Reserving using the Gaussian approximation to the Cox process with shot noise process

Overview

- the Cox process (or doubly stochastic Poisson process), N_t using the shot noise process, l_t as the intensity function.

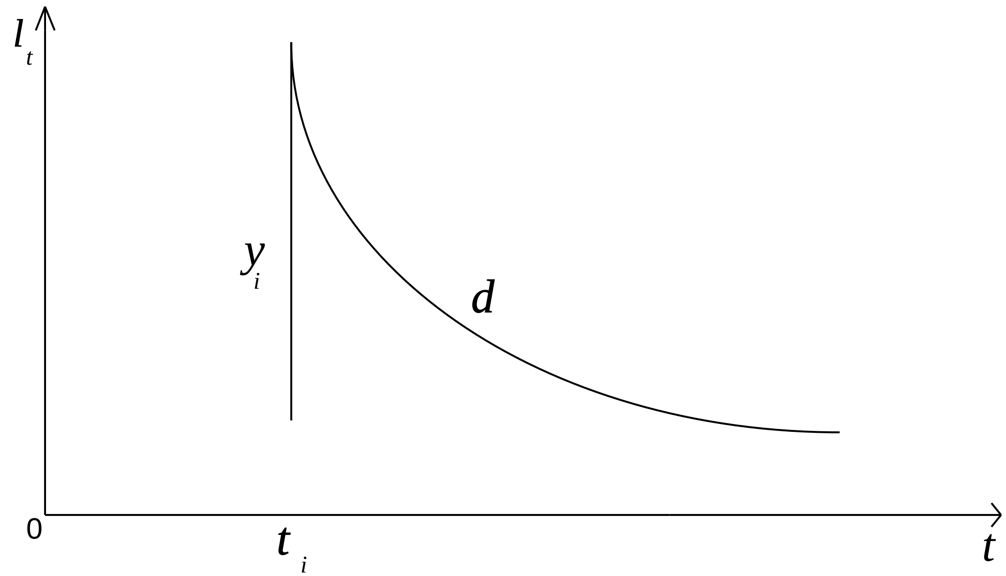
- transformation and approximation of the shot noise process, l_t, the Cox process, N_t and the total amount of claims up to time *t*, C_t, as Gaussian processes Z_t, W_t and U_t.

- Kalman-Busy filtering for linear system driven by the Cox process with shot noise intensity, i.e. deriving the conditional distribution of Z_t, given $\{W_s; 0 \leq s \leq t\}$.

- reserving based on the filter.

- illustration of this reserving model.

Illustration of the claim intensity function for a primary event



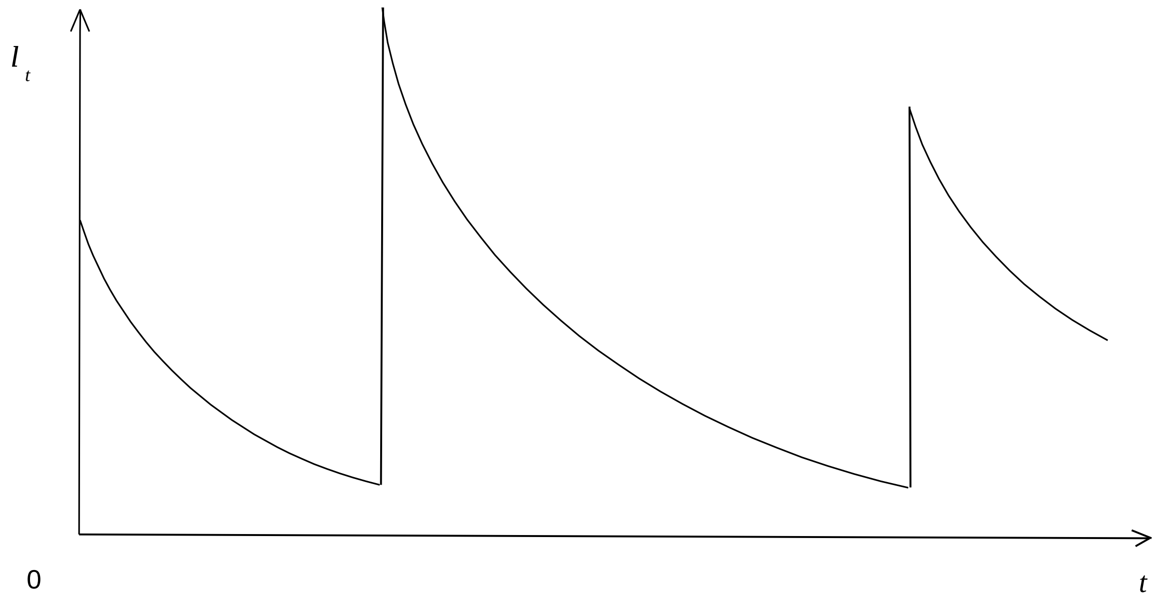
where:
i is a primary event, y_i is jump size of primary event I (i.e. magnitude of contribution of primary event i to intensity), t_i is time at which primary event i occurs and d is exponential decay which never reaches zero.

