STOCHASTIC MODELS FOR INFLATION, INVESTMENTS AND EXCHANGE RATES
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Introduction

I am very pleased to be here today to talk about stochastic models for inflation, investments and exchange rates. In particular I shall talk about the stochastic investment model I first developed some years ago, which has become known in Britain as the "Wilkie Investment Model". I have now fitted this model to data from a number of countries, and I have been able to use some data for Canada in preparation for this talk. I shall therefore describe the model both as it applies to United Kingdom data and to Canadian data. I shall discuss consumer prices, shares, share dividends and dividend yields, long-term bond yields and short-term bond or bill yields, wages and exchange rates.

The way I have approached stochastic modelling is to start with the data, and see what we can derive from it. Others seem to have started from a theoretical or ideal economic model, without being concerned about whether the data actually supports their model. I have in mind in particular the many proponents of pure random walk models for the prices of ordinary shares and many commodities, and in the course of this talk I shall explain why I agree with them in the short run, but not in the long run.

I like to start with as long a run of data as I can find. In some cases in Britain this is a very long run. One can learn something from this older data, but one has to interpret it in accordance with the circumstances at the time. The approach I then use is time-series modelling. But I have not simply used standard packages. One reason is that they were not available at the time I started my investigations. But also there are different objectives when using time-series modelling, which I can perhaps best explain by the difference between forecasting the weather and studying the climate.

For weather forecasting you wish to find the best possible estimate of what the weather is going to be like tomorrow or in the next few days. To do this you take as much information into account as you reasonably can, and you try to make the forecasting model as precise as possible, which in a statistical sense usually means reducing the residual or unexplained variance. But if you wish to study the climate,
perhaps in order to decide what size of reservoir you need to build, or how big flood banks to put alongside a river, you are interested in modelling the variability in the long run. You cannot use so much information, because although you may have a great deal of information today that you can use for short term forecasting, if you wish to use the same sort of information over a long period then you have to forecast that information as well. You are also not necessarily interested in the minimum variance forecast, because one of your objectives is to investigate the consequences of high variability. How often will there be a drought, how often will there be a flood?

So it is with the sort of stochastic modelling that I have been doing, which is intended for the use of actuaries when looking at long-run scenarios. I am not very interested in forecasting what the Consumer Price Index will be in three months time, but I am interested in knowing the variability in likely rates of inflation over the next 20 years or so. But if I were in the business of writing Consumer Price Index options or futures contracts on the market, then my approach would be quite different.

You will find that I introduce variables into the model one by one, in what I have called a cascade fashion. An alternative approach would be to model all variables simultaneously, allowing each one to affect any of the others. In fact my investigations started out this way, using conventional multivariate time series modelling. After my first investigation I found that simultaneous modelling was not necessary, and that it was actually possible to bring in the variables sequentially, because while the first variable A depended on previous values of itself, B depended on A and earlier values of B, but did not influence A, and C depended on A and B, but did not influence them.

This approach has the advantage that it makes it much easier to introduce additional variables, or sometimes to by-pass existing ones if you don't want them, or don't have data for them.

**Consumer price index**

Let me now start with some data. The first variable I have looked at is an index of consumer prices, what in Britain we call the Retail Prices Index. Here (Figure 1) is a graph of consumer prices in Britain starting in the year 1264 and coming up to date. Like most of my graphs, it is drawn with a vertical log scale. Of course the Government was not collecting statistics in 1264, or even until the late 19th Century. But economic historians have constructed a number of substitute indices, based on recorded prices of certain manufactured goods, foodstuffs and other commodities, and we have to make do with what data we can.
Figure 1. UK RPI 1264 - 1993

Figure 2. UK RPI 1264 - 1500
You can see the general pattern, and I shall show it more clearly in these successive
graphs. First (Figure 2) from 1200 to 1500 there was generally no upwards or
downwards drift in prices, but a great deal of annual fluctuation. Then during the
16th Century (Figure 3) there was a moderate rise in prices. Then (Figure 4) there
was another 300 years of fluctuation, again with zero drift. Then in the 20th Century
(Figure 5) we started with a sharp rise during the First World War, a sharp drop
immediately afterwards, a number of years of falling prices, and then about 60 years
of steadily rising prices.

Up to this century money was gold, and since the supply of gold was limited it was
not possible for prices to increase by very much. I believe that the 16th Century
inflation may have been caused by an increase in the supply of gold after the Spanish
discovery of America. But Britain went off gold during the First World War, back
onto it for a little period in the 1920s, and we have been using paper money, or now
more likely computerised money, ever since.

Here is a scatter diagram (Figure 6) of the rate of inflation in one year compared with
the rate of inflation in the previous year, for the period from 1600 to 1900. Strictly
I am using the force of inflation, the difference between the logarithms of the price
index, equivalent to a compound interest $\delta$ rather than $i$. There is very little
 correlation between the value in one year and the values in previous years, and
calculation of the autocorrelation coefficients at various lags confirms this. Although
prices fluctuated a lot, jumping up or down in any year by as much as 20% or more
either way, it was in fact reasonable to postulate a pure zero drift random walk for
the logarithm of the value of the price index.

Not so this century. Here is a scatter diagram (Figure 7) for the period from 1900
to 1993. There are some outliers in this graph, created by the substantial falls in
prices immediately after the First World War, and I have found it better to miss them
out from my investigations. So here is a scatter diagram (Figure 8) for the period
from 1923 to 1993. You can see how the points are not now symmetrically scattered
about zero, but have an upwards sloping tendency, and the centre of the distribution
is at about $+5\%$ a year. This means that the rate of inflation one year is connected
with the rate of inflation in the preceding year, and this allows us to postulate a
simple first order autoregressive model for inflation.

