Ruin probabilities for a Correlated Aggregate Claims Model

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Abstract

In this paper we consider a risk model having two disjoint classes of insurance business. Correlation may exist among the two claim number processes. Claim occurrences of both classes relate to Poisson and Erlang processes. We derive general solutions to the ultimate survival (ruin) probabilities for some risk processes generated from the assumed model when the claim sizes are exponentially distributed. In particular we study the correlated case in which both classes of claims occur as a mixture of Poisson and Erlang processes. We also examine the asymptotic property of the ruin probability for this special risk process with general claim size distributions.

Keywords

Bivariate compound Poisson; Correlated aggregate claims; Erlang process; Renewal process; Ruin probability.

1. Introduction

Various models for correlated aggregate claims have been proposed recently due to their potential usefulness in insurance industry. In this paper we consider a risk model involving two disjoint classes of insurance business. Let X_i be claim size random variables for the first class with common distribution function F_X and Y_i be those for the second class with common distribution function F_Y. Their means are denoted by μ_X and μ_Y. It is assumed that all X_i and Y_i are independent. Then the aggregate claim size process generated from the two classes of business is given by

$$U(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i, \tag{1.1}$$

where $N_i(t)$ is the claim number process for class $i, i = 1, 2$. Claim sizes are assumed to be independent of claim numbers. The two claim number processes are correlated in the way that

$$N_1(t) = K_1(t) + K_2(t), \qquad \text{and} \qquad N_2(t) = K_2(t) + K_3(t), \tag{1.2}$$

with $K_1(t)$, $K_2(t)$, and $K_3(t)$ being three independent renewal processes. As usual, we define the surplus process

$$S(t) = u + ct - U(t), \tag{1.3}$$

where u is the amount of initial surplus and c is the rate of premium. The ultimate survival probability is

$$\Phi(u) = \mathrm{P}(S(t) \geq 0; \text{ for all } t \geq 0). \tag{1.4}$$

Surplus process (1.3) is sometimes referred to as the common shock model in the literature. Many authors studied various aspect of the common shock model in recent years. For instance, Ambagaspitiya (1998,1999) considered a general method of constructing a vector of p ($p \geq 2$) dependent claim numbers from a vector of independent random variables, and derived formulas to compute the correlated aggregate claims distribution for the corresponding common shock model with p dependent classes of business; and Cossette and Marceau (2000) used a discrete-time approach to study how the common shock affects the finite-time ruin probabilities and the adjustment coefficient. In addition to the common shock model, there exist other kinds of correlated aggregate claims model in the literature. For example, Yuen and Guo (2001) studied the finite-time ruin probabilities for the compound binomial model with time-correlated claims; and Yuen and Wang (2001) considered a new risk model in which correlation comes from the thinning of Poisson claim number processes, and discussed the impact of the dependence structure on ruin probability.

This paper is to examine the ultimate survival (ruin) probability for the risk process (1.3). In Section 2 we briefly discuss the case that K_1, K_2, and K_3, are three independent Poisson processes. Dickson (1998) and Dickson and Hipp (1998) considered the probability of ruin when claims occur as an Erlang process. Their work motivates us to consider one of the three renewal processes as an Erlang process. We first study the case that K_1 is Poisson, $K_2 \equiv 0$, and K_3 is Erlang. From (1.2) it is easily seen that the two claim number processes are independent in this case. Under this independence assumption, Section 3 is devoted to deriving the ultimate survival (ruin) probability when claim sizes follow exponential distributions. We then investigate the correlated case in which K_2 is assumed to be Erlang. Given that K_1 and K_3 are Poisson, we obtain the required probabilities for exponential claim sizes in Section 4. When dealing with general claim size distributions, Section 5 presents an asymptotic result for the probability of ruin.

2. Three Poisson Processes

Let $K_1(t)$, $K_2(t)$, and $K_3(t)$ be three independent Poisson processes with parameters λ_1, λ_2, and λ_3 respectively. Then the moment generating function of (1.1) has the form

$$M_U(s) = \exp\left\{\lambda t\left(\frac{\lambda_1}{\lambda}M_X(s) + \frac{\lambda_2}{\lambda}M_X(s)M_Y(s) + \frac{\lambda_3}{\lambda}M_Y(s) - 1\right)\right\}, \tag{2.1}$$

where $\lambda = \lambda_1+\lambda_2+\lambda_3$. It is easily seen that $\lambda^{-1}(\lambda_1 M_X(s)+\lambda_2 M_X(s)M_Y(s)+\lambda_3 M_Y(s))$ is the moment generating function of the random variable

$$Z = XI(\xi = 0) + (X+Y)I(\xi = 1) + YI(\xi = 2),$$

where X, Y, and ξ are independent. The probability function of ξ is given by

$$\mathrm{P}(\xi = 0) = \frac{\lambda_1}{\lambda}, \qquad \mathrm{P}(\xi = 1) = \frac{\lambda_2}{\lambda}, \qquad \text{and} \qquad \mathrm{P}(\xi = 2) = \frac{\lambda_3}{\lambda}.$$

This shows that $U(t)$ is a compound Poisson process with parameter λ and the corresponding claim sizes follow the distribution of Z. Therefore the ruin probability can be calculated using the classical method. Such a result can be extended to a p-variate ($p > 2$) claim number process whose component is a linear combination of independent Poisson processes. The key step is to write the moment generating function of the aggregate claim size process in the form of (2.1).

