OPTIMAL DYNAMIC XL REINSURANCE

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ABSTRACT

We consider a risk process modelled as a compound Poisson process. We find the optimal dynamic unlimited excess of loss reinsurance strategy to minimize infinite time ruin probability, and prove the existence of a smooth solution of the corresponding Hamilton-Jacobi-Bellman equation as well as a verification theorem. Numerical examples with exponential, shifted exponential, and Pareto claims are given.

KEYWORDS

Stochastic control, Ruin probability, XL reinsurance

1. INTRODUCTION

Assume an insurance company has the possibility to choose and buy dynamically an unlimited excess of loss reinsurance. For this situation, stochastic control theory is used to derive the optimal reinsurance strategy which minimizes ruin probability when the reinsurer computes his premium according to the expected value principle. The corresponding problem has been solved by Schmidli (2000) for the case of dynamic proportional reinsurance.

We model the risk process R_t of an insurance company by a Lundberg process with claim arrival intensity λ and absolutely continuous claim size distribution Q. The number of claims A_t in a time interval $(0, t]$ is a Poisson process with intensity λ, and the claim sizes U_i, $i = 1, 2, \dots$ are positive iid variables independent of A_t. Let T_i be the occurrence time of the i-th claim, $i = 1, 2, \dots, c$ the premium intensity of the insurer which contains a positive safety loading

$$c > \lambda E[U_i],$$

and $s = R_0$ the initial reserve. Then – without reinsurance – the surplus of the insurance company at time t is

$$R_t = s + ct - \sum_{i=1}^{A_t} U_i .$$

The reinsurer uses the expected value principle with safety loading $\theta > 0$ for premium calculation. We assume $(1+\theta)\lambda E[U_i] > c$, because otherwise the insurer could get rid of all his risk by reinsuring his total portfolio.

Excess of loss reinsurance is a non proportional risk sharing contract in which, for a given retention level $b \geq 0$ a claim of size U is divided into the cedent's payment $\min\{U, b\}$ and the reinsurer's payment $(U-b)^+ = U - \min\{U, b\}$. In this paper the retention level is assumed to be chosen dynamically, i.e. the insurer adjusts the retention level b_t at every time $t \geq 0$, based on the information available just before time t: If $\mathcal{F}_t$ is the sigma-field generated by $R_u, u \leq t$, then b_t is assumed to be predictable (a pointwise limit of left continuous $\mathcal{F}_t$ adapted processes), i.e. it is a measurable function of s and the times and sizes of claims occurring before t. It can be represented by a sequence of functions $\pi_n, n = 0, 1, 2, \ldots$ with $\pi_n : \mathbb{R}^{2n+1} \to \mathbb{R}$ measurable and

$$b_t = \pi_n(T_1, \ldots, T_n, U_1, \ldots, U_n, t - T_n) \text{ for } T_n < t \leq T_{n+1}.$$

We will show that the optimal reinsurance strategy exists and is given via a feedback equation of the following form:

$$b_t = b(R^b_{t-}),$$

where R^b_t is the surplus process with strategy b_t, and $b(s)$ is a measurable function. In particular, the optimal strategy is Markovian, i.e. it depends on the actual surplus only and not on the history of the process. Let b_t be an arbitrary dynamic reinsurance strategy. Then with $\rho = (1+\theta)\lambda$,

$$R^b_t = s + ct - \rho \int_0^t E\left[(U - b_x)^+\right] dx - \sum_{i=1}^{A_t} \min\left\{U_i, b_{T_i}\right\} \tag{1}$$

is the surplus process under the strategy b_t.

Our aim is to minimize ruin probability which is the same as maximizing survival probability. The ruin time τ_b is the first time the surplus of the insurance company ever becomes negative using reinsurance strategy b_t. It is given by

$$\tau_b = \inf\left\{t \geq 0 : R^b_t < 0\right\}.$$

Then we can write the ruin probability as

$$\psi_b(s) = P(\tau_b < \infty).$$

With

$$\delta_b(s) = P\left(\tau_b = \infty \mid R_0 = s\right) = 1 - \psi_b(s)$$

we will compute the function

$$\delta(s) = \sup_b \{\delta_b(s)\},$$

and find an optimal strategy b_t^*, such that $\delta(s) = \delta_{b^*}(s)$.