Here is some notation. The Consumer Price Index in year $t$ is denoted $Q(t)$. The
force of inflation from $t-1$ to $t$ is denoted by $I(t)$, where:

$$I(t) = \ln Q(t) - \ln Q(t-1).$$

$I(t)$ is represented by the autoregressive model:

$$I(t) = QMU + QN(t).$$
Figure 3. UK RPI 1500 - 1600

Figure 4. UK RPI 1600 - 1900
Figure 5. UK RPI 1900 - 1993

Figure 6. UK Inflation 1600 - 1900
Figure 7. UK Inflation 1900 - 1993

Figure 8. UK Inflation 1923 - 1993
QMU represents the mean force of inflation and QN(t) has zero mean and is first order autoregressive so that:

\[ QN(t) = QA.QN(t-1) + QE(t). \]

QA is the autoregressive parameter, and the residuals QE(.) can also be expressed as:

\[ QE(t) = QSD.QZ(t). \]

QSD is the standard deviation of the residuals, and the standardised residuals QZ(.) are independently distributed, have zero mean, unit standard deviation, and turn out to be approximately normally distributed.

Here are the parameters of the model as it fits the United Kingdom. Don’t look at the figures for Canada yet. There has been roughly 5% a year inflation (QMU = 0.05). The autoregressive parameter QA is about 0.6. The standard deviation of the residuals QSD has been about 4% a year.

Now here are the corresponding graphs for Canada. First (Figure 9) is a graph of the Consumer Price Index in Canada from 1914 to 1993. This is on the same scale as the UK data, and if we put them on the same graph (Figure 10) we can see that they have behaved in rather a similar way. They differ simply because the base value I have used is different, but they are roughly a constant distance apart on this log scale graph, which means a constant ratio between the two indices.

Next is a scatter diagram (Figure 11) for the Canadian data also from 1923 to 1993, omitting the drops after the First World War. It has a similar appearance to the corresponding diagram for the UK. A similar statistical model would fit, and the parameters are not very different, though the average rate of inflation in Canada has been a bit lower than in Britain.

Here are the parameters for Canada and for the UK:

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<th>Canada</th>
<th>UK</th>
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<tbody>
<tr>
<td>QMU</td>
<td>0.034</td>
<td>0.05</td>
</tr>
<tr>
<td>QA</td>
<td>0.64</td>
<td>0.6</td>
</tr>
<tr>
<td>QSD</td>
<td>0.032</td>
<td>0.04</td>
</tr>
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They are not very different. The average rate of inflation in Canada has been lower than in the UK, and the different value of QMU shows this. The values of QA are about the same. The value of QSD in Canada is also lower than that for the UK. I have found that this is consistent across a lot of countries; where the overall rate of inflation is low, so also is the variability as expressed by the standard deviation.

I have recently read a very thorough report prepared for the Canadian Institute of Actuaries by Richard Deaves on "Modelling and Predicting Canadian Inflation and Interest Rates". He is talking later today and I shall be interested in listening. He
Figure 9. Canada CPI 1914 - 1993

Figure 10. UK and Canada CPI 1900 - 1993
Figure 11. Canada Inflation 1923 - 1993

Figure 12. Canada Inflation 1923 - 1993 monthly observed autocorrelation function
quotes work done by Keith Sharp on modelling inflation. I have not seen Sharp's paper, but the model that Richard Deaves quotes is the same model as mine, with rather similar parameters. Richard Deaves himself uses a different approach, modelling quarterly inflation. I want to do a little detour here to explain why I come back to modelling annual rates of inflation.

**Monthly inflation data**

In fact I have data available for Canada, as for the United Kingdom, at monthly intervals, so I start by calculating the force of inflation over each month, and then calculate the correlation coefficients between inflation in month $t$ and inflation in month $t-k$, for $k=1, 2, 3, \ldots$. This is what is known as the observed autocorrelation function. Here (Figure 12) is a chart of it. There are quite large correlations between inflation in any month and in a lot of the preceding months, with particular spikes at lags 12, 24, 36, 48 and 60 months, so that there is clear seasonal variation in price changes. It is, incidentally, rather odd that all the first 60 coefficients are positive; usually there would be some negative values by lag 60.

The same sort of pattern broadly applies in Britain. Two reasons are that certain foodstuffs are cheaper in the summer months, and that, at least until this year, we have had a Government Budget Statement in March, with specific tax rises coming through in April or May.

Such a complicated autocorrelation function shows that one would need a complicated autoregressive or possibly moving average model to represent monthly inflation.

If I look at inflation over periods of two months, I have two different series available, one that uses the price index in months 1, 3, 5, \ldots and takes differences, and the other that uses the price index in months 2, 4, 6, \ldots and takes differences. If I calculate the autocorrelation functions for these I get two patterns. These are similar to one another, and also show spikes at frequencies 6, 12, 18, \ldots, corresponding to annual seasonality.

The same happens when I take the three different 3-month series, the four different 4-month series and the six different 6-month series, and it is not until I look at the twelve different 12-month series that the seasonality effect disappears, and we are left with series that can be described with a simple first order autoregressive model. Further, the first autoregressive coefficient, the correlation coefficient between neighbouring changes, rises as the differencing interval increases, to a maximum for the annual series, falling again thereafter. Although I have twelve different series, the parameters are not very different for the different months. What I showed you earlier was the series using June values of the price index. Keith Sharp may have used values for some other month.
What would be interesting to investigate, and I have not done this, is to see the long run
difference between my model and Richard Deaves' quarterly model. We may produce
similar central forecasts, but our estimates of the variance of future forecasts might be
the same or might be quite different. As I have already said, I am not particularly interested
in forecasting inflation accurately over the next three months, but instead in making a
reasonable estimate of the uncertainty in inflation forecasts over the next 20 years or more.

Simulated futures

Here (Figure 13) is an example. This graph shows the actual Canadian Price Index
from 1914 to 1993, just as previously. From 1993 to 2050 I have produced a single
simulation, based on the model and the parameters I have just described. I hope that
it looks plausibly like the actual experience.