The number of claims arising from primary event i following the Poisson process is:

$$N_t^{(i)} \sim Poisson\left[y_i e^{-d(t-t_i)} \right]$$

where
$N_t^{(i)}$ number of claims arising from primary event i, $y_i e^{-d(t-t_i)}$ is claim intensity function, $t_i < t < \infty$ and $E(y_i) < \infty$.

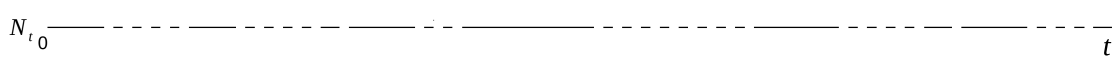
Illustration of the claim intensity function for primary events over a period of time



r the rate of primary event jump arrival.

The number of claims arising from primary events in time period t is: $N_t \sim Poisson\left[\sum_{\substack{\text{all } i \\ t_i<t}} y_i e^{-d(t-t_i)}\right]$. Let $l_t = \sum_{\substack{\text{all } i \\ t_i<t}} y_i e^{-d(t-t_i)}$

then $N_t \sim Poisson(l_t)$.



Probability of n claims for time period *t;*

$$\Pr\left\{N_t = n \middle| l_s; 0 \le s \le t\right\} = \frac{\exp(-\int_0^t l_s ds)(\int_0^t l_s ds)^n}{n!}.$$

Expected value of number of claims for time period *t;*

$$E(N_t) = \int_0^t E(l_s) ds.$$

Assuming that l_t is stationary (by letting $t \to \infty$), the asymptotic mean and variance of l_t and the mean of the number of claims for time period *t* are given by

$$E(l_t) = \frac{m_1 r}{d}, \quad Var(l_t) = \frac{m_2 r}{d} \quad \text{and} \quad E(N_t) = \frac{m_1 r}{d} t$$

where $m_1 = \int_0^\infty y dG(y)$, $m_2 = \int_0^\infty y^2 dG(y)$ and generally distributed jump size with distribution function $G(y)$ $(y > 0)$.

Let us assume r is large and transform the processes

$$Z_t^{(r)} = \frac{l_t - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} \qquad \text{i.e.} \qquad l_t = \frac{m_1 r}{d} + Z_t^{(r)} \sqrt{\frac{m_2 r}{2d}}$$

$$W_t^{(r)} = \frac{N_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} \qquad \text{i.e.} \qquad N_t = \frac{m_1 r}{d} t + W_t^{(r)} \sqrt{\frac{m_2 r}{2d}}$$

and

$$U_t^{(r)} = \frac{C_t - m_1 \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} \qquad \text{i.e.} \qquad C_t = m_1 \frac{m_1 r}{d} + U_t^{(r)} \sqrt{\frac{m_2 r}{2d}}$$

where C_t is the total amount of claims up to time t, $H(u)$ $(u > 0)$ is the claim size distribution function and $m_1 = \int_0^\infty u dH(u)$.

