Betas calculated with Garch models provides new parameters for a Portfolio selection with an Efficient Frontier

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Abstract
This paper is a summary of the appliance of the Arch models in the selection of as best Portfolio. Assumed that the returns not follows the behavior of the classical hypothesis applied to the capital markets, I consider the application of non-linear models to the CAPM to detect the presence of the heterocedasticity in the series with the objective to obtain robust estimators for calculate the best Portfolio with maximum benefit and minimum volatility.
The model of a selection a portfolio using the excess on beta, with beta estimated with traditional econometric model and betas estimated with the intervention of Garch effects. The comparison analysis with an Efficient Frontier is presented in this paper.
Keywords: Arch, Garch, Egarch, Tarch, Portfolio, Markowitz, CAPM, APT, Sharpe, betas, Efficient Frontier, Volatility,

This paper is presented for the 2002 ASTIN Colloquium, which takes place in Cancun, Mexico, for their acceptance.

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Introduction

In a many previous papers was been analyzed the behavior of the returns of a different stocks. In a brief resume the principals considerations are the following:

1) The returns of the stocks are not independent, and don't have normal and identical distributions
2) The returns of the stocks are auto-correlated and in consequence there are an structure than can be predicted using some econometric model
3) Consider the results of a fractal model, as Hurst coefficient, it can be determine that the series of returns are persistent and in consequence the series have a memory of a process and for that reason it cannot apply the law of $T^{0.5}$ for annualize the volatility

These results have been demonstrated in R. Tagliafichi "The Garch model and the applications to the VaR" (2001), and applied the demonstrations to different markets.

The Capital Asset Pricing Model proposed by Sharpe (1964) and Linter (1965) following the suggestions of mean variance optimization in Markowitz (1952), has provided a complete theory of asset-market pricing that in a simple form predicts that the expected return on an asset above the risk free rate is proportional to the non diversifiable risk which is measured by the covariance of the asset return with a portfolio composed of all the available assets in the market. The assumptions of these models are:

1) All investors choose mean-variance efficient portfolios with one period-horizon, although they need not have identical utility functions
2) All investors have the same subjective expectations on the means, variances, and covariances of returns.

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3) The market is fully efficient in that there are no transaction costs, indivisibilities, taxes or constraints on borrowing or lending at a risk-free rate.

Empirical tests of CAPM have found that the risk premium on individual assets can be explained by variables other then the estimated covariances, in particular the own variance, the firm size and the month of January. See for example Jensen (1972), Ross (1978) Schwartz (1983) for some surveys.

This paper applies the use of non-linear models, like GARCH models, to calculate best estimators with minimum variance used in the selection of a portfolio with maximum benefit and minimum volatility. In the first section it is developed the Efficient Frontier using the matrix of variances and covariances. In the second part it is explain how it can build a Portfolio with the use of Betas. In the third part it is developed de construction of best Betas using GARCH models. In the fourth part it is demonstrated that with the same assets included in the selection of portfolio using GARCH models and not using GARCH models, produce different composition of portfolio. In the fifth part I make some conclusions about the use of the Beta modified by a GARCH effect.

I. The Efficient Frontier

Analysis of the returns and volatilities of individual assets

For the use of the efficient frontier presented by Markowitz, it must be define certain conditions and considerations to the effect to construct a Portfolio. The returns of an asset are calculated as follows:

$$R_{1_t} = Ln\left({P_t}/{P_{t-1}} \right)$$

Where R_{A_t} is the return of an asset at day *t* and P_t is the price of the asset at day *t*. The days observed between *t* and *t-1* that is considered in this paper is a week. This decision is adopted to reduce the Monday effect and take a long horizon for the prediction. The observation between *t* and *t-1* are not superposed. The mean return is the following:

$$\overline{R}_1 = \frac{\sum_{t=1}^{n} R_{1_t}}{n}$$

Where n is the quantity of observations and $\overline{R}_1$ is the mean of the returns observed, and the volatility s_1 is as follows:

$$s_1 = \frac{\sqrt{\sum_{t=1}^{n} \left(R_{1_t} - \overline{R} \right)^2}}{n}$$

The returns and the volatility of a Portfolio

If it is defined X_i like a proportion of an asset i included in the portfolio, one condition is that $\sum_{i=1}^{k} X_i = 1$ then the return of a portfolio of k assets is:

$$\overline{R}_P = \sum_{i=1}^{k} X_i \overline{R}_i$$

Applying the properties of sum variances of random variables multiplied by a constant it can be obtained the volatility of a portfolio with k assets:

$$\sigma_P = \sqrt{\sum_{i=1}^{k} X_i^2 \sigma_i^2 + \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} X_i X_j \sigma_{ij}}$$

If it is considered that the volatility is a measure of risk, the volatility of portfolio returns depends on the variances and covariances of different assets included in the portfolio. In a Markowitz portfolio the sensitivities of the assets are measured by the variances and covariances. If we remember what is a volatility, we can observe that an asset x is more volatile than y if $P(\lfloor x \rfloor > c) > P(|y| > c)$ for all c. But when occurs that we have different number of observations we must annualize the volatility following the law of $T^{0.5}$ like the equation $\sigma_{\text{anual}} = \sigma_t A^{0.5}$ where A in the quantity of t observations[1]. Now it can be compared different volatilities with different number of observations.

