Reserving using the Gaussian approximation to the Cox process with shot noise intensity

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ABSTRACT

We employ the Cox process (or a doubly stochastic Poisson process) to model the claim arrival process for common events. The shot noise process is used for the claim intensity function within the Cox process. The Cox process with shot noise intensity is examined by piecewise deterministic Markov processes theory. Since the claim intensity is not observable we employ state estimation on the basis of the number of claims i.e. we obtain the Kalman-Bucy filter. In order to use the Kalman-Bucy filter, the claim arrival process (i.e. the Cox process) and the claim intensity (i.e. the shot noise process) should be transformed and approximated to two-dimensional Gaussian process. Based on this filter, we derive reserving formulae at any time for common events with and without stop-loss reinsurance contract. We also examine the effect on reserves caused by change in the values of the security loading and the retention limit.

KEYWORDS

The Cox process; Shot noise process; Piecewise deterministic Markov processes theory; The Kalman-Bucy filter; Reserving.

1. INTRODUCTION

Let $\aleph_i$ be the claim amount, which are assumed to be independent and identically distributed with distribution function $H(u)$ $(u > 0)$. The total loss up to time t, C_t is

$$C_t = \sum_{i=1}^{N_t} \aleph_i \tag{1.1}$$

where N_t is the number of claims up to time t. The risk premium (or net premium) at present time 0, assuming that interest rates to be constant, is

$$E(C_t) \tag{1.2}$$

and the gross premium (or risk-loaded premium) at time 0 is

$$(1+q)E(C_t) \tag{1.3}$$

where q (> 0) is the relative security loading.

The total loss excess over b, which is a retention limit, up to time t is

$$\left(C_t - b\right)^+ \tag{1.4}$$

where $\left(C_t - b\right)^+ = Max\left(C_t - b,\ 0\right)$. The stop-loss reinsurance premium at present time 0, assuming that interest rates to be constant, is

$$E\left\{(C_t - b)^+\right\} \tag{1.5}$$

and the gross reinsurance premium (or risk-loaded premium) at time 0 is

$$(1+x)E\left\{(C_t - b)^+\right\} \tag{1.6}$$

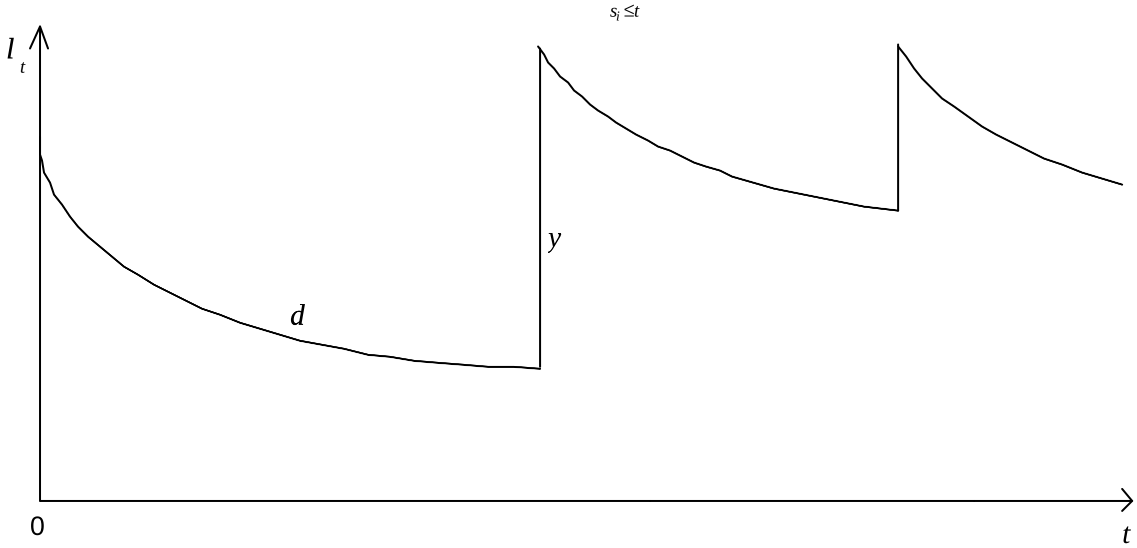
where x (> 0) is the relative security loading for reinsurance contract.

In insurance modelling, the Poisson process has been used as a claim arrival process. Extensive discussion of the Poisson process, from both applied and theoretical viewpoints, can be found in Cramér (1930), Cox & Lewis (1966), Bühlmann (1970), Cinlar (1975), and Medhi (1982). However, there has been a significant volume of literature that questions the appropriateness of the Poisson process in insurance modelling (Seal, 1983 and Beard *et al.*, 1984) and more specifically for rainfall modelling (Smith, 1980 and Cox & Isham, 1986).

For some events such as catastrophes, the assumption that resulting claims occur in terms of the Poisson process is inadequate as it has deterministic intensity. Therefore an alternative point process needs to be used to generate the claim arrival process. We will employ a doubly stochastic Poisson process, or the Cox process (Cox, 1955, Bartlett, 1963, Serfozo, 1972, Grandell, 1976, 1991, Bremaud, 1981 and Lando, 1994).

The shot noise process can be used as the parameter of doubly stochastic Poisson process to measure the number of claims (Cox & Isham, 1980,1986 and Klüppelberg & Mikosch, 1995). As time passes, the shot noise process decreases as more and more claims are settled. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. The shot noise process is particularly useful in the claim arrival process as it measures the frequency, magnitude and time period needed to determine the effect of the common events. Therefore we will use it as a claim intensity function to generate doubly stochastic Poisson process. We will adopt the shot noise process used by Cox & Isham (1980):

$$l_t = l_0 e^{-dt} + \sum_{\substack{\text{all } i \\ s_i \le t}} y_i e^{-d(t-s_i)}$$



where

i primary event

l_0 initial value of l

y_i jump size of primary event i (i.e. magnitude of contribution of primary event i to intensity) where $E(y_i) < \infty$

s_i time at which primary event *i* occurs where $s_i < t < \infty$

d exponential decay which never reaches zero

r the rate of primary event arrival.

The piecewise deterministic Markov processes theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. We present definitions and important properties of the Cox and shot noise processes with the aid of piecewise deterministic processes theory (Dassios, 1987 and Dassios & Embrechts, 1989). This theory is used to calculate the mean of the number of claims and the mean of the claim intensity. These are important factors in the reserving and pricing of any insurance products.

Since the claim intensity is unobservable, the state estimation is employed to derive the distribution of the claim intensity. One of the methods used is the Kalman-Bucy filter. Based on this filter, we derive reserving formulae at any time.