Let us also define

$$V_t^{(r)} = \frac{J_t - m_1 r t}{\sqrt{\frac{m_2 r}{2d}}},$$

$$L_t^{(r)} = \frac{N_t - \int_0^t l_s ds}{\sqrt{\frac{m_2 r}{2d}}}$$

and

$$Q_t^{(r)} = \frac{C_t - m_1 N_t}{\sqrt{\frac{m_2 r}{2d}}}$$

where and $J_t = \sum_{i=1}^{M_t} y_i$ and M_t is the total number of primary event jumps up to time *t*.

Now we will offer a proposition used by Ethier & Kurtz (1985).

Proposition For $n=1,2,\cdots$, let $\{\Im_t^n\}$ be a filtration and let M_n be an $\{\Im_t^n\}$-local martingale with sample paths in $D_{\Re^d}[0,\infty)$ and $M_n(0)=0$. Let $A_n=((A_n^{ij}))$ be symmetric $d\times d$ matrix-valued processes such that A_n^{ij} has sample paths in $D_{\Re^d}[0,\infty)$ and $A_n(t)-A_n(s)$ is nonnegative definite for $0\le s<t$. Assume

$$\lim_{n\to\infty} E\left[\sup_{t\le T} \left|A_n^{ij}(t)-A_n^{ij}(t-)\right| \right]=0,$$

$$\lim_{n\to\infty} E\left[\sup_{t\le T} \left|M_n(t)-M_n(t-)\right|^2 \right]=0,$$

and for $i,j=1,2,\cdots,d$,

$$M_n^i(t)M_n^j(t)-A_n^{ij}(t)$$

is an $\{\Im_t^n\}$-local martingale.

For each $t\ge 0$ and $i,j=1,2,\cdots,d$,

$$A_n^{ij}(t)\to c_{ij}(t)$$

in probability where $C=((c_{ij}))$ is a continuous, symmetric, $d\times d$ matrix-valued function, defined on $[0,\infty)$, satisfying $C(0)=0$ and $\sum(c_{ij}(t)-c_{ij}(s))x_i x_j\ge 0$, $x\in\Re^d$. Then

$$M_n \Rightarrow X$$

in law where X is a process with independent Gaussian increments such that $X_i X_j - c_{ij}$ are (local) martingales with respect to $\{\Im_t^X\}$.

From the proposition, if $r \to \infty$

$$V_t^{(r)} = \frac{J_t - m_1 r t}{\sqrt{\frac{m_2 r}{2d}}} \quad \Rightarrow \quad \sqrt{\frac{2m_1}{m_2}} B_t^{(2)}$$

$$L_t^{(r)} = \frac{N_t - \int_0^t l_s ds}{\sqrt{\frac{m_2 r}{2d}}} \quad \Rightarrow \quad \sqrt{\frac{2m_1}{m_2}} B_t^{(2)}$$

and

$$Q_t^{(r)} = \frac{C_t - m_1 N_t}{\sqrt{\frac{m_2 r}{2d}}} \quad \Rightarrow \quad \sqrt{m_2 \frac{2m_1}{m_2}} B_t^{(3)}$$

in law where $B_t^{(1)}$, $B_t^{(2)}$ and $B_t^{(3)}$ are three independent standard Brownian motions and $m_2 = \int_0^\infty u^2 dH(u) - \left(\int_0^\infty u dH(u) \right)^2$ (variance of claim size).