To calculate de portfolio variances it must be take different pairs of associated returns and we calculate de covariance as follows:

$$\alpha_{1,2} = \frac{\sum_{i=1}^{n} \left(R_{1_i} - \overline{R}_1\right)\left(R_{2_i} - \overline{R}_2\right)}{n}$$

The correlation coefficient as:

$$\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1 \sigma_2}$$

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[1] *The law of $T^{0.5}$ applied to the capital markets had been discuss by Edgar Peters in "Chaos and Order in the Capital Markets", and following the concepts of Peters in the presentation of Hurst coefficient, in my paper "The Garch models and their application to the VaR" it is demonstrated that this concept for annualized the volatility can't be applied.*

This coefficient does not need to be annualized, because it is in a standardized form, correlation always lies between +1 and –1. A negative value means that returns tend to move in opposite directions and a positive value indicate that the returns move in the same direction.

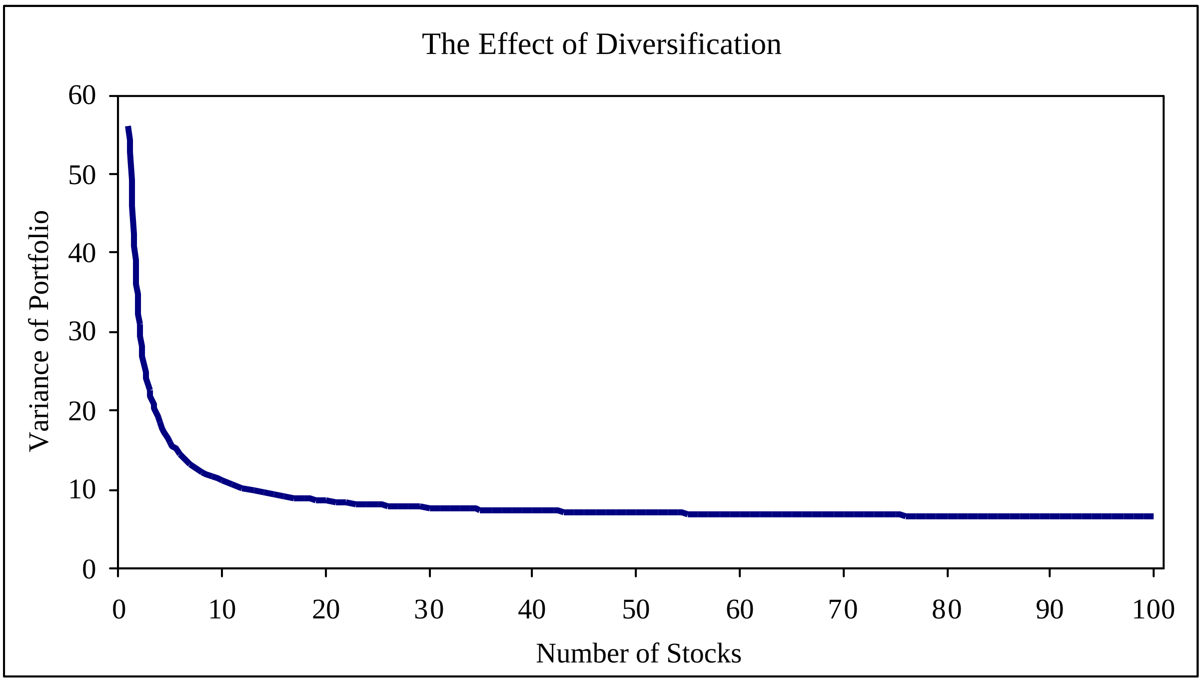
In most markets the correlation coefficient between different assets are positive and in consequence there are not independent variables for that reason $s_{1,2} \neq 0$.

The effect of a diversification

The effect of diversification it can be observed considering that we invest in equals parts of N assets included in the portfolio. In consequence like $X_i = 1/N$ the variance of a portfolio is the following:

$$s_P^2 = \frac{1}{N}\overline{s}_i^2 + \frac{N-1}{N}\overline{s}_{ij}$$

If we take an average variance of 56% and an average covariance of 6% we obtain the following graph in which we can observe that the variance of a portfolio decreases quickly with the first assets included and tend to the average covariance when the number of assets included in the portfolio is large as we can observe in the following chart



With this demonstration, it has shown how the risk of a portfolio of assets can be very different from the risk of the individual assets comprising in the portfolio.