Another interesting problem is to consider the two classes of business separately. Let the surplus process for class $i, i = 1, 2$ be

$$S_i(t) = u_i + c_i t - \sum_{j=1}^{N_i(t)} X_j,$$

where u_i and c_i are the initial surpluses and premium rates. Define the infinite time joint survival probability

$$\phi(u_1, u_2) = \mathrm{P}(S_1(t) \geq 0, S_2(t) \geq 0; \text{ for all } t \geq 0).$$

In a small time interval $(0, \Delta]$, there are five possible cases: no claim, one claim from class 1 and no claim from class 2, no claim from class 1 and one claim from class 2, two claims and one from each class, more than two claims. It follows that

$$\begin{aligned}
\phi(u_1,u_2) &= (1-\lambda_1\Delta-\lambda_2\Delta-\lambda_3\Delta+o(\Delta))\phi(u_1+c_1\Delta,u_2+c_2\Delta) \\
&\quad + (\lambda_1\Delta+o(\Delta))\int_0^{u_1+c_1\Delta}\phi(u_1+c_1\Delta-x,u_2+c_2\Delta)dF_X(x) \\
&\quad + (\lambda_3\Delta+o(\Delta))\int_0^{u_2+c_2\Delta}\phi(u_1+c_1\Delta,u_2+c_2\Delta-y)dF_Y(y) \\
&\quad + (\lambda_2\Delta+o(\Delta))\int_0^{u_1+c_1\Delta}\int_0^{u_2+c_2\Delta}\phi(u_1+c_1\Delta-x,u_2+c_2\Delta-y)dF_Y(y)dF_X(x) \\
&\quad + o(\Delta).
\end{aligned}$$

As Δ tends 0, we get

$$\begin{aligned}
& c_1\frac{\partial}{\partial u_1}\phi(u_1,u_2) \;+\; c_2\frac{\partial}{\partial u_2}\phi(u_1,u_2) \\
= \;& (\lambda_1+\lambda_2+\lambda_3)\phi(u_1,u_2) \;-\; \lambda_1\int_0^{u_1}\phi(u_1-x,u_2)dF_X(x) \\
& -\lambda_3\int_0^{u_2}\phi(u_1,u_2-y)dF_Y(y) \;-\; \lambda_2\int_0^{u_1}\int_0^{u_2}\phi(u_1-x,u_2-y)dF_X(x)dF_Y(y).
\end{aligned}$$

It is rather difficult to solve this two dimensional integro-differential equation. If we assume that $S_1(t)$ and $S_2(t)$ are two independent compound Poisson processes, that is, $\lambda_2=0$, and that X and Y are exponentially distributed, then one can show that

$$\phi(u_1,u_2)=\left(1-\frac{1}{\rho_1+1}\exp\left\{\frac{-\rho_1u_1}{\mu_X(1+\rho_1)}\right\}\right)\left(1-\frac{1}{\rho_2+1}\exp\left\{\frac{-\rho_2u_2}{\mu_Y(1+\rho_2)}\right\}\right),$$

where $\rho_1=c_1(\lambda_1\mu_X)^{-1}-1$ and $\rho_2=c_2(\lambda_3\mu_Y)^{-1}-1$.

3. Independent Poisson-Erlang Case

As shown in Section 2, the classical risk theory (see for example Gerber (1979) and Grandell (1991)) still holds for the process (1.3) with correlated aggregate claims. It is essentially due to the Poisson property of $K_1(t)$, $K_2(t)$, and $K_3(t)$. The derivation of (1.4) will become very complicated if the Poisson assumption is violated. Here we discuss the case that K_1 is a Poisson process with parameter λ_1, $K_2\equiv 0$, and K_3 is an Erlang(2) process with parameter λ_3. For a single class of business, Dickson (1998) and Dickson and Hipp (1998) considered the survival (ruin) probability for a risk process in which claim interarrival times having an Erlang(2) distribution. Actually Erlang distribution is one of the most commonly used distributions in queueing theory which is closely related to risk theory; see for example, Asmussen (1987, 1989) and Takács (1962).

Let $V_1,V_2,\cdots$ be the times between claims for the first class of business. They are independent and exponentially distributed with mean λ_1^{-1}. For the second class of business, the times between claims form a sequence of independent and identically distributed random variables, $L_1, L_2, \cdots$, following an Erlang(2,λ_3) distribution. Equivalently we write $L_1 = L_{11} + L_{12}, L_2 = L_{21} + L_{22}, \cdots$, where $L_{11}, L_{12}, L_{21}, L_{22}, \cdots$ are independent exponential random variables with mean λ_3^{-1}. Since $\lambda_1 \mu_X$ and $2^{-1}\lambda_3\mu_Y$ are the expected aggregate claims associated with N_1 and N_2 respectively over a unit time interval, the positive relative security loading condition implies that

$$c > \lambda_1 \mu_X + \frac{\lambda_3 \mu_Y}{2}. \tag{3.1}$$

With other things being the same, we consider a slight change in the distribution of L_1. Instead of being a sum of L_{11} and L_{12}, L_1 is equal to L_{12}. We denote the corresponding survival probability by $\Phi_1(u)$ which is very useful in the derivation of $\Phi(u)$.