2. DOUBLY STOCHASTIC POISSON PROCESS AND SHOT NOISE PROCESS

Under doubly stochastic Poisson process, or the Cox process, the claim intensity function is assumed to be stochastic. The Cox process is more appropriately used as a claim arrival process as some events should be based on a specific stochastic process.

The doubly stochastic Poisson process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore the doubly stochastic Poisson process can be viewed as a two step randomisation procedure. A process l_t is used to generate another process N_t by acting as its intensity. That is, N_t is a Poisson process conditional on l_t which itself is a stochastic process (if l_t is deterministic then N_t is a Poisson process).

Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Bremaud (1981).

Definition 2.1 *Let (Ω, F, P) be a probability space with information structure F. The information structure F is the filtration, i.e. $F = \{\Im_t,\ t \in [0, T]\}$. F consists of s-algebra's $\Im_t$ on Ω, for any point t in the time interval* $[0, T]$, *representing the information available at time t. Let N_t be a point process adapted to a history $\Im_t \supset \Im_t^N$, where $\Im_t^N$ is the s-algebra generated by the process $N = \{N_t;\ t \geq 0\}$ up to time t. Let l_t be a non- negative process and suppose that l_t is $\Im_t$-measurable, $t \geq 0$ and that*

$$\int_0^t l_s ds < \infty \quad \textit{almost surely (no explosions).}$$

If for all $0 \leq t_1 \leq t_2$ and $u \in \Re$

$$E\left\{ e^{iu\left(N_{t_2} - N_{t_1}\right)} \middle| \Im_{t_2} \right\} = \exp\left\{ \left(e^{iu} - 1\right) \int_{t_1}^{t_2} l_s ds \right\} \tag{2.1}$$

then N_t *is called a* $\Im_t$*-doubly stochastic Poisson process with intensity* l_t.

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Equation (2.1) gives us

$$\Pr\left\{N_{t_2} - N_{t_1} = k \middle| l_s; t_1 \le s \le t_2\right\} = \frac{e^{-\int_{t_1}^{t_2} l_s ds}\left(\int_{t_1}^{t_2} l_s ds\right)^k}{k!} \tag{2.2}$$

Now let us look at the shot noise process described in the previous section. The shot noise process can never reach 0, where the decay is exponential d which is a constant. The frequency of jump arrivals follows a Poisson distribution with r and we will have generally distributed jump sizes with distribution function $G(y)$ $(y > 0)$. If the jump size distribution is exponential, its density is $g(y) = ae^{-ay}$, $y > 0$, $a > 0$.

If l_t is a Markov process, the generator of the process (l_t, t) acting on a function $f(l,t)$ belonging to its domain is given by

$$\mathrm{A}\, f(l,t) = \frac{¶f}{¶t} - d\,l\frac{¶f}{¶l} + r\{\int_0^\infty f(l+y,t)dG(y) - f(l,t)\}. \tag{2.3}$$

It is sufficient that $f(l,t)$ is differentiable w.r.t. l, t for all l, t and that $\left|\int_0^\infty f(l+y,t)dG(y) - f(l,t)\right| < \infty$ for $f(l,t)$ to belong to the domain of the generatorA.

Let us derive the mean and variance of l_t assuming that l_0 is given.

Theorem 2.2 *Let* l_t *be a shot noise process. Assuming that we know* l_0,

$$E\left(l_t \middle| l_0\right) = \frac{m_1 r}{d} + (l_0 - \frac{m_1 r}{d})e^{-dt} \tag{2.4}$$

where $m_1 = \int_0^\infty y dG(y)$.

<u>Proof</u>

Set $f(l) = l$ in (2.3), then

$$\mathrm{A}\, l = -d\,l + m_1 r$$

where $m_1 = \int_0^\infty y dG(y)$.

From $E(l_t|l_0) - l_0 = E[\int_0^t \{\mathrm{A}\, f(l_s)| l_0\}ds]$

$$E\left(l_t \middle| l_0\right) = l_0 - d\int_0^t E\left(l_s \middle| l_0\right)ds + \int_0^t m_1 r\, ds.$$

Differentiate w.r.t t

$$\frac{dE\left(l_t \middle| l_0\right)}{dt} = -dE\left(l_t \middle| l_0\right) + \int_0^t m_1 r ds .$$

Solving the differential equation

$$E\left(l_t \middle| l_0\right) = \frac{m_1 r}{d} + (l_0 - \frac{m_1 r}{d})e^{-dt} .$$

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Lemma 2.3 *Let l_t be as defined. Assuming that we know l_0,*

$$E\left(l_t^2 \middle| l_0\right) = l_0^2 e^{-2dt} + \frac{2m_1 r}{d}(l_0 - \frac{m_1 r}{d})(e^{-dt} - e^{-2dt}) + (\frac{m_1^2 r^2}{d^2} + \frac{m_2 r}{d})(1 - e^{-2dt}) \tag{2.5}$$

where $m_2 = \int_0^\infty y^2 dG(y)$.

Proof

Set $f(l) = l^2$ in (2.3) then

$$\mathrm{A}\, l^2 = -2d l^2 + 2m_1 r l + m_2 r$$

where $m_2 = \int_0^\infty y^2 dG(y)$.

From $E(l_t^2 | l_0) - l_0^2 = E[\int_0^t (\mathrm{A}\, l_s^2 | l_0) ds]$

$$E(l_t^2 | l_0) = l_0^2 - 2d \int_0^t E(l_s^2 | l_0) ds + 2\int_0^t m_1 r E(l_s | l_0) ds + \int_0^t m_2 r ds .$$

Differentiate w.r.t t

$$\frac{dE\left(l_t^2 \middle| l_0\right)}{dt} = -2dE\left(l_t^2 \middle| l_0\right) + 2m_1 r E\left(l_t \middle| l_0\right) + m_2 r .$$

Multiply by e^{2dt}, then

$$\frac{d}{dt}\left[e^{2dt} E\left(l_t^2 \middle| l_0\right)\right] = e^{2dt}\left[2m_1 r E\left(l_t \middle| l_0\right) + m_2 r\right].$$

Solving the differential equation

$$E\left(l_t^2 \middle| l_0\right) = l_0^2 e^{-2dt} + \frac{2m_1 r}{d}(l_0 - \frac{m_1 r}{d})(e^{-dt} - e^{-2dt}) + (\frac{m_1^2 r^2}{d^2} + \frac{m_2 r}{d})(1 - e^{-2dt}).$$

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Corollary 2.4 *Let l_t be as defined. Assuming that we know l_0,*

$$Var\left(l_t \middle| l_0\right) = (1 - e^{-2dt})\frac{m_2 r}{2d}. \tag{2.6}$$

Proof

$$Var(l_t | l_0) = E(l_t^2 | l_0) - \{E(l_t | l_0)\}^2 .$$

Therefore (2.6) follows immediately from (2.5) and (2.4).