Note that $Z_t^{(r)}$, $W_t^{(r)}$ and $U_t^{(r)}$ can be written as

$$Z_t^{(r)} = \frac{l_t - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} = \frac{(l_0 - \frac{m_1 r}{d})e^{-dt}}{\sqrt{\frac{m_2 r}{2d}}} + \frac{J_t - m_1 r t}{\sqrt{\frac{m_2 r}{2d}}} - d\, e^{-d(t-u)} \int_0^t \frac{J_u - m_1 r u}{\sqrt{\frac{m_2 r}{2d}}} du$$

$$W_t^{(r)} = \frac{N_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} = \frac{N_t - \int_0^t l_s ds}{\sqrt{\frac{m_2 r}{2d}}} + \int_0^t \frac{l_s - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} ds$$

and

$$U_t^{(r)} = \frac{C_t - m_1 \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} = \frac{C_t - m_1 N_t}{\sqrt{\frac{m_2 r}{2d}}} + m_1 \left(\frac{N_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} \right)$$

Therefore by continuous mapping theorem (see Billingsley (1968)), if $r \to \infty$

$$Z_t^{(r)} = \frac{l_t - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} = \frac{(l_0 - \frac{m_1 r}{d})e^{-dt}}{\sqrt{\frac{m_2 r}{2d}}} + \frac{J_t - m_1 r t}{\sqrt{\frac{m_2 r}{2d}}} - d\, e^{-d(t-u)} \int_0^t \frac{J_u - m_1 r u}{\sqrt{\frac{m_2 r}{2d}}} du$$

$$\Rightarrow \quad Z_t = Z_0 e^{-dt} + \sqrt{2d} \int_0^t e^{-d(t-s)} dB_s^{(1)} ,$$

$$W_t^{(r)} = \frac{N_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} = \frac{N_t - \int_0^t l_s ds}{\sqrt{\frac{m_2 r}{2d}}} + \int_0^t \frac{l_s - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} ds$$

$$\Rightarrow \quad W_t = \int_0^t Z_s ds + \sqrt{\frac{2m_1}{m_2}} B_t^{(2)}$$

and

$$U_t^{(r)} = \frac{C_t - m_1 \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}} = \frac{C_t - m_1 N_t}{\sqrt{\frac{m_2 r}{2d}}} + m_1 \left(\frac{N_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}} \right)$$

$$\Rightarrow \quad U_t = m_1 W_t + \sqrt{m_2 \frac{2m_1}{m_2}} B_t^{(3)} .$$

Hence it can be written as in differential form

$$dZ_t = -d\,Z_t dt + \sqrt{2d}\,dB_t^{(1)}$$

$$dW_t = Z_t dt + \sqrt{\frac{2m_1}{m_2}}dB_t^{(2)}$$

and

$$dU_t = m_1 dW_t + \sqrt{m_2 \frac{2m_1}{m_2}}dB_t^{(3)} = m_1 Z_t dt + \sqrt{m_2 \frac{2m_1}{m_2}}dB_t^{(4)}$$

where $B_t^{(1)}$, $B_t^{(2)}$, $B_t^{(3)}$ are three independent standard Brownian motions and $B_t^{(4)} = \frac{m_1\sqrt{\frac{2m_1}{m_2}}B_t^{(2)} + \sqrt{m_2 \frac{2m_1}{m_2}}B_t^{(3)}}{\sqrt{(m_1^2 + m_2)\frac{2m_1}{m_2}}}$ (also a standard Brownian motion).

As a result of these, we obtained $\tilde{l}_t$, $\tilde{N}_t$ and $\tilde{C}_t$ which are Gaussian approximations of l_t, N_t and C_t;

$$\tilde{l}_t = \frac{m_1 r}{d} + Z_t \sqrt{\frac{m_2 r}{2d}} \quad \text{i.e.} \quad W_t = \frac{\tilde{l}_t - \frac{m_1 r}{d}}{\sqrt{\frac{m_2 r}{2d}}}$$

$$\tilde{N}_t = \frac{m_1 r}{d} t + W_t \sqrt{\frac{m_2 r}{2d}} \quad \text{i.e.} \quad W_t = \frac{\tilde{N}_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}}$$

and

$$\tilde{C}_t = m_1 \frac{m_1 r}{d} t + U_t \sqrt{\frac{m_2 r}{2d}} \quad \text{i.e.} \quad U_t = \frac{\tilde{C}_t - \frac{m_1 r}{d} t}{\sqrt{\frac{m_2 r}{2d}}}.$$

Let R_T denote the free reserve (or surplus) of the insurer at time *T*. We assume that premiums are received continuously at a constant rate $k\,(>0)$ and let R_T denote the reserve required on hand at time *t*.