But that was only one possible future. This graph (Figure 14) is the same as before,
with six more alternative futures. This gives some impression of the possible scatter
of results over the future. You can pick out some where inflation will have been
pretty high, others where there will have been periods of falling prices. If I were to
do 100 different simulations you could not pick them all out, but the stochastic bundle
would give quite a good plot of the distribution of possible futures.

In fact for this model the mean and variance of future values of the price index can
be calculated analytically. If the residuals, $QZ(.)$ are assumed to be normally
distributed, then each forecast value of $Q(.)$ is lognormally distributed. This has
been shown by Walter Hürlimann in a recent paper in Insurance: Mathematics and

Share prices

Forecasting inflation is all very well, but for investment purposes we wish to know
something about proper investments, and I shall first talk about ordinary shares or
common stocks. I prefer to talk about shares and bonds, because the word stocks
has different meanings on opposite sides of the Atlantic, as you probably all know.
This is probably not a problem in French, where I think actions and obligations are
the words everywhere.

Here is a graph of share prices (Figure 15) in the United Kingdom from 1918 to
1993, based on a series of different indices. Here is one for Canada (Figure 16) for
a rather shorter period. For most of the period I have been able to use the Toronto
Stock Exchange figures, but for the early years I have substituted the US Standard
and Poor's Composite Index, and for the most recent years I have used the Financial
Figure 13. Canada CPI 1914 - 2050

Figure 14. Canada CPI 1914 - 2050
Figure 15. UK Share Prices 1918 - 1993

Figure 16. Canada Share Prices 1936 - 1993
Times Actuaries World Index data for Canada, simply because I have that readily available.

Early thoughts about share prices were that they could be represented by a pure random walk, possibly with an upwards bias, but with successive changes independent. This remains a good model for short term changes.

When we first started looking at graphs like this it was observed that, if share prices really moved in a random walk manner, the walk was remarkably straight. Then we said that share prices surely depend on the level of company profits in some way, as expressed through the dividends paid. Instead of just looking at prices we should look at dividends and dividend yields. We could equally well have looked at earnings and P/E ratios if these had been available, but one can get a much longer dividend series than an earnings series, at least in the UK.

Share dividend yields

I want to look first at dividend yields. Here (Figure 17) is a graph of dividend yields for the UK, from 1918 to 1993. Here (Figure 18) is the same for Canada, starting in 1936. You see in both cases how yields fluctuate up and down around some sort of mean level. This suggest that yields are a stationary process. This means that we could extend the same graph indefinitely to the right, without having to change the scale of it, whereas you can easily imagine that if we were to continue the graph for consumer prices or for share prices into the future, we should in due course hit the top of the scale, and have to extend it upwards.

Here now is a scatter diagram (Figure 19) of the yield in the UK in June each year compared with the yield in June in the previous year, \( Y(t) \) against \( Y(t-1) \). You can see how the points lie on an upward sloping line. Here (Figure 20) is the same for Canada. The same sort of correlation can be seen. As it happens the correlation coefficients for both are about 0.6, rather similar to the correlation coefficients for the rate of inflation; but I think that this is just an accident.

This suggest that the first sort of model one might use for modelling share dividend yields is a first order autoregressive model, just as we had for inflation. But if we then look at the connection between dividend yields and inflation, we find that, in both countries, when inflation is high dividend yields are also high. This can be observed from the relevant statistics, and it seems to me to make sense. When inflation is rather high, it may also be the case that wages have risen sharply, and are taking an undue proportion of company profits; short term interest rates may also be high; and at least in Britain governments have brought in price controls and sometimes dividend controls. All this is bad news for shares, so prices fall and
Figure 17. UK Share Dividend Yields 1918 - 1993

Figure 18. Canada Share Dividend Yields 1936 - 1993
Figure 19. UK Share Dividend Yield 1923-1993

Figure 20. Canada Share Dividend Yield 1936 - 1993
dividend yields rise. We can therefore put an extra term into the autoregressive model, giving us a complete model that shows:
\[ \ln Y(t) = YW \cdot I(t) + \ln YMU + YN(t). \]

\( YW \) is the parameter multiplying \( I(t) \), the current rate of inflation, \( YMU \) is a constant mean, and \( YN(t) \) is a zero mean term subject to an autoregressive process:
\[ YN(t) = YA \cdot YN(t-1) + YE(t). \]

\( YA \) is the autoregressive parameter, and the residuals \( YE(.) \) can also be expressed as:
\[ YE(t) = YSD \cdot YZ(t). \]

\( YSD \) is the standard deviation of the residuals, and the standardised residuals \( YZ(.) \) are like \( QZ(.) \), independent and unit normal.

Parameter values for Canada and the UK are:

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<tr>
<th></th>
<th>Canada</th>
<th>UK</th>
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<tbody>
<tr>
<td>( YW )</td>
<td>1.17</td>
<td>1.95</td>
</tr>
<tr>
<td>( YA )</td>
<td>0.7</td>
<td>0.5</td>
</tr>
<tr>
<td>( YMU% )</td>
<td>3.75%</td>
<td>3.8%</td>
</tr>
<tr>
<td>( YSD )</td>
<td>0.19</td>
<td>0.16</td>
</tr>
<tr>
<td>Median</td>
<td>3.90%</td>
<td>4.19%</td>
</tr>
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They are reasonably similar. In fact the long term median (note that the mean is not the same as the median, but is a little bigger) for the yield does not depend just on \( YMU \), but is given by \( YMU \cdot \exp(YW \cdot QMU) \), and this is also shown in the table. It is a bit higher in the UK than in Canada.