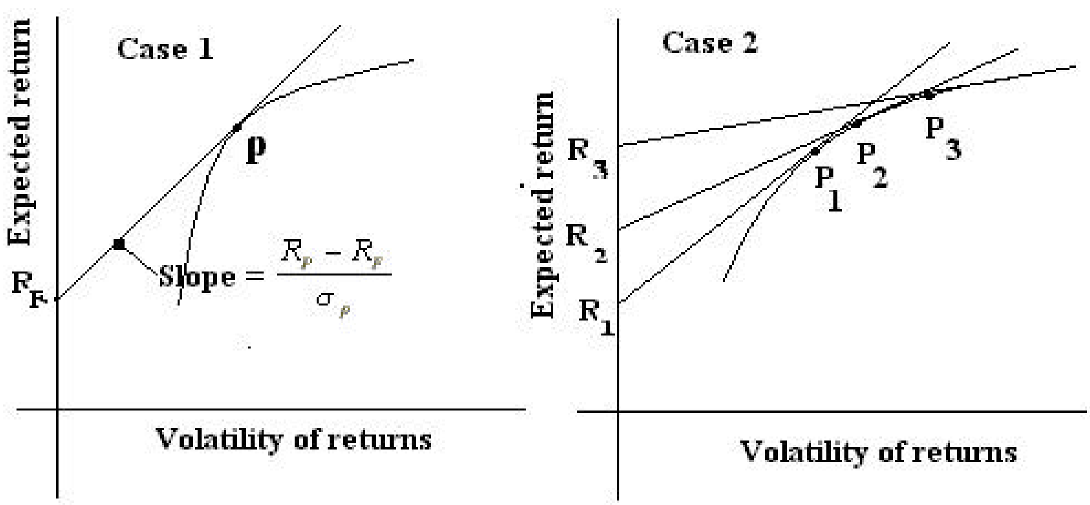
Delineating Efficient Portfolios

After making the previous calculus it can e delineating a subset of portfolios that will be preferred by all investors who exhibit risk avoidance and who preferred more return to less. This set is calling the efficient frontier.

If we consider what happened with the traders, the following situation ca be presented:

1) Short sales are allowed and riskless lending and borrowing is possible
   In this case the objective function to maximize is::
   $q = \frac{\overline{R}_P - R_F}{s_p}$ Where R_F is the risk free rate. This function is constraint to $\sum_{i=1}^{n} X_i = 1$ [2]
   The model developed for this case is showed in appendix A

2) Short sales are allowed but riskless lending or borrowing is not permitted
   There is necessary to introduce same changes to the previous model, because there are different portfolios at different rates allowed to lending and borrowing, in consequence we can obtain a full efficient frontiers al different rates



3) Short sales are disallowed but riskless lending and borrowing exists
   Maximize $q = \frac{\overline{R}_P - R_F}{s_p}$ Subject to a) $\sum_{i=1}^{n} X_i = 1$ and b) $X_i \geq 0$ for all i These two restrictions are linear and the objective function is not linear

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[2] *Lintner has advocated an alternative definition of short sales, one that is more realistic. The constraint.* $\sum_{i=1}^{n} |X_i| = 1$ *Because the investor receive an interest for the money invested in the guarantee to rent de assets.*

4) Neither short sales nor riskless lending and borrowing are allowed

Minimize $s_P = \sqrt{\sum_{i=1}^{k} X_i^2 s_i^2 + \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} X_i X_j s_{ij}}$

Subject to: a) $\sum_{i=1}^{n} X_i = 1$; b) $X_i \geq 0$ for all i and c) $\sum_{i=1}^{n} X_i \overline{R}_i = \overline{R}_P$

Considering this equations when there are n assets it is necessary to estimate n returns, n volatilities and $n(n-1)$ correlation coefficients. Then if a market has 100 assets it must calculate 100 returns and volatilities and 9900 correlation coefficients.

II. Delineating a Portfolio with Betas

When we analyze some market, it can be observe that when the prices goes up the price of the assets included in that market tend to increases and when the market goes down the price of the asset decrease. This suggests that there is a correlation between the responses of the market with the responses of the asset's price. Then the return of a stock can be written as:

$$R_i = a_i + b_i R_m$$

Where:

a_i is the component of the return of asset i, that is independent of the market performance
R_m is the rate of return of the index market
b_i is a value that measures the expected change in R_i given a change in R_m

This equation divides the returns on a stock in two parts, the part due to the market and the part independent of the market. The term a_i represents the component independent of the market returns. It is useful break the component a_i in two components as: $a_i = a_i + e_i$ where a_i is the expected value of a_i and e_i is the random variable of a_i that has an expected value of zero. Rearranging the previous relationship between the market returns and the assets returns the equation of a return of a stock can be written as:

$$R_i = a_i + b_i R_m + e_i$$

If we note that e_i and R_m are random variables in consequence both have a probabilistic distribution and a mean and volatility denoted as s_{ei} and s_m and it is convenient that these random variables are uncorrelated. To estimate the coefficients a_i, b_i and s_{ei} we use the time series regression analysis. This technique guarantees that the random variables e_i and R_m are uncorrelated at least over the period to which the equation has been fit. Other important characteristic of this single index model is that e_i is independent of e_j for all the values of i and j. This implies that the only reason that the stocks returns vary together, systematically, is because the stocks returns follow the movement of the market. How well this model works depends in part on how good (or bad) this approximation is.