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Similarly, the asymptotic (stationary) mean and variance can be obtained from theorem 2.2 and corollary 2.4.

Corollary 2.5 *Let N_t, l_t be as defined. Furthermore if l_t is stationary, that is l_0 has the stationary distribution, then*

$$E(l_t) = \frac{m_1 r}{d}. \tag{2.7}$$

Proof

Let $t \to \infty$ in (2.4) and the corollary follows immediately.

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Corollary 2.6 *Let l_t be as defined. If l_t is stationary then*

$$Var(l_t) = \frac{m_2 r}{2d}. \tag{2.8}$$

Proof

Let $t \to \infty$ in (2.6) and the corollary follows immediately.

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It will be of interest to examine to derive reserving formulae for common events based on the asymptotic (stationary) distribution of the claim intensity (Dassios, 1987, Dassios & Jang, 1998a, 1998b and Jang, 1998, 2000).

3. TRANSFORMATIONS AND APPROXIMATIONS

We have obtained $E(l_t)$ when l_t is stationary. Therefore the mean of the number of claims in a fixed time interval, $E(N_t)$, can be easily found;

$$E(N_t) = \frac{m_1 r}{d} t. \tag{3.1}$$

We can also obtain

$$Var(\int_0^t l_s ds | l_0) = Var(X_t | l_0) = \left\{ \frac{m_2}{d^2} t - \frac{2m_2}{d^3}(1 - e^{-dt}) + \frac{m_2}{2d^3}(1 - e^{-2dt}) \right\} r \tag{3.2}$$

and, when l_t is stationary,

$$Var(\int_0^t l_s ds) = Var(X_t) = \left(\frac{m_2}{d^2} t + \frac{m_2}{d^3} e^{-dt} - \frac{m_2}{d^3} \right) r \tag{3.3}$$

where $X_t = \int_0^t l_s ds$ (*the aggregated process*).

The shot noise process l_t has been taken to be unobservable. However, in practical situations, we observe claims and we want to filter the ‘noise’ out and ‘estimate’ the value of

l_t at any time. This is useful for reserving (and pricing) of insurance contract as it helps us estimate the distribution of l_0 from past data. In order to find an estimate of l_t based on the observations $\{N_s; 0 \le s \le t\}$ we assume r is large and start by transforming the processes l_t, N_t and C_t using

$$Z_t^{(r)} = \frac{l_t - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} \qquad \text{i.e.} \qquad l_t = \frac{m_1 r}{d} + Z_t^{(r)} \sqrt{\frac{m_2 r}{2d}} \tag{3.4}$$

$$W_t^{(r)} = \frac{N_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} \qquad \text{i.e.} \qquad N_t = \frac{m_1 r}{d} t + W_t^{(r)} \sqrt{\frac{m_2 r}{2d}} \tag{3.5}$$

and

$$U_t^{(r)} = \frac{C_t - m_1 \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} \qquad \text{i.e.} \qquad C_t = m_1 \frac{m_1 r}{d} t + U_t^{(r)} \sqrt{\frac{m_2 r}{2d}} \tag{3.6}$$

where $m_1 = \int_0^\infty u\,dH(u)$.

We start with a proposition used by Ethier & Kurtz (1985).

Proposition 3.1 *For $n = 1, 2, \ldots$, let $\{\Im_t^n\}$ be a filtration and let M_n be an $\{\Im_t^n\}$-local martingale with sample paths in $D_{\Re^d}[0,\infty)$ and $M_n(0) = 0$. Let $A_n = ((A_n^{ij}))$ be symmetric $d \times d$ matrix-valued processes such that A_n^{ij} has sample paths in $D_{\Re}[0,\infty)$ and $A_n(t) - A_n(s)$ is nonnegative definite for $0 \le s < t$. Assume that*

$$\lim_{n\to\infty} E\left[\sup_{t \le T} \left| A_n^{ij}(t) - A_n^{ij}(t-) \right| \right] = 0,$$

$$\lim_{n\to\infty} E\left[\sup_{t \le T} \left| M_n(t) - M_n(t-) \right|^2 \right] = 0,$$

and for $i, j = 1, 2, \ldots, d$,

$$M_n^i(t) M_n^j(t) - A_n^{ij}(t)$$

is an $\{\Im_t^n\}$-local martingale.

If for each $t \ge 0$ and $i, j = 1, 2, \ldots, d$,

$$A_n^{ij}(t) \to c_{ij}(t)$$

in probability where $C = ((c_{ij}))$ is a continuous, symmetric, $d \times d$ matrix-valued function, defined on $[0,\infty)$, satisfying $C(0) = 0$ and $\sum (c_{ij}(t) - c_{ij}(s)) x_i x_j \ge 0$, $x \in \Re^d$. Then

$$M_n \Rightarrow X$$

in law where X is a process with independent Gaussian increments such that $X_i X_j - c_{ij}$ are (local) martingales with respect to $\{\Im_t^X\}$.

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Let us now define $V_t^{(r)} = \dfrac{J_t - m_1 r t}{\sqrt{\frac{m_2 r}{2d}}}$, $L_t^{(r)} = \dfrac{N_t - \int_0^t l_s ds}{\sqrt{\frac{m_2 r}{2d}}} = \dfrac{N_t - X_t}{\sqrt{\frac{m_2 r}{2d}}}$ and $Q_t^{(r)} = \dfrac{C_t - m_1 N_t}{\sqrt{\frac{m_2 r}{2d}}}$,

where $J_t = \sum_{i=1}^{M_t} y_i$ and M_t is the total number of primary event's jumps up to time t.

Lemma 3.2 *Let $V_t^{(r)}$, $L_t^{(r)}$ and $Q_t^{(r)}$ be as defined and $r \to \infty$. Then*

$$\begin{bmatrix} V_t^{(r)} \\ L_t^{(r)} \\ Q_t^{(r)} \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{2d}\, B_t^{(1)} \\ \sqrt{\frac{2m_1}{m_2}}\, B_t^{(2)} \\ \sqrt{k_2 \frac{2m_1}{m_2}}\, B_t^{(3)} \end{bmatrix} \tag{3.7}$$

in law where $B_t^{(1)}$, $B_t^{(2)}$ and $B_t^{(3)}$ are three independent standard Brownian motions and $k_2 = \int_0^\infty u^2 dH(u) - \left(\int_0^\infty u dH(u) \right)^2$ (the variance of claim size).