Let W be the minimum of V_1 and L_{11}. If $W = L_{11} = t$, then no claim occurs in $(0, t]$. On the other hand, if $W = V_1 = t$, there is a claim at time t and no claim before t. Hence, we have

$$\begin{aligned} \Phi(u) &= \int_0^\infty P(W = t, W = L_{11}) \Phi_1(u + ct) dt \\ &\quad + \int_0^\infty P(W = t, W = V_1) \int_0^{u+ct} \Phi(u + ct - x) dF_X(x) dt. \end{aligned} \tag{3.2}$$

Note that

$$\begin{aligned} \mathrm{P}(W = V_1) &= \mathrm{P}(V_1 < L_{11}) = \frac{\lambda_1}{\lambda_1 + \lambda_3}, \\ \mathrm{P}(W = L_{11}) &= \mathrm{P}(V_1 > L_{11}) = \frac{\lambda_3}{\lambda_1 + \lambda_3}, \\ \mathrm{P}(W > t | W = V_1) &= \mathrm{P}(W > t | W = L_{11}) = \exp\{-(\lambda_1 + \lambda_3)t\}. \end{aligned}$$

It is obvious that the two conditional distributions are exponential with parameter $\lambda_1 + \lambda_3$. Using these probabilities, (3.2) can be rewritten as

$$\begin{aligned} \Phi(u) &= \frac{\lambda_3}{\lambda_1 + \lambda_3} \int_0^\infty (\lambda_1 + \lambda_3) \exp\{-(\lambda_1 + \lambda_3)t\} \Phi_1(u + ct) dt \\ &\quad + \frac{\lambda_1}{\lambda_1 + \lambda_3} \int_0^\infty (\lambda_1 + \lambda_3) \exp\{-(\lambda_1 + \lambda_3)t\} \int_0^{u+ct} \Phi(u + ct - x) dF_X(x) dt. \end{aligned}$$

By similar arguments, we obtain

$$\begin{aligned} \Phi_1(u) &= \frac{\lambda_3}{\lambda_1 + \lambda_3} \int_0^\infty (\lambda_1 + \lambda_3) \exp\{-(\lambda_1 + \lambda_3)t\} \int_0^{u+ct} \Phi(u + ct - x) dF_Y(x) dt \\ &\quad + \frac{\lambda_1}{\lambda_1 + \lambda_3} \int_0^\infty (\lambda_1 + \lambda_3) \exp\{-(\lambda_1 + \lambda_3)t\} \int_0^{u+ct} \Phi_1(u + ct - x) dF_X(x) dt. \end{aligned}$$

Putting $s = u + ct$ yields

$$
\begin{aligned}
c\Phi(u) &= \lambda_3 \int_u^\infty \Phi_1(s) \exp\left\{\frac{-(\lambda_1+\lambda_3)(s-u)}{c}\right\} ds \\
&\quad + \lambda_1 \int_u^\infty \exp\left\{\frac{-(\lambda_1+\lambda_3)(s-u)}{c}\right\} \int_0^s \Phi(s-x) dF_X(x) ds, \\
c\Phi_1(u) &= \lambda_3 \int_u^\infty \exp\left\{\frac{-(\lambda_1+\lambda_3)(s-u)}{c}\right\} \int_0^s \Phi(s-x) dF_Y(x) ds \\
&\quad + \lambda_1 \int_u^\infty \exp\left\{\frac{-(\lambda_1+\lambda_3)(s-u)}{c}\right\} \int_0^s \Phi_1(s-x) dF_X(x) ds.
\end{aligned}
$$

Differentiating with respect to u, we get the following integro-differential equations

$$
\begin{aligned}
c\Phi^{(1)}(u) &= -\lambda_3\Phi_1(u) - \lambda_1 \int_0^u \Phi(u-x) dF_X(x) + (\lambda_1+\lambda_3)\Phi(u), && (3.3) \\
c\Phi_1^{(1)}(u) &= -\lambda_3 \int_0^u \Phi(u-x) dF_Y(x) - \lambda_1 \int_0^u \Phi_1(u-x) dF_X(x) \\
&\quad + (\lambda_1+\lambda_3)\Phi_1(u). && (3.4)
\end{aligned}
$$

Assume that X and Y are exponentially distributed. Then differentiating (3.3) and (3.4) with respect to u once again yields

$$
\begin{aligned}
c\Phi^{(2)}(u) &= -\lambda_3\Phi_1^{(1)}(u) - \frac{\lambda_1}{\mu_X}\Phi(u) + \frac{\lambda_1}{\mu_X^2}\int_0^u \Phi(x)\exp\left\{\frac{-(u-x)}{\mu_X}\right\} dx \\
&\quad + (\lambda_1+\lambda_3)\Phi^{(1)}(u) \\
&= -\lambda_3\Phi_1^{(1)}(u) - \frac{\lambda_1}{\mu_X}\Phi(u) + \frac{1}{\mu_X}\Big((\lambda_1+\lambda_3)\Phi(u) - \lambda_3\Phi_1(u) - c\Phi^{(1)}(u)\Big) \\
&\quad + (\lambda_1+\lambda_3)\Phi^{(1)}(u) \\
&= (\lambda_1+\lambda_3-\frac{c}{\mu_X})\Phi^{(1)}(u) + \frac{\lambda_3}{\mu_X}\Phi(u) - \lambda_3\Phi_1^{(1)}(u) - \frac{\lambda_3}{\mu_X}\Phi_1(u), && (3.5) \\
c\Phi_1^{(2)}(u) &= (\lambda_1+\lambda_3)\Phi_1^{(1)}(u) - \frac{\lambda_3}{\mu_Y}\Phi(u) - \frac{\lambda_1}{\mu_X}\Phi_1(u) \\
&\quad + \frac{1}{\mu_Y}\left(\lambda_3\int_0^u \Phi(x)\frac{1}{\mu_Y}\exp\left\{\frac{-(u-x)}{\mu_Y}\right\} dx\right) \\
&\quad + \frac{1}{\mu_X}\left(\lambda_1\int_0^u \Phi_1(x)\frac{1}{\mu_X}\exp\left\{\frac{-(u-x)}{\mu_X}\right\} dx\right) \\
&= (\lambda_1+\lambda_3-\frac{c}{\mu_Y})\Phi_1^{(1)}(u) - \frac{\lambda_3}{\mu_Y}\Phi(u) - (\frac{\lambda_1}{\mu_X} - \frac{\lambda_1+\lambda_3}{\mu_Y})\Phi_1(u) \\
&\quad +(\frac{1}{\mu_X} - \frac{1}{\mu_Y})\lambda_1\int_0^u \Phi_1(x)\frac{1}{\mu_X}\exp\left\{\frac{-(u-x)}{\mu_X}\right\} dx. && (3.6)
\end{aligned}
$$