**Share dividends**

We now need to look at share dividends. Here (Figure 21) is a graph of share dividends in the UK from 1919 to 1993. It is clear that they have generally risen in the same sort of way as retail prices. This is hardly surprising: dividends depend on company earnings; company earnings depend on trading profits; trading profits depend on turnover; and the volume of turnover depends on the prices at which goods and services are sold.

Here (Figure 22) is a similar graph for Canada from 1936. Dividends here too have risen roughly in line with retail prices.

One way of looking at share dividends is to divide by the Consumer Price Index, to calculate dividends "in real terms". Here (Figure 23) is a graph of real dividends in the UK. You can see that there have been fairly long upswings and downswings in
Figure 21. UK Share Dividends 1918 - 1993

Figure 22. Canada Share Dividends 1936 - 1993
Figure 23. UK Real Share Dividends 1918 - 1993

Figure 24. Canada Real Share Dividends 1936 - 1993
this series. We are now at the top of a long upswing. One might wonder where we are going next, whether on up, back down, or staying roughly level.

Here (Figure 24) is a similar graph for Canada. There do not seem to have been the same strong upswings and downswings. The whole graph is more level than in the UK.

One way of modelling the influence of consumer prices on dividends is to introduce them into the model with a suitable time lag. There are plenty of ways in which one could do this, but I have chosen to represent the influence of price inflation in two ways: one in terms of an exponentially weighted moving average of past inflation, the other taking account of inflation in the current year.

This explains the first two terms of the rather lengthy formula I have used for modelling dividend increases.

The model is:

\[ \ln D(t) - \ln D(t-1) = DW \cdot DM(t) + (1-DW) \cdot I(t) + DMU + DY \cdot YE(t-1) + DB \cdot DE(t-1) + DE(t). \]

The term \( DM(t) \) represents an exponentially weighted moving average of current and past inflation, and is given by

\[ DM(t) = DD \cdot I(t) + (1-DD) \cdot DM(t-1). \]

DD is the smoothing parameter, DW is the weight given to the moving average of all past inflation and \( 1-DW \) is the extra weight given to current inflation.

The next term is DMU, the mean rate of growth of dividends in real terms. The last term DE(t) is the usual residual given by

\[ DE(t) = DSD \cdot DZ(t), \]

and DZ(t) is a sequence of independent unit normal variables.

The two previous terms represent influences from the residuals of the yield and the dividend series in the preceding year. The term DY.YE(t−1) represents the observation that changes in share prices anticipate changes in dividends, so share yields fall prior to a dividend rise. The term DB.DE(t−1) represents a carried forward effect of changes in dividends from one year to the next.
Estimated parameter values for Canada and for the UK are shown in this table.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Canada</th>
<th>UK</th>
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<tbody>
<tr>
<td>DW</td>
<td>0.19</td>
<td>0.8</td>
</tr>
<tr>
<td>DD</td>
<td>0.26</td>
<td>0.2</td>
</tr>
<tr>
<td>DMU</td>
<td>0.001</td>
<td>0.0135</td>
</tr>
<tr>
<td>DY</td>
<td>-0.11</td>
<td>-0.175</td>
</tr>
<tr>
<td>DB</td>
<td>0.58</td>
<td>0.55</td>
</tr>
<tr>
<td>DSD</td>
<td>0.07</td>
<td>0.06</td>
</tr>
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</table>

The response of dividends to inflation seems rather different in the two countries. In Canada there is a significant simultaneous effect of inflation, represented by the parameter $1 - DW$ being about 0.8. The way in which inflation has influenced dividends in the UK is rather more complicated, with a longer time lag.

As it happens the mean rate of growth of dividends in real terms has been higher in the UK than in Canada, but that might not be the case if we started both series in the same year. There was good real growth of dividends in Britain in the 1920s, because prices fell by more than dividends did.

**Dividend yields and share prices**

Since dividends do not change by very much in the short term, most of the short-term changes in share prices are reflected in changes in the dividend yield. The standard deviation of the $Y(.)$ series is about the same as the standard deviation of share price changes.

If we ignore the small influence of inflation on dividend yields we have a simple first order autoregressive model for dividend yields. Looking at the data at monthly instead of yearly intervals I also find that a first order autoregressive model is a good first approximation, but the autoregressive parameter is much closer to one, about 0.96. This means that dividend yields have a tendency to drift back towards their long term mean by about 4% a month or about 0.2% per working day. It is not surprising that those who have analysed daily share price changes, often over periods like two or three years, have decided that this drift towards the mean is too small to be significant, and have come to the conclusion, erroneously in my view, that dividend yields or share prices do not have this mean-reverting drift element. It is not until we look at the data over a sufficiently long period that we see that share dividend yields are stationary.

A vivid way of describing the movement of share prices is that it is a drunken stagger about a random walk, where it is dividends that are changing randomly. While this would be roughly true of share dividends if one did not look at their dependence on
inflation, the pattern when one does take inflation into account is more complicated, though one can still say that, as a first approximation, dividends measured in real terms can be modelled by something close to a random walk.

*Simulated future share prices*

We can combine the three models so far, those for inflation, dividend yields and share dividends, to generate simulated future scenarios, and we can calculate share prices from our simulated dividends and yields.

Here (Figure 25) is the graph of actual Canadian share prices up to 1993, with one possible simulation for the next 57 years up to 2050. Here (Figure 26) are another six simulations, making seven in all. Each simulation corresponds to one particular inflation simulation from the earlier graph. It seems to me particularly important that the various series are simulated jointly. It is not consistent if there is very high inflation and very low share price rises, or vice versa.

You may think that the graphs of possible futures jump about a bit more than the past has. But if you do not like the details of my model, you are free to change them. In particular, actuaries often have strong views about where the means of the different distributions lie, the mean rate of inflation, the mean real rate of dividend growth, the mean yield on shares and so on. Once you have had some experience in using stochastic models, you may decide that you would rather use a larger or a smaller standard deviation than has arisen from the past observations. This seems to me a proper use of actuarial judgement, just as has always been done in relation to future mortality rates and future interest rates.