<u>Proof</u>

The generator of the process $V_t^{(r)}$ acting on a function $f(v)$ is given by

$$\mathcal{A}\, f(v) = -\frac{m_1 r}{\sqrt{\frac{m_2 r}{2d}}} \frac{\partial f}{\partial v} + r \{ \int_0^\infty f(v + \frac{y}{\sqrt{\frac{m_2 r}{2d}}}) dG(y) - f(v) \}. \tag{3.8}$$

Set $f(v) = v^2$. Then

$$\mathcal{A}\, v^2 = 2d.$$

The generator of the process $(X_t, N_t, C_t, l_t, J_t, t)$ acting on a function $f(x, n, c, l, j, t)$ is given by

$$\begin{aligned} \mathcal{A}\, f(x,n,c,l,j,t) = {} & \frac{\partial f}{\partial t} + l \frac{\partial f}{\partial x} - d\, l \frac{\partial f}{\partial l} + r \{ \int_0^\infty f(x,n,c,l+y,j+y,t) dG(y) - f(x,n,c,l,j,t) \} \\ & + l \{ \int_0^\infty f(x,n+1,c+u,l,j,t) dH(u) - f(x,n,c,l,j,t) \}. \end{aligned} \tag{3.9}$$

Clearly, for $f(x,n,c,l,j,t)$ to belong to the domain of the generator $\mathcal{A}$, it is essential that $f(x,n,c,l,j,t)$ is differentiable w.r.t. x, c, l, t for all x, n, c, l, j, t and that $\left| \int_0^\infty f(\cdot, l+y, \cdot) dG(y) - f(\cdot, l, \cdot) \right| < \infty$ and $\left| \int_0^\infty f(\cdot, c+u, \cdot) dH(u) - f(\cdot, c, \cdot) \right| < \infty$.

Set $f(x,n,c,l,j,t) = \left(\dfrac{n - x}{\sqrt{\frac{m_2 r}{2d}}} \right)^2$ and $f(x,n,c,l,j,t) = \left(\dfrac{c - m_1 n}{\sqrt{\frac{m_2 r}{2d}}} \right)^2$. Then

$$\mathcal{A} \left(\frac{n - x}{\sqrt{\frac{m_2 r}{2d}}} \right)^2 = \frac{2d}{m_2} \frac{l}{r} \quad \text{and} \quad \mathcal{A} \left(\frac{c - m_1 n}{\sqrt{\frac{m_2 r}{2d}}} \right)^2 = k_2 \frac{2d}{m_2} \frac{l}{r}$$

where $m_1 = \int_0^\infty u dH(u)$, $m_2 = \int_0^\infty u^2 dH(u)$ and $k_2 = m_2 - m_1^2$.

$f(X_t) - \int_0^t \mathrm{A}\, f(X_s) ds$ is a martingale therefore $\mathrm{A} f$ is the solution to the 'martingale problem'. Hence $\left(V_t^{(r)}\right)^2 - 2dt$, $\left(L_t^{(r)}\right)^2 - \int_0^t \frac{2d}{m_2}\frac{l_s}{r} ds$ and $\left(Q_t^{(r)}\right)^2 - \int_0^t k_2 \frac{2d}{m_2}\frac{l_s}{r} ds$ are martingales.

As can be seen from (3.3), $Var(\int_0^t l_s ds) = K(t)r$. Therefore, by Chebyschev's inequality, as $r \to \infty$

$$\Pr\left\{\left|\int_0^t \frac{2d}{m_2}\frac{l_s}{r} ds - \frac{2m_1}{m_2} t\right| > e\right\} \le \frac{(\frac{2d}{m_2})^2 Var(\int_0^t l_s ds)}{r^2 e^2} = \frac{(\frac{2d}{m_2})^2 K(t) r}{r^2 e^2} \to 0 \tag{3.10}$$

and

$$\Pr\left\{\left|k_2 \int_0^t \frac{2d}{m_2}\frac{l_s}{r} ds - k_2 \frac{2m_1}{m_2} t\right| > e\right\} \le \frac{k_2^2 (\frac{2d}{m_2})^2 Var(\int_0^t l_s ds)}{r^2 e^2} = \frac{k_2^2 (\frac{2d}{m_2})^2 K(t) r}{r^2 e^2} \to 0. \tag{3.11}$$

Therefore from (3.10) and (3.11)

$$\int_0^t \frac{2d}{m_2}\frac{l_s}{r} ds \;\to\; \frac{2m_1}{m_2} t$$

and

$$\int_0^t k_2 \frac{2d}{m_2}\frac{l_s}{r} ds \;\to\; k_2 \frac{2m_1}{m_2} t$$

in probability.

Set $f(x,n,c,l,j,t) = \left(\frac{n-x}{\sqrt{\frac{m_2 r}{2d}}}\right)\left(\frac{j - m_1 rt}{\sqrt{\frac{m_2 r}{2d}}}\right)$, $f(x,n,c,l,j,t) = \left(\frac{c - m_1 n}{\sqrt{\frac{m_2 r}{2d}}}\right)\left(\frac{j - m_1 rt}{\sqrt{\frac{m_2 r}{2d}}}\right)$ and $f(x,n,c,l,j,t) = \left(\frac{c - m_1 n}{\sqrt{\frac{m_2 r}{2d}}}\right)\left(\frac{n-x}{\sqrt{\frac{m_2 r}{2d}}}\right)$. Then

$$\mathrm{A}\left(\frac{n-x}{\sqrt{\frac{m_2 r}{2d}}}\right)\left(\frac{j - m_1 rt}{\sqrt{\frac{m_2 r}{2d}}}\right) = 0,\ \mathrm{A}\left(\frac{c - m_1 n}{\sqrt{\frac{m_2 r}{2d}}}\right)\left(\frac{j - m_1 rt}{\sqrt{\frac{m_2 r}{2d}}}\right) = 0 \text{ and } \mathrm{A}\left(\frac{c - m_1 n}{\sqrt{\frac{m_2 r}{2d}}}\right)\left(\frac{n-x}{\sqrt{\frac{m_2 r}{2d}}}\right) = 0. \tag{3.12}$$

Hence from proposition 3.1,

$$V_t^{(r)} = \frac{J_t - m_1 rt}{\sqrt{\frac{m_2 r}{2d}}} \;\Rightarrow\; \sqrt{2d}\,B_t^{(1)} \tag{3.13}$$

$$L_t^{(r)} = \frac{N_t - \int_0^t l_s\,ds}{\sqrt{\frac{m_2 r}{2d}}} \;\Rightarrow\; \sqrt{\frac{2m_1}{m_2}}B_t^{(2)} \tag{3.14}$$

and

$$Q_t^{(r)} = \frac{C_t - m_1 N_t}{\sqrt{\frac{m_2 r}{2d}}} \;\Rightarrow\; \sqrt{k_2 \frac{2m_1}{m_2}}B_t^{(3)} \tag{3.15}$$

in law where $B_t^{(1)}$, $B_t^{(2)}$ and $B_t^{(3)}$ are three independent standard Brownian motions. Therefore (3.7) follows immediately from (3.12), (3.13), (3.14) and (3.15).