The basic equation is:

$R_i = a_i + b_i R_m + e_i$ For all stocks $i = 1, 2, 3. \ldots\ldots n$

Appling the considerations doing for the construction the model the mean of $e_i = E(e_i) = 0$ for all stocks $i = 1, 2, 3, \ldots\ldots, n$

The assumptions adopted for this model are: 1) the residuals e_i are uncorrelated with the market returns, for all stocks, and 2) the residuals of each stock are uncorrelated with other sock, for all pair of stocks.

By definition the variance of $e_i = s^2_{ei}$ and the variance of $R_m = s^2_m$

Using the model and applying the previous definitions the mean return is $\overline{R}_i = a_i + b_i \overline{R}_m$.(1) The variance of an asset return is $s_i^2 = b_i^2 s_m^2 + s_{e_i}^2$ (2). The covariance of returns between stocks i and j is $s_{ij} = b_i b_j s_m^2$ (3). The derivations of these coefficients are developed in appendix B and analyzing the value b_i is unique and separates the market return $(b_i R_m)$ from the unique return $(R_i - [b_i R_m])$.

Now considering (1) and substituting in the average return of a portfolio estimated en the first part it may be written the expected return of a portfolio as follows:

$$\overline{R}_P = \sum_{i=1}^{n} X_i a_i + \sum_{i=1}^{n} X_i b_i \overline{R}_m$$

If it is considering the equations (2) and (3) and the variance of a portfolio estimated in the previous part now using this single model the variance of a portfolio is the following:

$$s_P^2 = \sum_{i=1}^{n} X_i^2 b_i^2 s_m^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} X_i X_j b_i b_j s_m + \sum_{i=1}^{n} X_i^2 s_{e_i}^2$$

As we can observe there are less values to estimate for this single index model. For each asset it is necessary to estimate three values, alpha, beta, and the variance of the errors, plus two others values like the market return and the variance of the market return. Considering that there are N values in the market it is necessary calculate $3N + 2$ values against $2N + N(N-1)$ values used in Markowitz selection portfolio.

The characteristics of the single index model applied to the portfolio, it can be define that $b_P = \sum_{I=1}^{n} X_i b_i$ and $a_P = \sum_{I=1}^{n} X_i a_i$ then we can define the return of a portfolio as follows $\overline{R}_P = a_P + b_P \overline{R}_m$

If the risk portfolio is $s_P^2 = \sum_{i=1}^{n} X_i^2 b_i^2 s_m^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} X_i X_j b_i b_j s_m + \sum_{i=1}^{n} X_i^2 s_{e_i}^2$ and the double summation $i \neq j$, if $i = j$, then the terms would be $X_i X_j b^2{}_i s^2{}_{ei}$ that the term of the first summation. Rearranging the terms the risk investor portfolio is:

$$s_P^2 = \left(\sum_{I=1}^{n} X_i b_i \right) \left(\sum_{j=1}^{n} X_j b_j \right) s_m^2 + \sum_{i=1}^{n} X_i^2 s_{e_i}^2$$

$$s_P^2 = b_P^2 s_m^2 + \sum_{i=1}^{n} X_i^2 s_{e_i}^2$$

To estimate Beta and alpha we use the technique of the regression analysis and we obtain the coefficients using the following expression for a sample of *t* observations:

$$b_i = \frac{s_{im}}{s_m^2} \quad ; \quad a_i = \overline{R}_{i_t} - b_i \overline{R}_{m_t}$$

The variance of b_i is calculated as s_{ei} / s_m. The variance of a beta of portfolio is the sum of the errors weighted for the proportion of each stock included in the portfolio.

To simplify the use of models of selection portfolio, like was exposed in the first part, the following model make an assumption about why the assets co-vary together. This model simplifies the correlation matrix or covariance matrix between securities.

The calculation of a portfolio is based in the ratio call "excess on return to Beta". This ratio using the single index model to estimate the coefficient Beta of each asset, describes de co-movements of the asset with the market. The numerator is the excess of return or the deference between the asset return and the risk free rate, and the denominator is the nondiversificable risk or the risk that we cannot get rid of. This ratio is:

$$\textit{Excess on return to beta} = \frac{\overline{R}_i - R_F}{b_i}$$

If the assets are ranked by this ratio from highest to lowest, the ranking presents a preference to be included in the portfolio. If you select some particular ratio all the assets over this particular ratio will be included in the portfolio, and all the assets with a ratio under this particular value are excluded from the selection. This particular ratio is called as cut – off ratio C^*

The methods to select which stocks are included in the optimum portfolio is the follow:

1) Estimate the ratio of each stock under consideration, and rank from highest to lowest
2) The optimum portfolio consists of investing in all stocks for with the ratio of excess of return to beta is greater than a particular C^*

C^* is call the cut-off rate. All assets whose excess of return to beta is above C^* are selected and whose ratios are below are rejected. To estimate the cut-off ratio is necessary to rank the assets by the ratio of excess of return to beta and estimate the value of C_i. Now the value of i depends from this ranking and the value of C_i .is the following:

$$C_i = \frac{s_m^2 \sum_{j=1}^{i} \frac{\left(\overline{R}_j - R_F\right) b_j}{s_{e_j}^2}}{1 + s_m^2 \sum_{j=1}^{i} \frac{b_j^2}{s_{e_j}^2}}$$