The way in which you adjust the model may also depend on the use you are going to put it to. Allowing rather larger fluctuations than have happened in the past may be exactly what you need to do if you wish to be a little bit cautious about, for example, solvency margins. On the other hand, if high fluctuations are more favourable, for example by generating higher profits by trading in and out of shares, then you may think that the more cautious option is to allow for lower variability.

*Long-term bond yields*

I have just mentioned interest rates, and my model extends also to these. First I modelled yields on long-term government bonds, as represented in the UK by the long-standing irredeemable stock Consols. It is not strictly irredeemable: the Government has the option to redeem it at par; but since the coupon rate is only $2\frac{1}{4}\%$, the Government would not redeem the stock unless it could refinance it at less than $2\frac{1}{4}\%$. This seems unlikely in present circumstances.
Figure 25. Canada Share Prices 1936 - 2050

Figure 26. Canada Share Prices 1936 - 2050
Here first (Figure 27) is a graph of the Consols yield since 1756. It fluctuated mainly in the 3% to 5% band for a large part of this time, bursting out well above these levels only in the 1970s.

About the beginning of this century the American economist Irving Fisher put forward the principle that nominal interest rates can be seen as the sum of two elements: an allowance for expected future inflation over the term of the contract, and an interest rate in real terms, which seems often to have been somewhere around 3½%. It was not really until the 1950s that people in Britain imagined that inflation was here to stay, so I think it is reasonable to look at the 200 years from 1756 to 1956 as a period when expected inflation was virtually zero. We can look at the scatter diagram (Figure 28) of the Consols yield in one year against the Consols yield in the preceding year, \(C(t)\) against \(C(t-1)\), and we see a strong correlation. In fact the correlation coefficient is about 0.9. This suggests that we could have modelled long-term interest over this period by another simple first order autoregressive model:

\[
\ln C(t) = CMU + CN(t)
\]

As usual \(CN(t)\) is a zero-mean process:

\[
CN(t) = CA \cdot CN(t-1) + CE(t),
\]

and \(CE(t) = CSD \cdot CZ(t)\).

\(CMU\) is the mean rate of interest, real and nominal being the same, \(CA\) is the usual autoregressive parameter, and the residuals are distributed with standard deviation \(CSD\).

Parameters for the UK for this period can be estimated as:

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<tbody>
<tr>
<td>UK</td>
<td></td>
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<tr>
<td>1756-1956</td>
<td></td>
</tr>
<tr>
<td>CA</td>
<td>0.94</td>
</tr>
<tr>
<td>CMU%</td>
<td>3.5%</td>
</tr>
<tr>
<td>CSD</td>
<td>0.07</td>
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The mean is 3½%, similar to what one would expect from a study of the enormous amounts of long-term historic data that is available about interest rates in different countries - see Sidney Homer's *magnum opus* - and similar also to the real yields at which index-linked Government bonds in Britain have traded.

When we come to more recent decades we wish to make an appropriate allowance for expected inflation. What I have done is to assume that lenders and borrowers base their ideas of prospective inflation in the quite long distant future on what it has been over some suitably long distant past period.

Here (Figure 29) is a graph of long term government bond yields in Canada. Much the same pattern as in Britain can be seen. Interest rates rose sharply during the
Figure 27. UK Consols Yield 1756 - 1993

Figure 28. UK Consols Yield 1757 - 1956
Figure 29. Canada Long Term Government Bond Yields
Log Scale
1936 - 1993

Figure 30. Canada Long Term Government Bond Yields and Treasury Bill Rate 1936 - 1993
In the 1970s in response to the high inflation at that time, but long term bond yields did not rise nearly enough in either country to compensate precisely for the current very high inflation. Rather, lenders adjusted upwards their views of prospective future inflation to less than the current annual rate.

I represent this influence of past inflation by using an exponentially weighted moving average, and I set up a model as follows:

\[ C(t) = CW \cdot CM(t) + CMU \cdot \exp CN(t), \]

where the moving average of past inflation, \( CM(t) \), is given by

\[ CM(t) = CD \cdot I(t) + (1 - CD) \cdot CM(t-1). \]

CD is the smoothing parameter and the parameter CW allows for inflation not having a fully proportional effect. CMU is the mean real rate of interest, and the zero-mean autoregressive part, \( CN(t) \), is given by

\[ CN(t) = CA \cdot CN(t-1) + CY \cdot YE(t) + CE(t). \]

CE(t) is given by

\[ CE(t) = CSD \cdot CZ(t), \]

and CZ(t) is a sequence of independent unit normal variables.

The CY.YE(t) term represents the fact that there is a certain amount of simultaneous correlation between the movements of share yields and interest rates.

The parameter values I have estimated for the two countries are shown in the table.

<table>
<thead>
<tr>
<th></th>
<th>Canada</th>
<th>UK</th>
</tr>
</thead>
<tbody>
<tr>
<td>CW</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>CD</td>
<td>0.04</td>
<td>0.045</td>
</tr>
<tr>
<td>CA</td>
<td>0.95</td>
<td>0.9</td>
</tr>
<tr>
<td>CMU%</td>
<td>3.7%</td>
<td>3.1%</td>
</tr>
<tr>
<td>CY</td>
<td>0.1</td>
<td>0.15</td>
</tr>
<tr>
<td>CSD</td>
<td>0.185</td>
<td>0.175</td>
</tr>
<tr>
<td>Median</td>
<td>7.1%</td>
<td>8.1%</td>
</tr>
</tbody>
</table>

The median this time is given by \( CMU + 100 \cdot QMU \).