Example 3.1. If $\mu_X = \mu_Y = \mu$, then (3.5) and (3.6) become

$$
\begin{aligned}
c\Phi^{(2)}(u) &= (\lambda_1+\lambda_3-\frac{c}{\mu})\Phi^{(1)}(u)+\frac{\lambda_3}{\mu}\Phi(u)-\lambda_3\Phi_1^{(1)}(u)-\frac{\lambda_3}{\mu}\Phi_1(u), && (3.7)\\
c\Phi_1^{(2)}(u) &= (\lambda_1+\lambda_3-\frac{c}{\mu})\Phi_1^{(1)}(u)-\frac{\lambda_3}{\mu}\Phi(u)+\frac{\lambda_3}{\mu}\Phi_1(u), && (3.8)
\end{aligned}
$$

with boundary conditions

$$
\begin{cases}
c\Phi^{(1)}(0) = -\lambda_3\Phi_1(0)+(\lambda_1+\lambda_3)\Phi(0),\\
c\Phi_1^{(1)}(0) = (\lambda_1+\lambda_3)\Phi_1(0),\\
\Phi(\infty) = 1,\\
\Phi_1(\infty) = 1,
\end{cases} \tag{3.9}
$$

implied by (3.3), (3.4), and the fact that the survival probabilities are 1 if the company has an infinite initial surplus.

From (3.8) we have

$$
\Phi(u)=\Phi_1(u)+\frac{\mu}{\lambda_3}(\lambda_1+\lambda_3-\frac{c}{\mu})\Phi_1^{(1)}(u)-\frac{c\mu}{\lambda_3}\Phi_1^{(2)}(u). \tag{3.10}
$$

Then substituting (3.10) into (3.7) gives

$$
\begin{aligned}
\frac{c^2\mu}{\lambda_3}\Phi_1^{(4)}(u) &- \frac{2c\mu}{\lambda_3}(\lambda_1+\lambda_3-\frac{c}{\mu})\Phi_1^{(3)}(u)\\
&- \left(2c-\frac{\mu}{\lambda_3}(\lambda_1+\lambda_3-\frac{c}{\mu})^2\right)\Phi_1^{(2)}(u)+(2\lambda_1+\lambda_3-\frac{2c}{\mu})\Phi_1^{(1)}(u)=0.
\end{aligned}
$$

which has the general solution

$$
\Phi_1(u)=C_1+C_2\exp\{k_1u\}+C_3\exp\{k_2u\}, \tag{3.11}
$$

where k_1 and k_2 are the two negative roots of the characteristic equation

$$
\frac{c^2\mu}{\lambda_3}z^3-\frac{2c\mu}{\lambda_3}(\lambda_1+\lambda_3-\frac{c}{\mu})z^2-\left(2c-\frac{\mu}{\lambda_3}(\lambda_1+\lambda_3-\frac{c}{\mu})^2\right)z+(2\lambda_1+\lambda_3-\frac{2c}{\mu})=0.
$$

This equation has two negative solutions and one positive solution. Since $c > (\lambda_1 + 2^{-1}\lambda_3)\mu$ from condition (3.1), the left-hand side of the equation takes a negative value at $z = 0$. Thus a positive root exists. As for the existence of two negative roots, we only need to show that the left-hand side takes a positive value for some negative value of z. From (3.1) c can be written as $(1+\rho)(\lambda_1+\lambda_3/2)\mu$ where ρ is the relative security loading. Let $a = \lambda_3(2\lambda_1+\lambda_3)^{-1}$ and $y = (1+\rho)\mu z$. Then the above characteristic equation is equivalent to

$$
h(y)=y^3-2(a-\rho)y^2+\left((a-\rho)^2-4a(1+\rho)\right)y-4a\rho(1+\rho)=0.
$$

It is evident that $0 < a < 1$ and $\rho > 0$. The points of local maximum or minimum of $h(y)$ are the roots of equation $3y^2 - 4(a-\rho)y + (a-\rho)^2 - 4a(1+\rho) = 0$. One of the roots

$$y_0 = \frac{1}{3}\Big(2(a-\rho) - \big((a-\rho)^2 + 12a(1+\rho)\big)^{\frac{1}{2}}\Big)$$

is obviously less than zero. One can easily show that $h(y_0) > 0$.