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Let us now prove the main result of this section.

Theorem 3.3 *Let $Z_t^{(r)}$, $W_t^{(r)}$ and $U_t^{(r)}$ be as defined and $r \to \infty$. Then $Z_t^{(r)}$, $W_t^{(r)}$ and $U_t^{(r)}$ converge in law to Z_t, W_t and U_t where*

$$dZ_t = -dZ_t dt + \sqrt{2d}\,dB_t^{(1)} \tag{3.16}$$

$$dW_t = Z_t dt + \sqrt{\frac{2m_1}{m_2}}dB_t^{(2)} \tag{3.17}$$

$$dU_t = m_1 dW_t + \sqrt{k_2 \frac{2m_1}{m_2}}dB_t^{(3)} = m_1 Z_t dt + \sqrt{m_2 \frac{2m_1}{m_2}}dB_t^{(4)} \tag{3.18}$$

where $B_t^{(1)}$, $B_t^{(2)}$, $B_t^{(3)}$ are three independent standard Brownian motions and $B_t^{(4)} = \dfrac{m_1\sqrt{\frac{2m_1}{m_2}}B_t^{(2)} + \sqrt{k_2\frac{2m_1}{m_2}}B_t^{(3)}}{\sqrt{(m_1^2 + k_2)\frac{2m_1}{m_2}}}$ *(also a standard Brownian motion).*

<u>Proof</u>

$Z_t^{(r)}$, $W_t^{(r)}$ and $U_t^{(r)}$ can be written as

$$Z_t^{(r)} = \frac{l_t - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} = \frac{(l_0 - \frac{m_1 r}{d})e^{-dt}}{\sqrt{\frac{m_2 r}{2d}}} + \frac{J_t - m_1 rt}{\sqrt{\frac{m_2 r}{2d}}} - de^{-d(t-u)}\int_0^t \frac{J_u - m_1 ru}{\sqrt{\frac{m_2 r}{2d}}}du \tag{3.19}$$

$$W_t^{(r)} = \frac{N_t - \frac{m_1 r}{d}t}{\sqrt{\frac{m_2 r}{2d}}} = \frac{N_t - \int_0^t l_s ds}{\sqrt{\frac{m_2 r}{2d}}} + \int_0^t \frac{l_s - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}}ds \tag{3.20}$$

and

$$U_t^{(r)} = \frac{C_t - m_1 \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} = \frac{C_t - m_1 N_t}{\sqrt{\frac{m_2 r}{2d}}} + m_1\left(\frac{N_t - \frac{m_1 r}{d}t}{\sqrt{\frac{m_2 r}{2d}}}\right). \tag{3.21}$$

Therefore by continuous mapping theorem (see Billingsley (1968)) and lemma 3.2, (3.19), (3.20) and (3.21) converge to

$$Z_t = Z_0 e^{-dt} + \sqrt{2d}\int_0^t e^{-d(t-s)} dB_s^{(1)} \tag{3.22}$$

$$W_t = \int_0^t Z_s ds + \sqrt{\frac{2m_1}{m_2}} B_t^{(2)} \tag{3.23}$$

and

$$U_t = m_1 W_t + \sqrt{k_2 \frac{2m_1}{m_2}} B_t^{(3)}. \tag{3.24}$$

From (3.23) and (3.24), we have

$$dU_t = m_1 dW_t + \sqrt{k_2 \frac{2m_1}{m_2}} dB_t^{(3)} = m_1 Z_t dt + m_1 \sqrt{\frac{2m_1}{m_2}} dB_t^{(2)} + \sqrt{k_2 \frac{2m_1}{m_2}} dB_t^{(3)}. \tag{3.25}$$

Since the sum of two independent standard Brownian motions is also a standard Brownian motion this completes the proof of the theorem.

•

Theorem 3.3 has proved that Z_t, W_t and U_t are normally distributed. As a result of this, we define $\tilde{l}_t$, $\tilde{N}_t$ and $\tilde{C}_t$ as Gaussian approximations of l_t, N_t and C_t;

$$\tilde{l}_t = \frac{m_1 r}{d} + Z_t \sqrt{\frac{m_2 r}{2d}} \quad \text{i.e.} \quad Z_t = \frac{\tilde{l}_t - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} \tag{3.26}$$

$$\tilde{N}_t = \frac{m_1 r}{d} t + W_t \sqrt{\frac{m_2 r}{2d}} \quad \text{i.e.} \quad W_t = \frac{\tilde{N}_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} \tag{3.27}$$

and

$$\tilde{C}_t = m_1 \frac{m_1 r}{d} t + U_t \sqrt{\frac{m_2 r}{2d}} \quad \text{i.e.} \quad U_t = \frac{\tilde{C}_t - m_1 \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}}. \tag{3.28}$$

4. THE KALMAN-BUCY FILTER AND THE DISTRIBUTION OF Z_t

We will derive the conditional distribution of Z_t, given $\{W_s; 0 \le s \le t\}$, by the Kalman-Bucy filter where

$$dZ_t = -dZ_t dt + \sqrt{2d}\, dB_t^{(1)} \tag{4.1}$$

and

$$dW_t = Z_t dt + \sqrt{\frac{2m_1}{m_2}} dB_t^{(2)}. \tag{4.2}$$

Let us begin with a proposition used by Øksendal (1992) (see theorem 6.10 in chapter IV).