Then

$$R_T = R_t + k\,(T-t) - (C_T - C_t),\; 0 \le t \le T$$

where:

$\aleph_i$	claim amount
$N_T - N_t$	number of claims between time *T* and *t*,
$C_T - C_t = \sum_{i=1}^{N_T - N_t} \aleph_i$	total amount of claims between time *T* and *t*.

The surplus R_T must be non-negative if the insurer is able to meet his liabilities. Therefore it is very important that the probability that R_T becomes negative be small (e say), i.e.

$$\Pr(R_T < 0) = e$$

or

$$\Pr\left[C_T - C_t > R_t + k\,(T-t)\right] = e\;.$$

Since we have obtained $\tilde{C}_t$ which is Gaussian approximation of C_t, we will use this approximation, then

$$\Pr\left[\frac{\tilde{C}_T - \tilde{C}_t - E\left(\tilde{C}_T - \tilde{C}_t\right)}{\sqrt{Var\left(\tilde{C}_T - \tilde{C}_t\right)}} > \frac{R_t + k\,(T-t) - E\left(\tilde{C}_T - \tilde{C}_t\right)}{\sqrt{Var\left(\tilde{C}_T - \tilde{C}_t\right)}}\right] = e\,.$$

Set $k\,(T-t) = (1+q)\,E\left(\tilde{C}_T - \tilde{C}_t\right)$, $e = 5\%$ and consider the information available (i.e. the observed claims) up to time *t*,

$$R_t = 1.645\sqrt{Var\left(\tilde{C}_T - \tilde{C}_t \mid \tilde{N}_S; 0 \le s \le t\right)} - q\,E\left(\tilde{C}_T - \tilde{C}_t \mid \tilde{N}_S; 0 \le s \le t\right)$$

where $q > 0$ is a relative security loading.

Now let us examine how the above equation would be altered if the direct insurer purchase a stop-loss reinsurance contract that covers the total loss excess over *b*, a retention limit. Let R_T^h denote the free reserve of the insurer after purchasing reinsurance contract at time *T*. We assume that reinsurance premiums are paid continuously at a constant rate $k_h(>0)$ and let R_T^h denote the reserve required on hand at time *t*. Then

$$R_T^h = R_t^h + k\,(T-t) - k_h\,(T-t) - \left(C_T^I - C_t^I\right),\ \ 0 \le t \le T$$

where $C_T^I - C_t^I = \begin{cases} C_T - C_t, & C_T - C_t \le b \\ b, & C_T - C_t > b \end{cases}$.

Set $k_h(T-t) = (1+x)\,E\left[\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b\right\}^+\right]$, $e = 5\%$ and consider the information available (i.e. the observed claims) up to time *t*,

$$R_t^h = 1.645\sqrt{Var\left(\tilde{C}_T^I - \tilde{C}_t^I \mid \tilde{N}_s; 0 \le s \le t\right)} - q\,E\left(\tilde{C}_T - \tilde{C}_t \mid \tilde{N}_s; 0 \le s \le t\right) + x\,E\left[\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b\right\}^+ \mid \tilde{N}_s; 0 \le s \le t\right]$$

where $x > 0$ is the relative security loading for reinsurance contract and $\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b\right\}^+ = Max\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b,\ 0\right\}$.

If $x = q$, it becomes $R_t^h = 1.645\sqrt{Var\left(\tilde{C}_T^I - \tilde{C}_t^I \mid \tilde{N}_s; 0 \le s \le t\right)} - q\,E\left(\tilde{C}_T^I - \tilde{C}_t^I \mid \tilde{N}_s; 0 \le s \le t\right)$.