This formula is very easy to calculate, if you can observe are several summations and accumulated summations to solve the problem. To clarify the economic significance of the expression C_i it is change the previous expression by:

$$C_i = \frac{b_{iP}(\overline{R}_P - R_F)}{b_i}$$

Where b_{iP} is the expected change in the rate of return on stock i associated with 1% of change of the optimal portfolio and $\overline{R}_P$ is the expected return of the optimal portfolio. Both terms of course are unknown, but the expression is useful for interpreting the economic significance. If we consider the inclusion of a asset in the optimal portfolio as:

$$\frac{\overline{R}_i - R_F}{b_i} > C_i$$

The previous equation may be rearranging as:

$$(\overline{R}_i - R_F) > b_{iP}(\overline{R}_P - R_F)$$

The left hand side is the expected excess of return of an individual asset. The right hand side is the excepted excess of return on a particular stock based in the performance of the optimal portfolio. Now based in this relationship, is included all individual asset that perform better that the portfolio expected.

To construct the portfolio the percentage invested in each asset is $X_i = \frac{Z_i}{\sum_{included} Z_i}$

Where $Z_i = \frac{b_i}{s_{e_i}^2}\left(\frac{\bar{R}_i - R_F}{b_i} - C^*\right)$

If short sales are not allowed the values of Z_i must be all positives and the cut-off is the value that permits that the difference between the rate on excess of return to beta less the cut-of rate is positive.

If short sales are allowed then the cut-of rate is the last asset included in this selection for the portfolio. In this case $X_i = \frac{Z_i}{\sum_{i=1}^{n} |Z_i|}$ for instance $\sum_{I01}^{n} |X_i| = 1$

It can be estimate a portfolio with Betas. For an example I toke 5 assets, including more than 1200 weekly observations, from Merval Index and estimate the beta coefficient for each asset obtaining the following results:

Table I

A	C	D	E	Excess of return on Beta
Stocks	Excess of Returns in % daily	Beta	$s_{e_i}^2$	Rank (C/D)
Siderca	0.0682	1.1321	2.6074	0.060240
Perez Companc	0.0469	1.2123	1.3944	0.038708
Banco Galicia	0.0347	1.1506	4.1944	0.030172
Metrogas	0.0056	0.4781	4.2388	0.011807
Telefonica Arg	0.0118	1.0260	2.1932	0.011543

In base of the previous table it can be estimate the coefficients and the objective of the X_i for delineating a portfolio with an efficient frontier. Considering a selection of portfolio with short sales not allowed, the value of C^* *(the cut-of rate)* is 0.0444361 to obtain values of $Z_i > 0$

Table II

Stocks	C_i	Z_i	X_i
Siderca	0.0444361	0.00941713	1
Perez Companc	0.0409269	0	0
Banco Galicia	0.0392596	0	0
Metrogas	0.0385512	0	0
Telefonica Arg	0.0335068	0	0

The portfolio return, applying the values of X_i obtained is 0.0681 % per week and a volatility of 14.13 %

III. New Betas with Garch models

When we estimate a beta using the regression model, it is important to verify that the model comply with the homocedastic condition. The first test to use is the analysis of the squared residuals of the regression. The Q – Statistic for the first 10 lags must be guarantee the presence of a white noise for reject the probability of the presence of ARCH in the model.

The steps to evaluate the presence of heteroscedasticity in the model are the following:

1) Develop the best regression model and obtain the coefficients of the regression to estimate de Betas. In our case the values of the straight line, the constant a and the pendant b with the corresponding variances of this estimators.
2) Analyze the autocorrelation and partial autocorrelation function of the squared residuals of the regression model developed in 1). Calculate and plot the sample autocorrelations of the squared residuals as:

$$r_i = \frac{\sum_{t=i+1}^{T} \left(e_t^2 - s^2\right)\left(e_{t-i}^2 - s^2\right)}{\sum_{t=1}^{T} \left(e_t^2 - s^2\right)}$$

3) If the ACF and PAC are a white noise then it is rejected the presence of Garch. To accept the presence of Garch in the squared residuals it must be used the Q – Statistic, developed by Ljung – Box, to test a group of significant coefficients. The statistic is:

$$Q = T(T+2)\sum_{i=1}^{n} \frac{r_i^2}{(T-i)}$$

Where T is the number of observations, n is the number of lags included in the test. This statistic tends to c^2 with n degrees of freedom, if the squared residuals are uncorrelated. If it is rejected the null hypothesis that the squared residuals are uncorrelated is the same to accept the presence of Arch or Garch errors. In this case Engle (1982) proposed a more formal test like the Lagrange Multiplier test that involves the following step

1) Regress the squared residuals fitted in the regression on a constant and on the q lagged values like the following form:

$$e_t^2 = a_o + a_1 e_{t-1}^2 + a_2 e_{t-2}^2 + a_3 e_{t-3}^2 + \Lambda \; a_q e_{t-q}^2$$

If there are no Arch o Garch effect the coefficients a_i should be zero. The R^2 is very low, in consequence the determination of the model in very little. If it is proposed a null hypothesis that there are no ARCH or GARCH effects the coefficient TR^2 converges to a c^2 with q degrees of freedom. If TR^2 is sufficiently large, it can be rejected the null hypothesis that there is no presence of Arch or Garch errors. In the other hand if TR^2 is low we accept the null hypothesis that there is Arch or Garch effect and is equivalent to accept that the values of $a_1 \ldots\ldots a_q$ are equal to zero.