Using (3.10) and (3.11), we obtain

$$\Phi(u) = C_1 + C_2 l(k_1)\exp\{k_1 u\} + C_3 l(k_2)\exp\{k_2 u\}, \tag{3.12}$$

where

$$l(k) = 1 + (\lambda_1 + \lambda_3 - \frac{c}{\mu})\frac{\mu k}{\lambda_3} - \frac{c\mu k^2}{\lambda_3}.$$

Boundary conditions (3.9) imply that $C_1 = 1$ and that C_2 and C_3 are the solutions of

$$\begin{aligned} \lambda_1 &= (ck_1 l(k_1) + \lambda_3 - (\lambda_1+\lambda_3)l(k_1))\,C_2 + (ck_2 l(k_2) + \lambda_3 - (\lambda_1+\lambda_3)l(k_2))\,C_3, \\ \lambda_1 + \lambda_3 &= (ck_1 - \lambda_1 - \lambda_3)\,C_2 + (ck_2 - \lambda_1 - \lambda_3)\,C_3. \end{aligned}$$

As an illustration, let $\lambda_1 = 1$, $\lambda_3 = 2$, $\mu = 1$, and $c = 2.1$. Then $k_1 = -0.915174$, $k_2 = -0.05426$, $C_2 = 0.00184602$, $C_3 = -0.966323$. Hence

$$\begin{aligned} \Phi(u) &= 1 - 0.000537651\exp\{-0.915174u\} - 0.939741\exp\{-0.05426u\}, \\ \Phi_1(u) &= 1 + 0.00184602\exp\{-0.915174u\} - 0.966323\exp\{-0.05426u\}. \end{aligned}$$

Example 3.2. Suppose that $\mu_X \neq \mu_Y$. We differentiate (3.6) again to get

$$\begin{aligned} c\Phi_1^{(3)}(u) &= (\lambda_1 + \lambda_3 - \frac{c}{\mu_X} - \frac{c}{\mu_Y})\Phi_1^{(2)}(u) + (\frac{\lambda_1+\lambda_3}{\mu_Y} + \frac{\lambda_3}{\mu_X} - \frac{c}{\mu_X\mu_Y})\Phi_1^{(1)}(u) \\ &\quad + \frac{\lambda_3}{\mu_X\mu_Y}\Phi_1(u) - \frac{\lambda_3}{\mu_Y}\Phi^{(1)}(u) - \frac{\lambda_3}{\mu_X\mu_Y}\Phi(u). \end{aligned} \tag{3.13}$$

This together with (3.5) form a system of linear differential equations with boundary conditions

$$\begin{cases} c\Phi^{(1)}(0) = -\lambda_3\Phi_1(0) + (\lambda_1+\lambda_3)\Phi(0), \\ c\Phi_1^{(1)}(0) = (\lambda_1+\lambda_3)\Phi_1(0), \\ c\Phi_1^{(2)}(0) = (\lambda_1+\lambda_3 - \frac{c}{\mu_Y})\Phi_1^{(1)}(0) + (\frac{\lambda_1+\lambda_3}{\mu_Y} - \frac{\lambda_1}{\mu_X})\Phi_1(0) - \frac{\lambda_3}{\mu_Y}\Phi(0), \\ \Phi(\infty) = 1, \\ \Phi_1(\infty) = 1. \end{cases}$$

The characteristic equation for this differential system is thus

$$z^4 + b_3 z^3 + b_2 z^2 + b_1 z + b_0 = 0,$$

where

$$
\begin{aligned}
b_3 &= \frac{2}{\mu_X} + \frac{1}{\mu_Y} - \frac{2(\lambda_1+\lambda_3)}{c}, \\
b_2 &= \frac{(\lambda_1+\lambda_3)^2}{c^2} + \frac{1}{\mu_X^2} + \frac{2}{\mu_X\mu_Y} - \frac{2(\lambda_1+\lambda_3)}{c\mu_Y} - \frac{2(\lambda_1+2\lambda_3)}{c\mu_X}, \\
b_1 &= \frac{2\lambda_3(\lambda_1+\lambda_3)}{c^2\mu_X} + \frac{\lambda_1^2+2\lambda_1\lambda_3}{c^2\mu_Y} + \frac{1}{\mu_X^2\mu_Y} - \frac{2\lambda_3}{c\mu_X^2} - \frac{2(\lambda_1+2\lambda_3)}{c\mu_X\mu_Y}, \\
b_0 &= \frac{2\lambda_1\lambda_3}{c^2\mu_X\mu_Y} + \frac{\lambda_3^2}{c^2\mu_X^2} - \frac{2\lambda_3}{c\mu_X^2\mu_Y},
\end{aligned}
$$

from which $\Phi(u)$ and $\Phi_1(u)$ can be solved.

4. Dependent Poisson-Erlang Case

This section turns to the correlated case with $K_1(t)$ and $K_3(t)$ being Poisson and $K_2(t)$ being Erlang(2). As before the parameters associated with K_i are denoted by λ_i. From (1.1) and (1.2), it is easy to see that the surplus process (1.3) is distributed the same way as

$$S'(t) = u + ct - \sum_{i=1}^{K_{13}(t)} X_i' - \sum_{i=1}^{K_2(t)} Y_i',$$

where $K_{13}(t) = K_1(t) + K_3(t)$. Also we have

$$X_i' = X_i I(\eta_i = 0) + Y_i I(\eta_i = 1) \qquad \text{and} \qquad Y_i' = X_i + Y_i,$$

where $\{\eta_i, i = 1, 2, \cdots\}$ are binary random variables having probability function

$$\mathrm{P}(\eta_i = 0) = \frac{\lambda_1}{\lambda_1+\lambda_3}, \qquad \text{and} \qquad \mathrm{P}(\eta_i = 1) = \frac{\lambda_3}{\lambda_1+\lambda_3}.$$

Note that $\{X_i', i = 1, 2, \cdots\}$, $\{Y_i', i = 1, 2, \cdots\}$ and $\{\eta_i, i = 1, 2, \cdots\}$ are independent random variables. Denote the distribution functions of X_i' and Y_i' by $F_{X'}$ and $F_{Y'}$, where

$$F_{X'}(x) = \frac{\lambda_1}{\lambda_1+\lambda_3} F_X(x) + \frac{\lambda_3}{\lambda_1+\lambda_3} F_Y(x) \qquad \text{and} \qquad F_{Y'}(x) = F_X * F_Y(x).$$

Thus, in this setup, the risk process with two correlated classes of business can be converted back to a risk process with two independent claim number processes.