Proposition 4.1 *The solution* $\hat{Z}_t = E(Z_t | W_s; 0 \le s \le t)$ *of the 1-dimensional linear filtering problem*

$$dZ_t = F(t)Z_t dt + C(t)dB_t^{(1)}; \;\; F(t), C(t) \in \Re \tag{4.3}$$

$$dW_t = G(t)Z_t dt + D(t)dB_t^{(2)}; \;\; G(t), D(t) \in \Re \tag{4.4}$$

satisfies the stochastic differential equation

$$d\hat{Z}_t = \{F(t) - \frac{G^2(t)S(t)}{D^2(t)}\}\hat{Z}_t\, dt + \frac{G(t)S(t)}{D^2(t)} dW_t; \quad \hat{Z}_0 = E(Z_0) \tag{4.5}$$

where $S(t) = E\left\{\left(Z_t - \hat{Z}_t\right)^2\right\}$ *satisfies the Riccati equation*

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)} S^2(t) + C^2(t), \;\; S(0) = E\left[\{Z_0 - E(Z_0)\}^2\right] = Var(Z_0). \tag{4.6}$$

•

Theorem 4.2 *Let* (Z_t, W_t) *be a two-dimensional normal process satisfying the system of equations of* (4.1) *and* (4.2). *Then the estimate of* Z_t *based on the observed* $\{W_s; 0 \le s \le t\}$ *is*

$$\hat{Z}_t = E(Z_t | W_s; 0 \le s \le t) = \exp\{\int_0^t \Psi(s)ds\}\hat{Z}_0 + \frac{m_2}{2m_1}\int_0^t \exp\{\int_s^t \Psi(u)du\} S(s) dW_s \tag{4.7}$$

where

$$S(0) = a^2,$$

$$S(s) = \frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}\left\{1 + \frac{a^2 + \frac{2dm_1}{m_2} + \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2 + \frac{2dm_1}{m_2} - \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}}s\right)\right\}}{\frac{a^2 + \frac{2dm_1}{m_2} + \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2 + \frac{2dm_1}{m_2} - \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}}s\right) - 1} - 2d\frac{m_1}{m_2} \tag{4.8}$$

and

$$\Psi(s) = -\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}\left\{1 + \frac{a^2 + \frac{2dm_1}{m_2} + \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2 + \frac{2dm_1}{m_2} - \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}}s\right)\right\}}{\frac{2m_1}{m_2}\left\{\frac{a^2 + \frac{2dm_1}{m_2} + \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2 + \frac{2dm_1}{m_2} - \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}}s\right) - 1\right\}}. \tag{4.9}$$

Proof

Let $S(0) = a^2$. Then from (4.6), the Riccati equation has the solution

$$S(t)=\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}\left\{1+\frac{a^2+\frac{2dm_1}{m_2}+\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2+\frac{2dm_1}{m_2}-\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}}t\right)\right\}}{\frac{a^2+\frac{2dm_1}{m_2}+\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2+\frac{2dm_1}{m_2}-\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}}t\right)-1}-2d\,\frac{m_1}{m_2}. \tag{4.10}$$

Therefore from (4.5), (4.10) offers the solution for $\hat{Z}_t$ of the form

$$\hat{Z}_t=E\left(Z_t\middle|W_s;0\le s\le t\right)=\exp\{\int_0^t\Psi(s)ds\}\hat{Z}_0+\frac{m_2}{2m_1}\int_0^t\exp\{\int_s^t\Psi(u)du\}S(s)dW_s$$

where

$$\Psi(s)=-\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}\left\{1+\frac{a^2+\frac{2dm_1}{m_2}+\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2+\frac{2dm_1}{m_2}-\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}}s\right)\right\}}{\frac{2m_1}{m_2}\left\{\frac{a^2+\frac{2dm_1}{m_2}+\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2+\frac{2dm_1}{m_2}-\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}}s\right)-1\right\}}.$$

•

We have obtained $E\left(Z_t\middle|W_s;0\le s\le t\right)=\hat{Z}_t$. Now let us try to find the distribution of Z_t.

Corollary 4.3 *Let Z_t, W_t, $\hat{Z}_t$ and $S(t)$ be as defined. Then*

$$E\left(e^{-gZ_t}\middle|W_s;0\le s\le t\right)=\exp\left\{-g\,\hat{Z}_t+\frac{1}{2}g^2S(t)\right\}. \tag{4.11}$$

Proof

From theorem 4.2 and the fact that $Var\left(Z_t\middle|W_s;0\le s\le t\right)=S(t)$ and Z_t is normally distributed, given $W_s;0\le s\le t$, the result follows immediately.

•

5. RESERVING BASED ON THE KALMAN-BUCY FILTER

Let R_T denote the free reserve (or surplus) of the insurer at time T. We assume that premiums are received continuously at a constant rate k (>0) and let R_t denote the reserve required on hand at time t. Then

$$R_T=R_t+k(T-t)-(C_T-C_t),\quad 0\le t\le T \tag{5.1}$$

where $C_T-C_t=\sum_{i=1}^{N_T-N_t}\aleph_i$.

The surplus R_T must be non-negative if the insurer is able to meet his liabilities. Therefore it is very important that the probability that R_T becomes negative be small (e say), i.e.

$$\Pr(R_T < 0) = e \tag{5.2}$$

or

$$\Pr\left[C_T - C_t \ > \ R_t + k(T-t)\right] = e\ . \tag{5.3}$$

Since we have obtained $\tilde{C}_t$ which is Gaussian approximation of C_t, we will use this approximation, then

$$\Pr\left[\frac{\tilde{C}_T - \tilde{C}_t - E(\tilde{C}_T - \tilde{C}_t)}{\sqrt{Var(\tilde{C}_T - \tilde{C}_t)}} \ > \ \frac{R_t + k(T-t) - E(\tilde{C}_T - \tilde{C}_t)}{\sqrt{Var(\tilde{C}_T - \tilde{C}_t)}}\right] = e\ . \tag{5.4}$$

Set $k(T-t) = (1+q)E(\tilde{C}_T - \tilde{C}_t)$ and $e = 5\%$, then

$$R_t = 1.645\sqrt{Var(\tilde{C}_T - \tilde{C}_t)} - q\,E(\tilde{C}_T - \tilde{C}_t)\,. \tag{5.5}$$

Considering the information available (i.e. the observed claims) up to time t,

$$R_t = 1.645\sqrt{Var\left(\tilde{C}_T - \tilde{C}_t \,\middle|\, \tilde{N}_s; 0 \le s \le t\right)} - q\,E\left(\tilde{C}_T - \tilde{C}_t \,\middle|\, \tilde{N}_s; 0 \le s \le t\right). \tag{5.6}$$

Now let us examine how the above equation would be altered if the direct insurer purchase a stop-loss reinsurance contract that covers the total loss excess over b, a retention limit. Let R_T^h denote the free reserve of the insurer after purchasing reinsurance contract at time T. We assume that reinsurance premiums are paid continuously at a constant rate $k_h\ (>0)$ and let R_t^h denote the reserve required on hand at time t. Then

$$R_T^h = R_t^h + k(T-t) - k_h(T-t) - (C_T^I - C_t^I), \quad 0 \le t \le T \tag{5.7}$$

where $C_T^I - C_t^I = \begin{cases} C_T - C_t, & C_T - C_t \le b \\ b, & C_T - C_t > b \end{cases}$.