In fact these parameters are not joint least squared estimates, which is what a statistical computer package would fit. Rather I have juggled the parameters, keeping a roughly 3½% mean real yield, and allowing inflation to be recognised gradually over 20 years or so. If one allows all the parameters to be optimised, the residual standard deviation can be brought down quite a lot, but the parameters become quite unrealistic, well outside their sensible ranges. Fitting this sort of model requires a fair amount of judgement; you might even say pre-judgement or prejudice. But I think the resulting model nevertheless is sensible and realistic.
Now let us consider short-term bond yields. For this I have used yields on 3-month Treasury bills for Canada, as shown in this graph (Figure 30). There was a period during the 1930s and 1940s of extremely low bill rates, and I have started my analysis in 1956.

I model short-term bond yields by reference to the long-term or irredemables rate. Note that I am not using zero-coupon bond yields, even though that would in some ways be nicer; but they are just not available for a long period. I look at the ratio of the short-term bond yield, B(t), to the long-term bond yield, C(t), (Figure 31). I then write:

\[ \ln B(t) = \ln C(t) + BMU + BN(t). \]

BMU is a mean difference in the logs, which is negative, because short-term yields are usually lower than long-term ones. BN(.) is a zero mean part, modelled again as a first order autoregressive series:

\[ BN(t) = BA.BN(t-1) + BC.CE(t) + BE(t) \]

where as usual

\[ BE(t) = BSD.BZ(t), \]

and BZ(t) is i.i.d unit normal. BC.CE(t) is an extra influence from the residuals of the consols yield. In fact this is zero in Britain. There could well be other ways of representing short-term interest rates, but this model fits the data quite well.

Parameter values for the two countries are:

<table>
<thead>
<tr>
<th></th>
<th>Canada</th>
<th>UK</th>
</tr>
</thead>
<tbody>
<tr>
<td>BMU</td>
<td>-0.26</td>
<td>-0.185</td>
</tr>
<tr>
<td>BA</td>
<td>0.38</td>
<td>0.75</td>
</tr>
<tr>
<td>BC</td>
<td>0.73</td>
<td>0.0</td>
</tr>
<tr>
<td>BSD</td>
<td>0.21</td>
<td>0.175</td>
</tr>
<tr>
<td>Median</td>
<td>5.47%</td>
<td>6.73%</td>
</tr>
</tbody>
</table>

The median short-term yield is given by (Mean C).exp(BMU), and this too is shown in the table.
Figure 31. Canada Long Term Government Bond Yields and Treasury Bill Rate (Log Ratio) 1936 - 1993

Figure 32. Exchange Rate: Canada v UK 1972 - 1993
**Some results**

It is now time to show some results of simulations created from the model so far. I have calculated the following tables by simulation, from the Canadian parameters, using what I call 'neutral' starting values, that is the stationary values we would get if the residuals were always zero. An alternative would be to use market conditions at some chosen date, either those for an actual date or an arbitrarily chosen set of values.

I have presented the results in a series of tables. In each case I have simulated 1,000 possible futures, calculated the relevant price index or total return index for $n$ years; taken the $n$th root to calculate the percentage internal return, and then calculated the means and standard deviations, and selected correlation coefficients of the resulting variables.

In the first table I show the mean rate of inflation $E(GQ)$ over 1, 2, 5, 10, 20 and 50 years, and also the standard deviation, $SD(GQ)$. The mean rate of inflation is about 3.4%, and the standard deviation reduces as the period increases.

Also shown are the results for a total return or accumulation index for shares. The mean is $E(GP)$, the standard deviation is $SD(GP)$ and the correlation coefficient between $GQ$ and $GP$, inflation and share returns, is $C(GQ,GP)$. You can see how the mean rate of return on shares is very high over one year; but the standard deviation over one year is also high, and the high mean results from a feature of the lognormal distribution. If the median is $\exp(\mu)$ then the mean is $\exp(\mu + \frac{1}{2}\sigma^2)$, so a high value of $\sigma$ puts up the mean. The mean reduces as the period increases, as does the standard deviation.

There is a virtually zero correlation between share returns and inflation over one year, but quite a strong correlation as the period lengthens. This is consistent both with what has been observed by the short-term analysts, and with the widely held belief that shares are a hedge against inflation in the long run.
Mean rate of inflation - GQ

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(GQ)</td>
<td>3.53</td>
<td>3.53</td>
<td>3.46</td>
<td>3.40</td>
<td>3.45</td>
<td>3.40</td>
</tr>
<tr>
<td>SD(GQ)</td>
<td>3.21</td>
<td>3.12</td>
<td>2.84</td>
<td>2.47</td>
<td>1.88</td>
<td>1.27</td>
</tr>
</tbody>
</table>

Mean rate of total return on shares - nominal - GP

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(GP)</td>
<td>9.56</td>
<td>8.77</td>
<td>7.88</td>
<td>7.67</td>
<td>7.72</td>
<td>7.59</td>
</tr>
<tr>
<td>SD(GP)</td>
<td>21.31</td>
<td>14.48</td>
<td>7.90</td>
<td>5.01</td>
<td>3.39</td>
<td>2.20</td>
</tr>
<tr>
<td>C(GP,GQ)</td>
<td>-0.01</td>
<td>0.05</td>
<td>0.24</td>
<td>0.43</td>
<td>0.56</td>
<td>0.60</td>
</tr>
</tbody>
</table>

The next table shows similar results for a total return index based on long bonds. In fact I have assumed that long bonds are irredeemables. The mean returns are lower than from shares, though they are very close by 50 years out, and the standard deviations are also lower. This is consistent with the notion that more variable investments carry higher expected returns. The correlation coefficients with inflation are initially negative, but become positive after a long enough period, and the correlation coefficients with share returns are quite small, but move back and forth between positive and negative.