Suppose that $F_X(x)$ and $F_Y(x)$ are exponentially distributed with the same mean μ. Then $F_{X'}(x)$ is an exponential distribution with mean μ and $F_{Y'}(x)$ follows an Erlang distribution with mean 2μ. Parallel to (3.3) and (3.5), we obtain

$$\begin{aligned}
c\Phi^{(1)}(u) &= -\lambda_2\Phi_1(u) - (\lambda_1+\lambda_3)\int_0^u \Phi(u-x)\frac{1}{\mu}\exp\left\{\frac{-x}{\mu}\right\}dx + \lambda\Phi(u),\\
c\Phi^{(2)}(u) &= (\lambda - \frac{c}{\mu})\Phi^{(1)}(u) + \frac{\lambda_2}{\mu}\Phi(u) - \lambda_2\Phi_1^{(1)}(u) - \frac{\lambda_2}{\mu}\Phi_1(u), \qquad (4.1)
\end{aligned}$$

where $\lambda = \lambda_1+\lambda_2+\lambda_3$. Similarly, in accordance with (3.4), (3.6), and (3.13), we have

$$\begin{aligned}
c\Phi_1^{(1)}(u) &= -\int_0^u \Phi(u-x)\left(\frac{\lambda_2 x}{\mu^2} + \frac{\lambda_1+\lambda_3}{\mu}\right)\exp\left\{\frac{-x}{\mu}\right\}dx + \lambda\Phi_1(u),\\
c\Phi_1^{(2)}(u) &= (\lambda - \frac{c}{\mu})\Phi_1^{(1)}(u) + \frac{\lambda_2}{\mu}\Phi_1(u) - \frac{\lambda_2}{\mu^2}\int_0^u \Phi(x)\exp\left\{\frac{-(u-x)}{\mu}\right\}dx,\\
c\Phi_1^{(3)}(u) &= (\lambda - \frac{c}{\mu})\Phi_1^{(2)}(u) + \frac{\lambda_2}{\mu}\Phi_1^{(1)}(u) + \frac{\lambda_2}{\mu^3}\int_0^u \Phi(x)\exp\left\{\frac{-(u-x)}{\mu}\right\}dx - \frac{\lambda_2}{\mu^2}\Phi(u)\\
&= (\lambda - \frac{2c}{\mu})\Phi_1^{(2)}(u) + (\frac{\lambda+\lambda_2}{\mu} - \frac{c}{\mu^2})\Phi_1^{(1)}(u) + \frac{\lambda_2}{\mu^2}(\Phi_1(u) - \Phi(u)). \qquad (4.2)
\end{aligned}$$

Hence (4.1) and (4.2) form a linear differential system with boundary conditions

$$\left\{\begin{array}{lcl}
c\Phi^{(1)}(0) &=& -\lambda_2\Phi_1(0) + \lambda\Phi(0),\\
c\Phi_1^{(1)}(0) &=& \lambda\Phi_1(0),\\
c\Phi_1^{(2)}(0) &=& (\lambda - \frac{c}{\mu})\Phi_1^{(1)}(0) + \frac{\lambda_2}{\mu}\Phi_1(0),\\
\Phi(\infty) &=& 1,\\
\Phi_1(\infty) &=& 1.
\end{array}\right. \qquad (4.3)$$

Equations (4.1) and (4.2) yield

$$\begin{aligned}
&c^2\mu^2\Phi_1^{(5)}(u) + c\mu(3c - 2\lambda\mu)\Phi_1^{(4)}(u) + \Big((\lambda\mu - c)(\lambda\mu - 3c) - 2c\mu\lambda_2\Big)\Phi_1^{(3)}(u)\\
&\quad + \Big((\lambda\mu - c)(\lambda + \lambda_2 - \frac{c}{\mu}) - 2c\lambda_2\Big)\Phi_1^{(2)}(u) + 2(\lambda - \frac{c}{\mu})\Phi_1^{(1)}(u) = 0. \qquad (4.4)
\end{aligned}$$

Its characteristic equation

$$\begin{aligned}
&c^2\mu^2 z^5 + c\mu(3c - 2\lambda\mu)z^4 + \Big((\lambda\mu - c)(\lambda\mu - 3c) - 2c\mu\lambda_2\Big)z^3\\
&\quad + \Big((\lambda\mu - c)(\lambda + \lambda_2 - \frac{c}{\mu}) - 2c\lambda_2\Big)z^2 + 2(\lambda - \frac{c}{\mu})z = 0
\end{aligned}$$

has five roots, namely, $z_1 = 0$, $z_2 = -\mu^{-1}$, $z_3 = (c\mu)^{-1}(\lambda\mu - c)$,

$$\begin{aligned}
z_4 &= \frac{1}{2c\mu}\Big((\lambda\mu - c - (8c\mu\lambda_2 + (c - \lambda\mu)^2)^{\frac{1}{2}}\Big),\\
z_5 &= \frac{1}{2c\mu}\Big((\lambda\mu - c + (8c\mu\lambda_2 + (c - \lambda\mu)^2)^{\frac{1}{2}}\Big).
\end{aligned}$$