Similar to the equation (5.4), set $k_h(T-t) = (1+x)E\left[\left\{(\tilde{C}_T - \tilde{C}_t) - b\right\}^+\right]$ and $e = 5\%$, then considering the information available (i.e. the observed claims) up to time t,

$$\begin{aligned} R_t^h = 1.645\sqrt{Var\left(\tilde{C}_T^I - \tilde{C}_t^I \,\middle|\, \tilde{N}_s; 0 \le s \le t\right)} - q\,E\left(\tilde{C}_T - \tilde{C}_t \,\middle|\, \tilde{N}_s; 0 \le s \le t\right) \\ + x\,E\left[\left\{(\tilde{C}_T - \tilde{C}_t) - b\right\}^+ \,\middle|\, \tilde{N}_s; 0 \le s \le t\right]. \end{aligned} \tag{5.8}$$

If $x = q$, the equation (5.8) becomes

$$R_t^h = 1.645\sqrt{Var\left(\tilde{C}_T^I - \tilde{C}_t^I \,\middle|\, \tilde{N}_s; 0 \le s \le t\right)} - q\,E\left(\tilde{C}_T^I - \tilde{C}_t^I \,\middle|\, \tilde{N}_s; 0 \le s \le t\right). \tag{5.9}$$

From (5.6) and (5.8), we can see that mean and variance of $\tilde{C}_T - \tilde{C}_t$ need to be determined to obtain the reserve required at any time t. Therefore set $\tilde{C}_t = m_1 \frac{m_1 r}{d} t + U_t \sqrt{\frac{m_2 r}{2d}}$ in the expectation then

$$E\left[\tilde{C}_T - \tilde{C}_t \,\middle|\, \tilde{N}_s; 0 \le s \le t\right] = E\left[\sqrt{\frac{m_2 r}{2d}}(U_T - U_t) + m_1 \frac{m_1 r}{d}(T-t) \,\middle|\, W_s; 0 \le s \le t\right]. \tag{5.10}$$

From (5.10), we can also see that mean and variance of $U_T - U_t$ need to be determined to obtain the above expectation.

Lemma 5.1 *Let* Z_t, W_t, U_t *and* $\hat{Z}_t$ *be as defined. Then*

$$\Gamma = E\left(U_T - U_t \middle| W_s; 0 \le s \le t\right) = m_1 \frac{1 - e^{-d(T-t)}}{d} \hat{Z}_t \tag{5.11}$$

and

$$\Theta = Var\left(U_T - U_t \middle| W_s; 0 \le s \le t\right)$$

$$= \left(\frac{m_1}{d}\right)^2 \left\{\left(1 - e^{-d(T-t)}\right)^2 S(t) - e^{-2d(T-t)} + 4e^{-d(T-t)} - 3\right\} + 2\left(\frac{m_1^2}{d} + \frac{m_2 m_1}{m_2}\right)(T-t). \tag{5.12}$$

Proof

From (3.18)

$$U_T - U_t = m_1 \int_t^T Z_s ds + \sqrt{m_2 \frac{2m_1}{m_2}} \int_t^T dB_s^{(4)}. \tag{5.13}$$

Set (3.22) in (5.13) then

$$U_T - U_t = m_1 \frac{1 - e^{-d(T-t)}}{d} Z_t + m_1 \sqrt{2d} \int_t^T \frac{1 - e^{-d(T-u)}}{d} dB_u^{(1)} + \sqrt{m_2 \frac{2m_1}{m_2}} \int_t^T dB_s^{(4)}. \tag{5.14}$$

Take expectation in (5.14) then (5.11) follows immediately.

$$Var\left(U_T - U_t \middle| W_s; 0 \le s \le t\right) = E\left\{\left(U_T - U_t\right)^2 \middle| W_s; 0 \le s \le t\right\} - E\left\{\left(U_T - U_t \middle| W_s; 0 \le s \le t\right)\right\}^2. \tag{5.15}$$

Therefore (5.12) follows immediately from (5.14) and (5.11).

•

As results of lemma 5.1, we can easily obtain

$$\Omega = E\left[\tilde{C}_T - \tilde{C}_t \,\middle|\, \tilde{N}_s; 0 \le s \le t\right] = \sqrt{\frac{m_2 r}{2d}} \Gamma + \frac{m_1 m_1 r}{d}(T-t) \tag{5.16}$$

and

$$\Sigma = Var\left[\tilde{C}_T - \tilde{C}_t \,\middle|\, \tilde{N}_s; 0 \le s \le t\right] = \frac{m_2 r}{2d} \Theta. \tag{5.17}$$

Therefore we can obtain the reserve required on hand at time t by

$$R_t = 1.645\sqrt{\Sigma} - q\Omega. \tag{5.18}$$

Now let us find the stop-loss reinsurance premium at time t based on the observations $\left\{\tilde{N}_s; 0 \le s \le t\right\}$ to obtain the reserve required on hand at time t after purchasing a stop-loss reinsurance contract.

Theorem 5.2 *Let* Z_t *,*W_t*,* $\hat{Z}_t$ *and* $S(t)$ *be as defined. Then*

$$\Omega^h = E\left[\{(\tilde{C}_T - \tilde{C}_t) - b\}^+ \middle| \tilde{N}_s; 0 \le s \le t\right] = \sqrt{\frac{m_2 r\Theta}{4dp}} e^{-\frac{1}{2}L^2} + \left\{\sqrt{\frac{m_2 r}{2d}}\Gamma + \frac{m_1 m_1 r}{d}(T-t) - b\right\}\Phi(-L) \tag{5.19}$$

where $L = \dfrac{K - \Gamma}{\sqrt{\Theta}}$, $K = \dfrac{b - m_1 \frac{m_1 r}{d}(T-t)}{\sqrt{\frac{m_2 r}{2d}}}$ *and* $\Phi(\cdot)$ *is the cumulative normal distribution function.*

<u>Proof</u>

Similar to (5.10),

$$E\left[\{(\tilde{C}_T - \tilde{C}_t) - b\}^+ \middle| W_s; 0 \le s \le t\right] = E\left[\left\{\sqrt{\frac{m_2 r}{2d}}(U_T - U_t) + m_1 \frac{m_1 r}{d}(T-t) - b\right\}^+ \middle| W_s; 0 \le s \le t\right]$$

$$= \int_K^{\infty} \left\{\sqrt{\frac{m_2 r}{2d}} u + m_1 \frac{m_1 r}{d}(T-t) - b\right\} \frac{1}{\sqrt{2p\Theta}} e^{-\frac{1}{2}\frac{(u-\Gamma)^2}{\Sigma}} du \tag{5.20}$$

where $K = \dfrac{b - m_1 \frac{m_1 r}{d}(T-t)}{\sqrt{\frac{m_2 r}{2d}}}$.