Mean rate of total return on bonds - nominal - GC

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(GC)</td>
<td>7.14</td>
<td>7.22</td>
<td>7.18</td>
<td>7.24</td>
<td>7.32</td>
<td>7.56</td>
</tr>
<tr>
<td>SD(GC)</td>
<td>9.55</td>
<td>6.42</td>
<td>3.58</td>
<td>2.04</td>
<td>1.20</td>
<td>1.56</td>
</tr>
<tr>
<td>C(GC,GQ)</td>
<td>-0.20</td>
<td>-0.29</td>
<td>-0.37</td>
<td>-0.42</td>
<td>-0.17</td>
<td>0.23</td>
</tr>
<tr>
<td>C(GC,GP)</td>
<td>0.06</td>
<td>0.06</td>
<td>-0.01</td>
<td>-0.11</td>
<td>-0.02</td>
<td>0.19</td>
</tr>
</tbody>
</table>

The results for bills are more or less as expected. I have calculated these by assuming that the bill rate is in fact a one-year rate; this is not precise, but it gives an idea of the results. The mean returns are lower than those on long bonds, but the standard deviation increases with duration until it exceeds that for long bonds by 20 years.

Mean rate of total return on bills - nominal - GB

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(GB)</td>
<td>5.47</td>
<td>5.66</td>
<td>5.79</td>
<td>5.91</td>
<td>6.04</td>
<td>6.20</td>
</tr>
<tr>
<td>SD(GB)</td>
<td>0.0</td>
<td>0.94</td>
<td>1.24</td>
<td>1.42</td>
<td>1.63</td>
<td>1.73</td>
</tr>
<tr>
<td>C(GB,GQ)</td>
<td>0.0</td>
<td>0.04</td>
<td>0.09</td>
<td>0.19</td>
<td>0.23</td>
<td>0.32</td>
</tr>
<tr>
<td>C(GB,GP)</td>
<td>0.0</td>
<td>-0.03</td>
<td>0.00</td>
<td>0.08</td>
<td>0.16</td>
<td>0.21</td>
</tr>
<tr>
<td>C(GB,GC)</td>
<td>0.0</td>
<td>-0.45</td>
<td>-0.58</td>
<td>-0.48</td>
<td>0.23</td>
<td>0.87</td>
</tr>
</tbody>
</table>

The next set of tables shows the same sort of results calculated for total returns in real terms, that is after dividing the resulting index by the consumer price index. I
shall not go through them in detail, but note that although the mean real returns can be calculated approximately just by subtracting the mean rate of inflation, the standard deviations and the correlation coefficients do not behave so tidily. In particular, shares have a lower standard deviation than bonds, measured in real terms, over a long enough period.

Mean rate of total return on shares - real - JP

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(JP)</td>
<td>5.93</td>
<td>5.14</td>
<td>4.30</td>
<td>4.14</td>
<td>4.12</td>
<td>4.05</td>
</tr>
<tr>
<td>SD(JP)</td>
<td>20.87</td>
<td>14.17</td>
<td>7.49</td>
<td>4.38</td>
<td>2.71</td>
<td>1.71</td>
</tr>
<tr>
<td>C(JP,GQ)</td>
<td>-0.17</td>
<td>-0.17</td>
<td>-0.14</td>
<td>-0.09</td>
<td>-0.02</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Mean rate of total return on bonds - real - JC

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(JC)</td>
<td>3.64</td>
<td>3.72</td>
<td>3.72</td>
<td>3.80</td>
<td>3.77</td>
<td>4.03</td>
</tr>
<tr>
<td>SD(JC)</td>
<td>10.40</td>
<td>7.74</td>
<td>5.25</td>
<td>3.78</td>
<td>2.37</td>
<td>1.74</td>
</tr>
<tr>
<td>C(JC,GQ)</td>
<td>-0.49</td>
<td>-0.64</td>
<td>-0.78</td>
<td>-0.88</td>
<td>-0.87</td>
<td>-0.54</td>
</tr>
<tr>
<td>C(JC,JP)</td>
<td>0.13</td>
<td>0.17</td>
<td>0.16</td>
<td>0.12</td>
<td>0.06</td>
<td>0.06</td>
</tr>
</tbody>
</table>

Mean rate of total return on bills - real - JB

<table>
<thead>
<tr>
<th>Term</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(JB)</td>
<td>1.97</td>
<td>2.15</td>
<td>2.33</td>
<td>2.49</td>
<td>2.52</td>
<td>2.71</td>
</tr>
<tr>
<td>SD(JB)</td>
<td>3.16</td>
<td>3.19</td>
<td>2.94</td>
<td>2.57</td>
<td>2.14</td>
<td>1.75</td>
</tr>
<tr>
<td>C(JB,GQ)</td>
<td>-1.0</td>
<td>-0.96</td>
<td>-0.91</td>
<td>-0.85</td>
<td>-0.70</td>
<td>-0.42</td>
</tr>
<tr>
<td>C(JB,JP)</td>
<td>0.17</td>
<td>0.16</td>
<td>0.12</td>
<td>0.08</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>C(JB,JW)</td>
<td>0.49</td>
<td>0.51</td>
<td>0.57</td>
<td>0.63</td>
<td>0.71</td>
<td>0.89</td>
</tr>
</tbody>
</table>

Wages index

I have developed a model for a wages or earnings index, but only for the United Kingdom. I don’t happen to have data for a Canadian wages index for a long period.

I have modelled the wages index, $W(t)$, as depending only on current inflation and on the previous year’s inflation, plus a mean term, plus an autoregressive part, thus:

$$\ln W(t) - \ln W(t-1) = WW.I(t) + (1 - WW).I(t-1) + WMU + WN(t)$$
where $WW$ and $1 - WW$ are the effects of this year's and last year's inflation, and 
$WMU$ is the mean rate of growth of wages in real terms. The autoregressive part, 
$WN(t)$, is given by:

$$WN(t) = WA \cdot WN(t-1) + WE(t)$$

with

$$WE(t) = WSD \cdot WZ(t)$$

and $WZ(t)$ i.i.d. normal as usual.