The procedure to estimate the best betas with Garch effect, is first estimate the equation by OLS, and with the squared residuals obtained new coefficients may be obtained using the appropriate method. To estimate both equations with full efficiency it must be use the nonlinear maximum likehood routines.

The numerical procedures used by software to calculate the maximum likelihood estimation of Garch models are based in the following premises. Suppose that the values of certain random variable has normal distribution having a mean m and a constant variance s^2 then the log likelihood function using T independent observations is:

$$\lambda = -\left(\frac{T}{2}\right)\ln(2p) - \left(\frac{T}{2}\right)\ln s^2 - \left(\frac{1}{2}s^2\right)\sum_{t=1}^{T}(y_t - m)^2$$

Where λ is the log likelihood function and y_t is a random variable. The system to estimate the parameters in maximum likehood function is maximizing the likehood of drawing the observed sample. In the previous equation the problem is to maximize the λ with respect to m and s^2 as:

$$\left[\frac{\partial\lambda}{\partial m}\right] = \left(\frac{1}{s^2}\right)\sum_{t=1}^{T}(y_t - m)$$

$$\left[\frac{\partial\lambda}{\partial s^2}\right] = -\left(\frac{T}{2}s^2\right) + \left(\frac{1}{2}s^4\right)\sum_{t=1}^{T}(y_t - m)^2$$

These partial derivatives are equal to zero and solving the values of m and s^2 we obtain that the maximum value of log likelihood as:

$$\hat{m} = \frac{\sum_{t=1}^{T} y_t}{T} \quad ; \quad \hat{s}^2 = \sum_{t=1}^{T}\frac{(y_t - \hat{m})^2}{T}$$

Applying the same principles for a regression analysis to a model $e_t = y_t - b\,x_t$, assuming that the mean e_t is equal zero, and the variance is constant end the different realizations of $|e_t|$ are independent. This are the conditions applied to errors of the regression analysis. Now the log likelihood is:

$$\lambda = -\left(\frac{T}{2}\right)\ln(2p) - \left(\frac{T}{2}\right)\ln s^2 - \left(\frac{1}{2}s^2\right)\sum_{t=1}^{T}(y_t - bx_t)^2$$

Following the steps to maximize the log likelihood function, respect s^2 and b yields:

$$\left[\frac{\partial \log \lambda}{\partial s^2}\right] = -\left(\frac{T}{2} s^{2}\right) + \left(\frac{1}{2} s^{4}\right) \sum_{t=1}^{T} (y_t - b x_t)^2$$

$$\left[\frac{\partial \log \lambda}{\partial b}\right] = \left(\frac{1}{s^2}\right) \sum_{t=1}^{T} (y_t x_t - b x_t^2)$$

Setting these partial derivatives equals zero and solving the system respect the values of b and s^2 yields the maximum value of the log likelihood function of the regression analysis, estimating the values of the variance regression and Beta. These values are the following for: $\hat{s}^2 = \sum (e_t)^2 / T$ and $\hat{b} = \sum x_t y_t / \sum (x_t)^2$. This calculus was been presented in many econometrics books, but is necessary to explain what happens when we are in presence of an Arch model. The log likelihood for a regression models is very simple, because the first conditions are linear. Unfortunately the Arch models are nonlinear as we can observe with an Arch(1) model. Assuming that the errors, e_t, was generated by $e_t = y_t - b x_t$, now e_t, is given by:

$$e_t = v_t \sqrt{(a_0 + a_1 e_{t-1}^2)}$$

and the conditional variance of e_t, is:

$$h_t = a_0 + a_1 e_{t-1}^2$$

Since the realizations of e_t has a conditional variance of h_t the appropriate log likelihood function is the following:

$$\lambda = -\left(\frac{T}{2}\right) \ln (2p) - \left(\frac{T}{2}\right) \ln h_t - \left(\frac{1}{2} h_t\right) \sum_{t=1}^{T} (y_t - b x_t)^2$$

$$\text{where:}$$

$$h_t = a_0 + a_1 (y_{t-1} - b\, x_{t-1})^2$$

Finally it is possible to combine the above and obtain by some numerical method the values of a_0, $a1$, and b to maximize the log likelihood function. Then the new values obtained applying the Garch model to a linear regression provides a best beta coefficient with minimum variance and with different $| e_t^2 |$ uncorrelated.

New Betas with Garch models

Using the same base of data toke for prepare de Table II and I and apply the estimation of Betas with Garch models, including the asymmetric models (see *Tagliafichi Ricardo The Garch models and the application to the VaR*), we obtain a new betas with minimum variance as we can observe in Table III. Then we can analyze the results of the regression with both methods, with the traditional econometric model and this traditional econometric model solving the presence of heteroscedasticity or the presence of Garch effect.