Since we have obtained $\tilde{C}_t$ and $\tilde{N}_t$ which are Gaussian approximations of C_t and N_t, we will use these approximations. Therefore set $\tilde{C}_t = m_1 \frac{m_1 r}{d} t + U_t \sqrt{\frac{m_2 r}{2d}}$, then the risk (net) insurance premium at time *t* is

$$E\left[\tilde{C}_T - \tilde{C}_t \mid \tilde{N}_s; 0 \le s \le t\right] = E\left[\sqrt{\frac{m_2 r}{2d}}\left(U_T - U_t\right) + m_1 \frac{m_1 r}{d}\left(T-t\right) \middle| W_s; 0 \le s \le t\right]$$

and the stop-loss reinsurance premium at time *t* is

$$E\left[\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b\right\}^+ \mid \tilde{N}_s; 0 \le s \le t\right] = E\left[\left\{\sqrt{\frac{m_2 r}{2d}}\left(U_T - U_t\right) + m_1 \frac{m_1 r}{d}\left(T-t\right) - b\right\}^+ \mid W_s; 0 \le s \le t\right].$$

We can see that mean and variance of $U_T - U_t$ given $\{W_s; 0 \le s \le t\}$, need to be determined to obtain insurance / reinsurance premium. Therefore we derive them, i.e.

$$E\left(U_T - U_t \mid W_s; 0 \le s \le t\right) = m_1 \frac{1 - e^{-d(T-t)}}{d} \cdot E\left(Z_t \mid W_s; 0 \le s \le t\right)$$

and

$$Var\left(U_T - U_t \mid W_s; 0 \le s \le t\right) = \left(\frac{m_1}{d}\right)^2 \left\{\left(1 - e^{-d(T-t)}\right)^2 \cdot Var\left(Z_t \mid W_s; 0 \le s \le t\right) - e^{-2d(T-t)} + 4e^{-d(T-t)} - 3\right\} + 2\left(\frac{m_1^2}{d} + \frac{m_2 m_1}{m_2}\right)\left(T-t\right).$$

The Filtering Problem:

Suppose the process Z_t is not accessible for observation and one can observe only the values $\{W_s; 0 \le s \le t\}$. Therefore, at any time *t*, it is required to estimate the unobservable state Z_t. This problem of estimating Z_t based on $\{W_s; 0 \le s \le t\}$ is called the *filtering* problem.

If we take the optimality of estimation in the mean square sense, the optimal estimate for Z_t, given $\{W_s; 0 \le s \le t\}$ coincides with the conditional expectation

$$E\left(Z_t \mid W_s; 0 \le s \le t\right).$$

One of the methods to solve the filtering problem, where the process is Gaussian, is the Kalman-Bucy filter.

Therefore we will derive the conditional distribution of Z_t, given $\{W_s; 0 \le s \le t\}$, by the linear system driven by the Cox process with shot noise intensity where

$$dZ_t = -d\, Z_t dt + \sqrt{2d}\, dB_t^{(1)}$$

$$dW_t = Z_t dt + \sqrt{\frac{2m_1}{m_2}} dB_t^{(2)}.$$

Now we will offer a proposition used by Oksendal (1992).

Proposition The solution $\hat{Z}_t = E\left(Z_t \,|\, W_s; 0 \le s \le t\right)$ of the 1-dimensional linear filtering problem

$$dZ_t = F(t)Z_t dt + X(t)dB_t^{(1)};\; F(t), C(t) \in \Re$$
$$dW_t = G(t)Z_t dt + D(t)dB_t^{(2)};\; G(t), D(t) \in \Re$$

satisfies the stochastic differential equation

$$d\hat{Z}_t = \left\{F(t) - \frac{G^2(t)S(t)}{D^2(t)}\right\}\hat{Z}_t dt + \frac{G(t)S(t)}{D^2(t)} dW_t;\; \hat{Z}_0 = E\left(Z_0\right)$$

where

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t) + C^2(t),$$
$$S(0) = E\left[\left\{Z_0 - E\left(Z_0\right)\right\}^2\right] = Var\left(Z_0\right).$$