Set $y = \dfrac{u - \Gamma}{\sqrt{\Theta}}$ in (5.20) and put $L = \dfrac{K - \Gamma}{\sqrt{\Theta}}$ then (5.19) follows immediately.

•

Let us also find the second moment of $\tilde{C}_T^I - \tilde{C}_t^I$ to obtain the variance of $\tilde{C}_T^I - \tilde{C}_t^I$.

Corollary 5.3 *Let* Z_t *,*W_t*,* $\hat{Z}_t$ *and* $S(t)$ *be as defined. Then*

$$E\left[\left(\tilde{C}_T^I - \tilde{C}_t^I\right)^2 \middle| \tilde{N}_s; 0 \le s \le t\right]$$

$$= \frac{m_2 r\Theta}{2d\sqrt{2p}}(Ae^{-\frac{1}{2}A^2} - Le^{-\frac{1}{2}L^2}) + \left\{\frac{m_2 r\Gamma\sqrt{\Theta}}{d} + 2\sqrt{\frac{m_2 r\Theta}{2d}}\frac{m_1 m_1 r}{d}(T-t)\right\}(e^{-\frac{1}{2}A^2} - e^{-\frac{1}{2}L^2})$$

$$+ \left\{\frac{m_2 r}{2d}\left(\Gamma^2 + \Theta\right) + 2\sqrt{\frac{m_2 r}{2d}}\Gamma\frac{m_1 m_1 r}{d}(T-t) + \left(\frac{m_1 m_1 r}{d}\right)^2 (T-t)^2\right\}\{\Phi(-A) - \Phi(-L)\}$$

$$+b^2\Phi(-L) \tag{5.21}$$

where $A=\frac{B-\Gamma}{\sqrt{\Theta}}$, $B=\frac{-m_1\frac{m_1 r}{d}(T-t)}{\sqrt{\frac{m_2 r}{2d}}}$ *and* $\Phi(\cdot)$ *is the cumulative normal distribution function.*

Proof

From $C_T^I - C_t^I = \begin{cases} C_T - C_t, & C_T - C_t \le b \\ b, & C_T - C_t > b \end{cases}$,

$$E\left[\left(\tilde{C}_T^I - \tilde{C}_t^I\right)^2 \middle| \tilde{N}_s; 0 \le s \le t\right]$$

$$=\int_B^K \left\{\sqrt{\frac{m_2 r}{2d}}u + m_1\frac{m_1 r}{d}(T-t)\right\}^2 \frac{1}{\sqrt{2p\Theta}} e^{-\frac{1}{2}\frac{(u-\Gamma)^2}{\Sigma}} du + \int_K^\infty b^2 \frac{1}{\sqrt{2p\Theta}} e^{-\frac{1}{2}\frac{(u-\Gamma)^2}{\Sigma}} du \tag{5.22}$$

where $B=\frac{-m_1\frac{m_1 r}{d}(T-t)}{\sqrt{\frac{m_2 r}{2d}}}$.

Set $y=\frac{u-\Gamma}{\sqrt{\Theta}}$ in (5.22) and put $A=\frac{B-\Gamma}{\sqrt{\Theta}}$, $L=\frac{K-\Gamma}{\sqrt{\Theta}}$ then (5.21) follows immediately.

•

As the result of corollary 5.3, the variance of $\tilde{C}_T^I - \tilde{C}_t^I$, denoted by Σ^I, is

$$\Sigma^I = Var\left(\tilde{C}_T^I - \tilde{C}_t^I \middle| \tilde{N}_s; 0 \le s \le t\right) = E\left[\left(\tilde{C}_T^I - \tilde{C}_t^I\right)^2 \middle| \tilde{N}_s; 0 \le s \le t\right] - \left\{E\left(\tilde{C}_T^I - \tilde{C}_t^I \middle| \tilde{N}_s; 0 \le s \le t\right)\right\}^2$$

$$= E\left[\left(\tilde{C}_T^I - \tilde{C}_t^I\right)^2 \middle| \tilde{N}_s; 0 \le s \le t\right] - \left[E\left(\tilde{C}_T - \tilde{C}_t \middle| \tilde{N}_s; 0 \le s \le t\right) - E\left[\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b\right\}^+ \middle| \tilde{N}_s; 0 \le s \le t\right]\right]^2.$$

Therefore the reserve required on hand at time t, R_t^h, after purchasing a stop-loss reinsurance contract can be obtained by

$$R_t^h = 1.645\sqrt{\Sigma^I} - q\Omega + x\Omega^h \tag{5.23}$$

and if $x = q$, it becomes

$$R_t^h = 1.645\sqrt{\Sigma^I} - q\Omega^I \tag{5.24}$$

where $\Omega^I = E\left(\tilde{C}_T^I - \tilde{C}_t^I \middle| \tilde{N}_s; 0 \le s \le t\right)$.

Now let us illustrate the calculation of reserves using the formulae derived above.

Example 5.1

The numerical values used to simulate the claim arrival process are $d = 0.5$, $l_0 = 200$. We will assume that $r = 100$ i.e. the interarrival time between jumps is exponential with mean 0.01 and that the jump size follows exponential with mean 1, i.e. $y \sim Exponential(1)$. *S-Plus* was used to generate random values and to simulate the claim arrival process.

The numerical values used to calculate (4.7) and (5.18) are $\hat{Z}_0 = 0$, $S(0) = 0$, $m_1 = 1$, $m_2 = 2$, $m_1 = 1$, $m_2 = 3$, $t = 1$, $T = 2$ and $q = 0, 0.1, 0.2, 0.2129$. By computing (4.7) and (5.18) using *MAPLE* and *S-Plus*, where $\hat{Z}_1 = 0.5579152$, the calculation of the reserves at each security loading q are shown in Table 5.1.

Table 5.1

Security loading q	Reserve R_t at $e = 5\%$
0.0	43.903
0.1	23.282
0.2	2.661
0.2129	0

Example 5.2

We will now examine the effect on reserves caused by purchasing a stop-loss reinsurance contract. The reserves at the retention limit $b = 0, 270, 280, \infty$ are shown in Table 5.2 ($q = x = 0.1$) and in Table 5.3 ($q = 0.1$ and $x = 0.2$).

Table 5.2

Retention level b	Reserve R_t^h at $e = 5\%$
0	0
270	8.874
280	18.442
∞	23.282

Table 5.3

Retention level b	Reserve R_t^h at $e = 5\%$
0	20.621
270	8.889
280	18.447
∞	23.282

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