} b_{i\beta}(s,\vartheta)\psi_i \cdot a_{\vartheta - s]r} +$$

$$\sum_{\beta \in E} \sum_{\vartheta = s}^{t} b_{i\beta}(s,\vartheta) V_\beta(\vartheta,t)(1+r)^{s-\vartheta}$$

- m.p.v. payments done from $s$ up $t$
- reward paid for $t$-$s$ periods
- rewards paid in the state $i$ up to the first transition
- payments in the states visited after first transition

# Fixed interest rate and variable rewards.

$$V_i(s,t)=(1-S_i(s,t))\sum_{\nu=s+1}^{t}\psi_i(\nu)\cdot(1+r)^{s-\nu}+$$

$$\sum_{\beta\in E}\sum_{\vartheta=s}^{t}b_{i\beta}(s,\vartheta)\sum_{\nu=s}^{\vartheta}\psi_i(\nu)\cdot(1+r)^{s-\nu}+$$

$$+\sum_{\beta\in E}\sum_{\vartheta=s}^{t}b_{i\beta}(s,\vartheta)V_\beta(\vartheta,t)(1+r)^{s-\vartheta}$$

# Variable reward and interest rate

- implied forward rates term structure $r_1, r_2, \ldots, r_t$

$$V_i(s,t) = (1 - S_i(s,t)) \sum_{\nu=s+1}^{t} \psi_i(\nu) \cdot \upsilon_{s\nu} +$$

$$\sum_{\beta \in E} \sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta) \sum_{\nu=s}^{\vartheta} \psi_i(\nu) \cdot \upsilon_{s\nu} +$$

$$\sum_{\beta \in E} \sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta) V_\beta(\vartheta,t) \cdot \upsilon_{s\vartheta}$$

# Duration reward and variable interest rate

$$V_i(s,t) = (1 - S_i(s,t)) \sum_{\nu=s+1}^{t} \psi_i(s,\nu - s) \cdot \upsilon_{s\nu} +$$

$$\sum_{\beta \in E} \sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta) \sum_{\nu=s}^{\vartheta} \psi_i(s,\nu - s) \cdot \upsilon_{s\nu}$$

$$+ \sum_{\beta \in E} \sum_{\vartheta=s}^{t} b_{i\beta}(s,\vartheta) \cdot \upsilon_{s\vartheta} \cdot \left(\gamma_{i\beta}(\vartheta) + V_\beta(\vartheta,t)\right)$$

- reward, paid or received at moment state change
- reward taking in account the duration in the state

# Stochastic interest rate structure.

- stochastic interest rates $F = \{\rho_1, \rho_2, \ldots, \rho_k\}$

$$\phi_{ij}(s,t) = \delta_{ij}(1 - S_i(s,t)) + \sum_{\beta=1}^{k} \sum_{\vartheta=1}^{t} \phi_{\beta j}(\nu,t) b_{i\beta}(s,\vartheta)$$

$$\upsilon_i(s,h) = \prod_{\mu=s+1}^{h} \left( 1 + \left( \sum_{j=1}^{k} \phi_{ij}(s,\mu) \rho_j \right) \right)^{-1}$$

$\upsilon_i(s,h)$

- mean discounting factor for a time *h*-*s*

# Duration reward, stochastic interest rate

$$V_i^{\varepsilon}(s,t) = (1 - S_i(s,t)) \sum_{\nu=s+1}^{t} \psi_i(s,\nu - s) \cdot \upsilon_{\varepsilon}(s,\nu) +$$

$$\sum_{\beta \in E} \sum_{\vartheta = s}^{t} b_{i\beta}(s,\vartheta) \sum_{\nu = s}^{\vartheta} \psi_i(s,\nu - s) \cdot \upsilon_{\varepsilon}(s,\nu) +$$

$$+ \sum_{\beta \in E} \sum_{\vartheta = s}^{t} b_{i\beta}(s,\vartheta) \cdot \upsilon_{\varepsilon}(s,\vartheta) \cdot \left( \gamma_{i\beta}(\vartheta) + V_{\beta}^{\varepsilon}(\vartheta,t) \right)$$

$$V_i(s,t) = \sum_{\varepsilon=1}^{k} V_i^{\varepsilon}(s,t) \phi_{\eta\varepsilon}(0,s)$$

# General DTNHSMRP insurance model.

- Insurance environment two time variables
- time
- age

$$
{}^{\mu}V_i^{\varepsilon}(s,t) = (1 - {}^{\mu}S_i(s,t)) \sum_{\nu=s+1}^{t} \psi_i(s,\nu - s) \cdot \upsilon_{\varepsilon}(s,\nu) +
$$

$$
\sum_{\beta \in E} \sum_{\vartheta=s}^{t} {}^{\mu}b_{i\beta}(s,\vartheta) \sum_{\nu=s}^{\vartheta} \psi_i(s,\nu - s) \cdot \upsilon_{\varepsilon}(s,\nu) +
$$

$$
+ \sum_{\beta \in E} \sum_{\vartheta=s}^{t} {}^{\mu}b_{i\beta}(s,\vartheta) \cdot \upsilon_{\varepsilon}(s,\vartheta) \cdot \left( \gamma_{i\beta}(\vartheta) + {}^{\mu+\vartheta-s}V_{\beta}^{\varepsilon}(\vartheta,t) \right)
$$

# Meaning of the formula

- (e.p.v.) at time *s* all the sums in *t-s* $\quad {}^{\mu}V^{\varepsilon}{}_{i}(s,t)$

•without transition $\quad (1-{}^{\mu}S_i(s,t))\sum_{\nu=s+1}^{t}\psi_i(s,\nu-s)\cdot\upsilon_{\varepsilon}(s,\nu)$

•Before I transition $\quad \sum_{\beta\in E}\sum_{\vartheta=s}^{t}{}^{\mu}b_{i\beta}(s,\vartheta)\sum_{\nu=s}^{\vartheta}\psi_i(s,\nu-s)\cdot\upsilon_{\varepsilon}(s,\nu)$

•After I transit. $\quad \sum_{\beta\in E}\sum_{\vartheta=s}^{t}{}^{\mu}b_{i\beta}(s,\vartheta)\cdot\upsilon_{\varepsilon}(s,\vartheta)\cdot\left(\gamma_{i\beta}(\vartheta)+{}^{\mu+\vartheta-s}V_{\beta}^{\varepsilon}(\vartheta,t)\right)$

# Data necessary to apply the model

- increasing d.f. $\quad {}^{\mu}G_{ij}(s,*)$

- non-homogeneous Markov chain $\quad {}^{\mu}\mathbf{P}(s)$

# Conclusions.

- General model can be applied to many insurance problems


A New Millennium.
A New Challenge for Actuaries