Table III

Stock	a	b	Ranking	Econometric Model Used
Perez Companc 1216 observations	0.058	1.2123 (0.0142)	0.0387	Traditional
	0.0146	0.925 (0.0078)	0.0507	Tarch (1,1)
Metro gas 1200 observations	0.5912	0.478125 (0.0248)	0.0118	Traditional
	-0.0481	0.4467 (0.01632)	0.0126	Garch (2,1)
Galicia 1216 observations	0.0473	1.15058 (0.0245)	0.0302	Traditional
	0.0102	1.05954 (0.02026)	0.0328	Tarch (1,1)
Siderca 1216 observations	0.0805	1.1321 (0.0194)	0.0602	Traditional
	0.0774	1.1142 (0.0145)	0.0612	Garch (2,2)
Tear 1216 observations	0.02307	1.026 (0.018)	0.0116	Traditional
	-0.013	1.054 (0.013)	0.0112	Garch (1,1)

There is strong evidence that there is difference between the traditional econometric model, with the same model but corrected by the presence of Garch effects. The new Beta values are more efficient because this estimator has the minimum variance as we can observe in table III on the values marked in brackets.

What happened with the use of a traditional econometric model? If we developed a regression model to calculate the values of beta with the traditional form, we obtain a value of beta and the variance of this estimator. But the question is what happens with the squared residuals? In effect if we follow the steps recommended in the third part of this paper we must analyze the autocorrelation function and the partial autocorrelation function of the squared residuals. These functions denote the absence of white noise, and in consequence it must be accept the presence of Garch effects, and that Garch effects change the previous results as we can observe in Table III.

IV Different Portfolios are calculated with different Betas

In the second section was presented the Tables I and II with the estimation of the values of $X_{i,}$. This indicates the proportion of participation of each asset in the portfolio. This mentioned values are obtained using the procedure explained to get the best values that maximizes the returns and minimizes the volatility.

The presence of Garch effects in the estimations of Beta, produce new value of Beta with less variance than the traditional Beta, and in consequence a new values of X_i are obtained as we can observe in the Tables IV and V.

Table IV

A	**C**	**D**	**E**	Excess of Return on Beta
Stocks	Excess of Returns in % daily	Beta	$s^2_{e_i}$	Rank (C/D)
Siderca	0.0682	1.1142	2.6092	0.0612
Perez Companc	0.0469	0.9250	1.4490	0.0507
Banco Galicia	0.0347	1.0595	4.2440	0.0328
Metrogas	0.0056	0.4467	4.2473	0.0126
Telefonica Arg	0.0118	1.5040	2.1990	0.0112

In base of the previous table it can be estimate the coefficients and the objective of the X_i for delineating a portfolio with an efficient frontier. Considering a selection of portfolio with short sales not allowed, the value of C^* *(the cut-of rate)* is 0.0476015 to obtain values of $Z_i > 0$

Table V

Stocks	**Ci**	**Zi**	**Xi**
Siderca	0.044761	0.00581	0.74415
Perez Companc	0.047601	0.00199	0.25585
Banco Galicia	0.044994	0	0
Metrogas	0.044015	0	0
Telefonica Arg	0.035968	0	0

As a result of the new coefficients the new portfolio is formed by two assets with a participation of 75% of Siderca and a 25% of Perez Companc. The new portfolio return is 0.063 % per weekly and the volatility is 10.40%. With this new portfolio the expected return decrease in a 8% but the risk decrease in a 40%.

V Conclusions

The presence of Garch effects in the model used for calculates the values of Beta, permit to enforce the idea of obtain a best coefficients, with minimum variance. In all the cases under study, this premises was been confirmed by the practice or the reality. This results may by reproduced in several assets of different markets.

Accepting the necessity of the building a portfolio with minimum variance and the best results, the model of obtain an efficient frontier using the Beta values, is a best solution when we manage a lot of assets in the market due to the number of coefficients to be estimate.

Combining the two aspects, in one hand the estimate the efficient frontier with Betas, and in the other hand the presence of Garch effects in the estimation of beta coefficients, it can be obtain a best portfolio which complies the restrictions that the model have in their develop.

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Appendix A

To solve the constraints it can be used the method of Lagrangian multipliers. The alternative is substitute into the objective function the objective function maximized as in an unconstrained problem. If we write $R_F = 1\ R_F$ then we obtain:

$$R_F = 1R_F = \left(\sum_{I01}^{N} X_i \right) R_F = \sum_{I=1}^{N} X_i R_F$$

then

$$q = \sum_{i=1}^{N} X_i \left(\overline{R}_i - R_F \right) \left[\sum_{i=1}^{N} X_i^2 s_i^2 + \sum_{i=1}^{N} \sum_{\substack{i=1 \\ j=1}}^{N} X_i X_j s_{ij} \right]^{-\frac{1}{2}}$$