From the proposition, we can obtain the estimate of Z_t based on the observed $\{W_s; 0 \le s \le t\}$:

$$\hat{Z}_t = E\left(Z_t \middle| W_s; 0 \le s \le t\right) = e^{\int_0^t \Psi(s)ds} \cdot \hat{Z}_0 + \frac{m_2}{2m_1} \cdot \int_0^t e^{\int_s^t \Psi(u)du} S(s) dW_s$$

where

$$S(0) = a^2,$$

$$S(s) = \frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}\left\{1 + \frac{a^2 + \frac{2dm_1}{m_2} + \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2 + \frac{2dm_1}{m_2} - \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}} e^{\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}} s}\right\}}{\frac{a^2 + \frac{2dm_1}{m_2} + \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2 + \frac{2dm_1}{m_2} - \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}} e^{\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}} s} - 1} - 2d\frac{m_1}{m_2}$$

and

$$\Psi(s) = -\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}\left\{1 + \frac{a^2 + \frac{2dm_1}{m_2} + \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2 + \frac{2dm_1}{m_2} - \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}} e^{\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}} s}\right\}}{\frac{2m_1}{m_2}\left\{\frac{a^2 + \frac{2dm_1}{m_2} + \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{a^2 + \frac{2dm_1}{m_2} - \sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}} e^{\frac{\sqrt{\frac{2m_1}{m_2}}\sqrt{d\left(\frac{2dm_1}{m_2}+2\right)}}{\frac{m_1}{m_2}} s} - 1\right\}}.$$

From the fact that Z_t, given $\{W_s; 0 \le s \le t\}$, is normally distributed, with $Var\left(Z_t \middle| W_s; 0 \le s \le t\right) = S(t)$, we can find the distribution of Z_t:

$$E\left(e^{-g Z_t} \middle| W_s; 0 \le s \le t\right) = e^{-g \hat{Z}_t + \frac{1}{2} g^2 S(t)}$$

where Z_t , W_t, $\hat{Z}_t$ and $S(t)$ as defined

Therefore the net premium for insurance/stop-loss reinsurance contract at time t, based on the observations $\{W_s; 0 \le s \le t\}$ is as follows:

$$E\left[\tilde{C}_T - \tilde{C}_t \mid \tilde{N}_s; 0 \le s \le t\right] = E\left[\sqrt{\frac{m_2 r}{2d}}(U_T - U_t) + m_1 \frac{m_1 r}{d}(T-t) \middle| W_s; 0 \le s \le t\right] = \sqrt{\frac{m_2 r}{2d}}\Gamma + \frac{m_1 m_1 r}{d}(T-t)$$

and

$$E\left[\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b\right\}^+ \mid \tilde{N}_s; 0 \le s \le t\right] = E\left[\left\{\sqrt{\frac{m_2 r}{2d}}(U_T - U_t) + m_1 \frac{m_1 r}{d}(T-t) - b\right\}^+ \mid W_s; 0 \le s \le t\right]$$

$$= \left[\sqrt{\frac{m_2 r \Theta}{4dp}} e^{-\frac{1}{2}L^2} + \left\{\sqrt{\frac{m_2 r}{2d}}\Gamma + \frac{m_1 m_1 r}{d}(T-t) - b\right\}\Phi(-L)\right]$$

where

$$\Gamma = E\left(U_T - U_t \middle| W_s; 0 \le s \le t\right) = m_1 \frac{1 - e^{-d(T-t)}}{d} \hat{Z}_t,$$

$$\Theta = Var\left(U_T - U_t \middle| W_s; 0 \le s \le t\right) = \left(\frac{m_1}{d}\right)^2 \left\{\left(1 - e^{-d(T-t)}\right)^2 S(t) - e^{-2d(T-t)} + 4e^{-d(T-t)} - 3\right\} + 2\left(\frac{m_1^2}{d} + \frac{m_2 m_1}{m_2}\right)(T-t),$$

$L = \dfrac{K - \Gamma}{\sqrt{\Theta}}$, $K = \dfrac{b - m_1 \frac{m_1 r}{d}(T-t)}{\sqrt{\frac{m_2 r}{2d}}}$ and $\Phi(\cdot)$ is the cumulative normal distribution function.