The positive relative security loading condition, $c > \lambda\mu$, implies that only z_5 is positive. Therefore the general solution of (4.4) is

$$\Phi_1(u) = C_1 + C_2 \exp\{z_2\} + C_3 \exp\{z_3 u\} + C_4 \exp\{z_4 u\}. \tag{4.5}$$

Using (4.1) and (4.5), we get

$$\Phi(u) = C_1 + C_2 q(x_2) \exp\{z_2 u\} + C_3 q(z_3) \exp\{z_3 u\} + C_4 q(z_4) \exp\{z_4 u\},$$

where

$$q(z) = 1 + \frac{\mu}{\lambda_2}(\lambda + \lambda_2 - \frac{c}{\mu})z + \frac{\mu^2}{\lambda_2}(\lambda - \frac{2c}{\mu})z^2 - \frac{c\mu^2}{\lambda_2}z^3.$$

From the boundary conditions (4.3), we immediately get $C_1 = 1$ and the remaining coefficients can be computed by the following equations

$$\begin{aligned}
\lambda &= (cz_2 - \lambda)C_2 + (cz_3 - \lambda)C_3 + (cz_4 - \lambda)C_4, \\
\frac{\lambda_2}{\mu} &= (cz_2^2 - \lambda - \frac{\lambda_2 - c}{\mu})C_2 + (cz_3^2 - \lambda - \frac{\lambda_2 - c}{\mu})C_3 + (cz_4^2 - \lambda - \frac{\lambda_2 - c}{\mu})C_4, \\
\lambda_1 + \lambda_3 &= (cz_2 - \lambda + \lambda_2)q(z_2)C_2 + (cz_3 - \lambda + \lambda_2)q(z_3)C_3 + (cz_4 - \lambda + \lambda_2)q(z_4)C_4.
\end{aligned}$$

Example 4.1. Let $\lambda_1 + \lambda_3 = 3$, $\lambda_2 = 1$, $\mu = 1$, $c = 6$, and the relative security loading, $\rho = 0.5$, then the survival probabilities are

$$\begin{aligned}
\Phi(u) &= 1 - 0.0196911 \exp\{-0.767592u\} - 0.596063 \exp\{-0.333333u\}, \\
\Phi_1(u) &= 1 + 0.0635439 \exp\{-u\} + 0.0847262 \exp\{-0.767592u\} \\
&\quad - 0.894093 \exp\{-0.333333u\}.
\end{aligned}$$

5. Asymptotic Result For General Claim Sizes

In the previous section we have shown how to get the survival (ruin) probabilities for the dependent Poisson-Erlang case when X and Y are exponential. Here we are interested in studying the asymptotic behaviour of the ruin probability when dealing with general claim size distributions. The notations used in Section 4 are still valid in the present situation. Furthermore the means of X' and Y' are denoted by $\mu_{X'}$ and $\mu_{Y'}$.

Integrating (3.3) both sides from 0 to u, we have

$$\Phi(u) = \Phi(0) - \frac{\lambda_2}{c}\int_0^u \Phi_1(s)ds + \frac{\lambda}{c}\int_0^u \Phi(s)ds + \frac{\lambda_1 + \lambda_3}{c}\int_0^u \int_0^s \Phi(s-x)d(1 - F_{X'}(x))ds.$$

Since integration by parts yields

$$\int_0^u \int_0^s \Phi(s-x)d(1 - F_{X'}(x))ds = \int_0^u \Phi(u-x)(1 - F_{X'}(x))dx - \int_0^u \Phi(x)dx,$$

$\Phi(u)$ can be rewritten as

$$\Phi(u) = \Phi(0) + \frac{\lambda_2}{c}\int_0^u (\Phi(x) - \Phi_1(x))dx + \frac{\lambda_1+\lambda_3}{c}\int_0^u \Phi(u-x)(1-F_{X'}(x))dx. \quad (5.1)$$

By the monotone convergence theorem, it follows from (5.1), as $u \to \infty$, that

$$\Phi(\infty) = \Phi(0) + \frac{\lambda_2}{c}\int_0^\infty (\Phi(x) - \Phi_1(x))dx + \frac{(\lambda_1+\lambda_3)\mu_{X'}}{c}\Phi(\infty).$$

Denote the ruin probabilities by $\Psi(u) = 1 - \Phi(u)$ and $\Psi_1(u) = 1 - \Phi_1(u)$. Noting that $\Phi(\infty) = 1$, we obtain

$$\Psi(0) = \frac{(\lambda_1+\lambda_3)\mu_{X'}}{c} + \frac{\lambda_2}{c}\int_0^\infty (\Psi_1(x) - \Psi(x))dx. \quad (5.2)$$

Using (5.1) and (5.2), the ruin probability takes the form

$$\begin{aligned}\Psi(u) &= \frac{\lambda_1+\lambda_3}{c}\left(\int_u^\infty (1-F_{X'}(x))dx + \int_0^u \Psi(u-x)(1-F_{X'}(x))dx\right) \\ &\quad + \frac{\lambda_2}{c}\int_u^\infty (\Psi_1(x) - \Psi(x))dx. \quad (5.3)\end{aligned}$$

Parallel to (5.2) and (5.3), one can use (3.4) to derive

$$\Psi_1(0) = \frac{(\lambda_1+\lambda_3)\mu_{X'}}{c} + \frac{\lambda_2}{c}\left(\mu_{Y'} - \int_0^\infty (\Psi_1(x) - \Psi(x))dx\right), \quad (5.4)$$

and

$$\begin{aligned}\Psi_1(u) &= \frac{\lambda_1+\lambda_3}{c}\left(\int_u^\infty (1-F_{X'}(x))dx + \int_0^u \Psi_1(u-x)(1-F_{X'}(x))dx\right) \\ &\quad + \frac{\lambda_2}{c}\Big(\int_u^\infty (1-F_{Y'}(x))dx + \int_0^u \Psi(u-x)(1-F_{Y'}(x))dx \\ &\qquad - \int_u^\infty (\Psi_1(x) - \Psi(x))dx\Big). \quad (5.5)\end{aligned}$$