An alternative way of modelling prices and wages jointly would be to treat them in
a vector autoregressive (VAR) model, so that each might depend on previous values
of itself and of the other. I have not done this, first because I already had a model
for inflation, and secondly because the model I have just shown fits the data
adequately.

Parameters for this model in the United Kingdom are:

**UK**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$WW$</td>
<td>0.7</td>
</tr>
<tr>
<td>$WMU$</td>
<td>0.0165</td>
</tr>
<tr>
<td>$WA$</td>
<td>0.12</td>
</tr>
<tr>
<td>$WSD$</td>
<td>0.025</td>
</tr>
</tbody>
</table>

That is, wages this year increase by 70% of this year's price inflation, plus 30% of
last year's price inflation, plus about 1.65% real increase, plus 12% of last year's
extra change, plus a residual with a 2½% standard deviation.

I expect that a similar model might well fit Canadian data.

*Other countries*

I should say that I have recently been fitting these sorts of model to data from a lot
of different countries. There are differences in the parameters, but the inflation
model fits many countries quite well. Although the means are different, the
autoregressive parameters and the standard deviations are reasonably similar in each
of the OECD countries that I have looked at. I have not considered countries that
suffer from extremely high inflation rates. One also needs to take into account any
simultaneous correlations between countries, i.e. correlations between the residuals
$QE(\cdot)$, $YE(\cdot)$, $DE(\cdot)$, etc, which are certainly significant, as well as investigating
lagged cross-correlations, and hoping that they are not significant, because it would
make the model much more complicated if they were.
Exchange rates

In order to combine models for two or more countries it is necessary to bring in exchange rates. In my paper to the Montreal Congress in 1992 I showed how one could model exchange rates for three countries, in that case the United Kingdom, United States, and France. Today I shall just discuss the exchange rate between Canadian dollars and UK pounds, which I shall call $X(t)$.

A hypothesis put forward by economists is that exchange rates conform to 'purchasing power parity', that is they adjust exactly for changes in the price levels in the two countries. If we call the countries b and c, we can put:

$$X(t).Q_b(t)/Q_c(t) = \text{constant}.$$ 

Study of the data soon makes it clear that the left hand side of this equation is not constant, so instead I put:

$$\ln X(t) = X_{MU} + \ln Q_b(t) - \ln Q_c(t) + X_N(t)$$

that is the actual exchange rate depends on a fixed mean, which represents both the scale of the two currencies, and the base values of the relevant price indices $Q_b(.)$ and $Q_c(.)$, and it adjusts for changes in those price indices. As usual, $X_N(t)$ is a zero mean autoregressive part:

$$X_N(t) = X_A.X_N(t-1) + X_E(t)$$

with autoregressive parameter $X_A$, and residuals $X_E(t)$:

$$X_E(t) = X_{SD}.X_Z(t)$$

with $X_Z(t)$ i.i.d. unit normal.

This model fits the data reasonably well for a lot of countries. Here (Figure 32) is a graph for Canada and the UK, showing the actual exchange rate wandering up and down, and the 'PPP' rate cutting through the middle of the wanderings. Provided that one's estimate of the mean is correct, which is reflected in the vertical position of the PPP rate on the graph, one can estimate whether the exchange rate is high or low relative to the PPP rate. But it may easily be a long time before this apparent anomaly corrects itself, so I would not recommend this method of analysis for short-term trading on the exchange. But it ought to be good in the long term.

Parameter values for this exchange rate are:

<table>
<thead>
<tr>
<th>Country</th>
<th>$X_{MU}$</th>
<th>$X_A$</th>
<th>$X_{SD}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canada v. UK</td>
<td>0.63</td>
<td>0.62</td>
<td>0.12</td>
</tr>
</tbody>
</table>

The $X_A$ parameter is typically in the range 0.6 to 0.7 for most exchange rates, but the standard deviations vary quite a lot because certain currencies are much more closely connected than others.
The paper I prepared for the 4th AFIR International Colloquium in Orlando in April 1994 discusses more about exchange rates and consumer price indices in many countries.

Uses of stochastic models

I hope that what I have said so far, in particular the tables of results that I have presented, gives some indication of the sorts of way in which one can use such a comprehensive stochastic investment model. An early version of this model was first used from about 1980 by British life insurance companies which had written equity-linked contracts with minimum money guarantees to assess the contingency reserves that might be needed to cover such guarantees. Others in Britain have used my model or similar ones for asset-liability modelling, investigation of solvency requirements for both life insurance and general insurance (casualty) companies, investigation of bonus policy for life offices, and investigation of various structures for pension funds.

There are those who will adhere to the pure random walk, or pure diffusion process school of thought, but if you write the usual diffusion equation in its full form:

$$dP = \mu(P,t).P.dt + \sigma(P,t).P.dz$$

you can see that this allows both $\mu$ and $\sigma$ to vary with $P$ and $t$. I in fact assume a constant $\sigma$, although it would be possible to introduce models with varying standard deviations, such as autoregressive conditional heteroscedastic (ARCH) models. But I allow the $\mu$ term to vary, not just with the value of $P$, the share price or the consumer price index, but with the other features, the current dividend yield, the recent or carried forward values of inflation, and so on.

My model does not allow arbitrage, but it does mean that at times, for instance when dividend yields are unusually high or low, one can reasonably confidently say that the expected returns on shares are also unusually high or low. Expected returns are not necessarily constant at all times, and the market is not necessarily perfectly efficient when profits from inefficiencies can only be realised over several years.

Some recent investigators in North America seem to have noticed that dividend yields are tolerable predictors of share price performance, and of course the various stochastic models for interest rates that have been proposed are not all consistent with a pure random walk model for total returns on bonds. But I should like to hear what others have to say about models developed like mine, on the basis of empirical observation combined with what I hope is a suitable dash of common sense.
BIBLIOGRAPHY