Similarly,

$$E\left\{\left(\tilde{C}_T^I-\tilde{C}_t^I\right)^2 \mid \tilde{N}_s;0\le s\le t\right\}=\frac{m_2 r\Theta}{2d\sqrt{2p}}(Ae^{-\frac{1}{2}A^2}-Le^{-\frac{1}{2}L^2})+\left\{\frac{m_2 r\Gamma\sqrt{\Theta}}{d}+2\sqrt{\frac{m_2 r\Theta}{2d}}\frac{m_1 m_1 r}{d}(T-t)\right\}(e^{-\frac{1}{2}A^2}-e^{-\frac{1}{2}L^2})$$
$$+\left\{\frac{m_2 r}{2d}\left(\Gamma^2+\Theta\right)+2\sqrt{\frac{m_2 r}{2d}}\Gamma\frac{m_1 m_1 r}{d}(T-t)+\left(\frac{m_1 m_1 r}{d}\right)^2(T-t)^2\right\}\{\Phi(-A)-\Phi(-L)\}+b^2\Phi(-L)$$

where $A=\dfrac{B-\Gamma}{\sqrt{\Theta}}$ and $B=\dfrac{-m_1\frac{m_1 r}{d}(T-t)}{\sqrt{\frac{m_2 r}{2d}}}$.

Therefore we can obtain

$$Var\left(\tilde{C}_T^I-\tilde{C}_t^I \mid \tilde{N}_s;0\le s\le t\right)=E\left\{\left(\tilde{C}_T^I-\tilde{C}_t^I\right)^2 \mid \tilde{N}_s;0\le s\le t\right\}-\left\{E\left(\tilde{C}_T^I-\tilde{C}_t^I \mid \tilde{N}_s;0\le s\le t\right)\right\}^2$$
$$=E\left\{\left(\tilde{C}_T^I-\tilde{C}_t^I\right)^2 \mid \tilde{N}_s;0\le s\le t\right\}-\left[E\left(\tilde{C}_T-\tilde{C}_t \mid \tilde{N}_s;0\le s\le t\right)-E\left[\left\{\left(\tilde{C}_T-\tilde{C}_t\right)-b\right\}^+ \mid \tilde{N}_s;0\le s\le t\right]\right]^2.$$

The parameter values used to calculate the reserves required are $\hat{Z}_0 = 0$, $S(0) = a^2 = 0$, $d = 0.5$, $r = 100$, $m_1 = 1$, $m_2 = 2$, $m_1 = 1$, $m_2 = 3$, $t = 1$, $T = 2$, $q = 0,\ 0.1,\ 0.2,\ 0.2129$. By computing, using *MAPLE* and *S-Plus*, where $\hat{Z}_1 = 0.5579152$, the calculation of the reserves for insurance contract, at each security loading q, are shown in Table 1,

Table 1

Security loading q	Reserve R_t at $e = 5\%$
0.0	43.903
0.1	23.282
0.2	2.661
0.2129	0

and the reserves after purchasing stop-loss reinsurance contracts at the retention limit $b = 0,\ 270,\ 280,\ \infty$ are shown in Table 2 ($q = x = 0.1$) and in Table 3 ($q = 0.1$ and $x = 0.2$).

Table 2

Retention level b	Reserve R_t^h at $e = 5\%$
0	0
270	8.874
280	18.442
∞	23.282

Table 3

Retention level b	Reserve R_t^h at $e = 5\%$
0	20.621
270	8.890
280	18.448
∞	23.282