From (5.2) and (5.4), it is easy to see that

$$\Psi(0) + \Psi_1(0) = \frac{\lambda_2\mu_{Y'} + 2(\lambda_1+\lambda_3)\mu_{X'}}{c}.$$

Combining (5.3) and (5.5), we get

$$
\begin{aligned}
&\frac{1}{2}(\Psi(u)+\Psi_1(u)) \\
= \;& \frac{1}{2}\Big(\int_u^\infty \Big(\frac{2(\lambda_1+\lambda_3)}{c}(1-F_{X'}(x))+\frac{\lambda_2}{c}(1-F_{Y'}(x))\Big)dx \\
&+\int_0^u \Big(\frac{\lambda_1+\lambda_3}{c}(1-F_{X'}(x))+\frac{\lambda_2}{c}(1-F_{Y'}(x))\Big)\Psi(u-x)dx \\
&+\int_0^u \frac{\lambda_1+\lambda_3}{c}(1-F_{X'}(x))\Psi_1(u-x)dx\Big) \\
\leq \;& \frac{1}{2}\int_u^\infty \Big(\frac{2(\lambda_1+\lambda_3)}{c}(1-F_{X'}(x))+\frac{\lambda_2}{c}(1-F_{Y'}(x))\Big)dx \\
&+\frac{1}{4}\int_0^u \Big(\frac{2(\lambda_1+\lambda_3)}{c}(1-F_{X'}(x))+\frac{\lambda_2}{c}(1-F_{Y'}(x))\Big)\Big(\Psi(u-x)+\Psi_1(u-x)\Big)dx.
\end{aligned}
$$

By the net profit condition,

$$
\frac{1}{2}\int_0^\infty \left(\frac{2(\lambda_1+\lambda_3)}{c}(1-F_{X'}(x))+\frac{\lambda_2}{c}(1-F_{Y'}(x))\right)dx<1.
$$

Hence we can obtain an upper bound for $\Psi(u)+\Psi_1(u)$ by the renewal theorem. Define

$$
h_1(r)=\int_0^\infty \exp\{rx\}dF_{X'}(x)-1, \qquad \text{and} \qquad h_2(r)=\int_0^\infty \exp\{rx\}dF_{Y'}(x)-1.
$$

Assume that there exist $r_1>0$ and $r_2>0$ such that $h_1(r)\uparrow\infty$ when $r\uparrow r_1$ and $h_2(r)\uparrow\infty$ when $r\uparrow r_2$. Then there exists a $R>0$ such that

$$
\frac{1}{2}\int_0^\infty \exp\{Rx\}\left(\frac{2(\lambda_1+\lambda_3)}{c}(1-F_{X'}(x))+\frac{\lambda_2}{c}(1-F_{Y'}(x))\right)dx=1.
$$

In other words, R is the positive solution of the equation

$$
\frac{1}{2}\left(2(\lambda_1+\lambda_3)h_1(r)+\lambda_2h_2(r)\right)=cr.
$$

where ρ is again the relative security loading. Therefore we have the following renewal type of inequality

$$
\begin{aligned}
&\frac{1}{2}\exp\{Rx\}(\Psi(u)+\Psi_1(u)) \\
\leq \;& \frac{1}{2}\exp\{Ru\}\int_u^\infty \left(\frac{2(\lambda_1+\lambda_3)}{c}(1-F_{X'}(x))+\frac{\lambda_2}{c}(1-F_{Y'}(x))\right)dx \\
&+\frac{1}{4}\int_0^u \exp\{Rx\}\Big(\frac{2(\lambda_1+\lambda_3)}{c}(1-F_{X'}(x))+\frac{\lambda_2}{c}(1-F_{Y'}(x))\Big)\cdot \\
&\qquad \exp\{R(u-x)\}\left(\Psi(u-x)+\Psi_1(u-x)\right)dx. \qquad (5.6)
\end{aligned}
$$

Denote

$$H(x) = \frac{1}{2}\left(\frac{2(\lambda_1+\lambda_3)}{c}(1-F_{X'}(x)) + \frac{\lambda_2}{c}(1-F_{Y'}(x))\right).$$

It can be shown by direct integration that

$$\frac{\rho}{1+\rho}\frac{c}{(\lambda_1+\lambda_3)h_1^{(1)}(R)+2^{-1}\lambda_2 h_2^{(1)}(R)-c} = \frac{\int_0^\infty \exp\{Ru\}\int_u^\infty H(x)dxdu}{\int_0^\infty x\exp\{Rx\}H(x)}$$

where $h_1^{(1)}$ and $h_2^{(1)}$ are the first derivatives of h_1 and h_2. Then an application of the renewal theorem to (5.6) gives

$$\lim_{u\to\infty}\exp\{Ru\}\left(\frac{\Psi(u)+\Psi_1(u)}{2}\right) \le \frac{\rho}{1+\rho}\frac{c}{(\lambda_1+\lambda_3)h_1^{(1)}(R)+2^{-1}\lambda_2 h_2^{(1)}(R)-c}.$$

Furthermore it is clear that $\Psi(u) \le \Psi_1(u)$.

Acknowledgements

This research was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 7202/99H).

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