The solution of a maximization problem is to derive with respective to each variable and equals zero. After must be solve the system of simultaneous equations with n equations and n variables as:

$$1.\quad \frac{\partial q}{\partial X_1} = 0$$

$$2.\quad \frac{\partial q}{\partial X_1} = 0$$

$$3.\quad \frac{\partial q}{\partial X_3} = 0$$

$$\text{M}$$

$$N.\quad \frac{\partial q}{\partial X_N} = 0$$

Defining $l = \sum_{i=1}^{N} X_i \left(\overline{R}_i - R_F \right) \left[\sum_{i=1}^{N} X_i^2 s_i^2 + \sum_{i=1}^{N} \sum_{\substack{i=1 \\ j=1}}^{N} X_i X_j s_{ij} \right]$

The derivative of the product for the asset k is $-\left[lX_k s_k^2 + \sum_{\substack{j=1 \\ j \neq k}}^{N} lX_j s_{kj} \right] + (\overline{R}_k - R_f) = 0$

Then $\frac{\partial J}{\partial X_i} = -\left(lX_1 s_{1i} + lX_2 s_{2i} + lX_3 s_{3i} + \Lambda + lX_i s_i^2 + \text{K} + lX_N s_{Ni} \right) + \overline{R}_i - R_F = 0$

Like l is a constant we can define a new variable $Z_i = lX_i$ then Z_k is a proportion of X_k in consequence the values of X_k may be obtained after solve the values of Z_k. Replacing the new variable in the derivative we obtain the following system of equations:

$$
\begin{aligned}
\overline{R}_1 - R_F &= Z_1 s_1^2 + Z_2 s_{12} + Z_3 s_{13} + \Lambda + Z_N s_{1N} \\
\overline{R}_1 - R_F &= Z_1 s_{12} + Z_2 s_2^2 + Z_3 s_{23} + \Lambda + Z_N s_{2N} \\
\overline{R}_1 - R_F &= Z_1 s_{13} + Z_2 s_{23} + Z_3 s_3^2 + \Lambda + Z_N s_{3N} \\
\mathrm{M} \quad & \quad \mathrm{M} \\
\overline{R}_1 - R_F &= Z_1 s_{1N} + Z_2 s_{2N} + Z_3 s_{3N} + \Lambda + Z_N s_N^2
\end{aligned}
$$

To solve this system we use the matrix form that is represented by $\left|\overline{\mathbf{R}}_\mathbf{i} - \mathbf{R}_\mathbf{F}\right| = \left|\mathbf{Z}_\mathbf{i}\right|\left|\mathrm{s}\right|$ and in consequence the solution is $\left|\mathbf{Z}_\mathbf{i}\right| = \left|\overline{\mathbf{R}}_\mathbf{i} - \mathbf{R}_\mathbf{F}\right|\left|\mathrm{s}\right|^{-1}$

Finally the values of $X_k = \dfrac{Z_k}{\sum_{included} Z_i}$

Appendix B

The expected return on an asset is

$$
\begin{aligned}
& E(R_i) = E\left[a_i + b_i R_m + e_i\right] \\
& E(R_i) = E(a_i) + E(b_i R_m) + E(e_i)
\end{aligned}
$$

like a and b are constants and the expected value of e_i is zero by constructi on

$$E(R_i) = a_\mathrm{i} + b_\mathrm{i}\overline{\mathrm{R}}_\mathrm{m}$$

The variance of an asset return is the following:

$$s_i^2 = E(R_i - \overline{R})^2$$

sustitutin g from the expression above yields

$$s_\mathrm{i}^2 = E\left[(a_i + b_i R_m + e_i) - (a_i + b_i \overline{R}_m\right]^2$$

Rearrangin g and cancel de alpha

$$
\begin{aligned}
& s_\mathrm{i}^2 = E\left[b_i\left(R_m - \overline{R}_m\right) - e_i\right]^2 \\
& s_i^2 = b_i^2 E(R_m - \overline{R}_m)^2 + 2 b_i E\Big(e_i (R_m - \overline{R}_{M)}\Big) + E(e_i)^2
\end{aligned}
$$

if by assumption $E\Big(e_i (R_m - \overline{R}_{M)}\Big) = 0$

$$s_\mathrm{i}^2 = b_i^2 s_m^2 + s_{e_i}^2$$

The covariances between any two different assets can bu written as follows:

$$s_{ij} = E\left[(R_i - \overline{R}_i)(R_j - \overline{R}_j)\right]$$

Substituti ng for $R_i, \overline{R}_i, R_j, \overline{R}_j$ yields

$$s_{\text{ij}} = E\left[(a_i + b_i R_m + e_i) - (a_i + b_i \overline{R}_m\right] \bullet E\left[(a_j + b_j R_m + e_j) - (a_j + b_j \overline{R}_m\right]$$

simplifyin g the alpha and combinig the terms with betas

$$s_{ij} = b_i b_j E(R_m - \overset{\rho}{R}_m)^2 + b_j E\left[e_i (R_m - \overset{\rho}{R}_m)\right] + b_i E\left[e_{j\text{The three}} (R_m - \overset{\rho}{R}_m)\right] + E\left(e_i e_j\right)$$

The three last terms are zero by assumption

$$s_{ij} = b_i b_j s_m^2$$