A more realistic problem would have a loading of the reinsurer which varies with the retention level (e.g. if instead of the expected value principle one would use the variance principle). Furthermore, one should also consider limited XL-covers, and then both, the retention and the limit, will be considered as control variables, see [7].

2. Hamilton-Jacobi-Bellman equation

The computation of the optimal reinsurance strategy is based on the classical Hamilton-Jacobi-Bellman equation which can be derived heuristically considering (1) on a short time interval $[0, \Delta]$ in which a constant strategy b is used. One of the following two cases can occur:

1. There is no claim in $[0, \Delta]$, which happens with probability $1 - \lambda\Delta + o(\Delta)$. Then the reserve of the company at time Δ is given by

$$R_\Delta = s + (c - \rho E[(U-b)^+])\Delta.$$

2. There is exactly one claim with claim size $U \sim Q$ in $(0, \Delta]$ and this happens with probability $\lambda\Delta + o(\Delta)$. Then the reserve can be written as

$$R_\Delta = s + (c - \rho E[(U-b)^+])\Delta - \min\{U, b\}.$$

Taking expectations and averaging over all possible claim sizes, we arrive at the equation

$$\begin{aligned}\delta_b(s) = (\lambda\Delta + o(\Delta))\, E\left[\delta\left(s + \left(c - \rho E\left[(U-b)^+\right]\right) - \min\{U,b\}\right)\right] \\ + (1 - \lambda\Delta + o(\Delta))\,\delta\left(s + \left(c - \rho E\left[(U-b)^+\right]\right)\Delta\right) + o(\Delta).\end{aligned}$$

For $\Delta \to 0$ we obtain for a smooth function $\delta(s)$

$$0 = \lambda E[\delta(s - \min\{U,b\})] - \lambda\delta(s) + \left(c - \rho E[(U-b)^+]\right)\delta'(s)$$

and finally by maximizing over all possible values for b the Hamilton-Jacobi-Bellman equation for our optimization problem:

$$0 = \sup_{b>0}\left\{\lambda E[\delta(s - \min\{U,b\}) - \delta(s)] + (c - \rho E[(U-b)^+])\delta'(s)\right\} \qquad (2)$$

An optimal strategy is derived from a solution $(\delta(s), b^*(s))$ of the equation (2), where $b^*(s)$ is the point at which the supremum in (2) is attained.

The insurance company has a non negative net premium income if

$$c \geq \rho E[(U-b)^+].$$

Let $\underline{b}$ be the value where equality holds:

$$c = \rho E[(U-\underline{b})^+].$$

Since we are looking for a nondecreasing solution of equation (2) we can rewrite it as

$$\delta'(s) = \inf_{b > \underline{b}} \left\{ \lambda \frac{\delta(s) - E\left[\delta(s - \min\{U, b\})\right]}{c - \rho E\left[(U-b)^+\right]} \right\}. \tag{3}$$

3. Existence of a solution

In this section we shall prove the existence of a solution of equation (2). This will be done through a monotonicity argument, similar to the approach in [2].

Theorem 1 *Assume the claim size distribution Q is absolutely continuous. There exists a nondecreasing solution $V(s)$ of the Hamilton-Jacobi-Bellman equation (2) which is continuous on $[0, \infty)$, continuously differentiable on $(0, \infty)$, with $V(s) = 0$ for $s < 0$, and $V(s) \to 1$ for $s \to \infty$.*

Proof. Define a sequence $V_n(s)$ via $V_0(s) = \delta_0(s)$, the ruin probability without reinsurance (which means $b = \infty$ or $b = M$ if $P(U \geq M) = 0$) for $n = 0$, and through the recursion

$$V'_{n+1}(s) = \inf_{b > 0} \left\{ \lambda \frac{V_n(s) - E\left[V_n(s - \min\{U, b\})\right]}{c - \rho E[(U-b)^+]} \right\}, n = 0, 1, \ldots. \tag{4}$$

We show by induction that $V_n'(s)$, $n = 0, 1, 2, \ldots$ is a decreasing sequence. For $n = 0$ we have

$$V_0'(s) = \lambda \frac{V_0(s) - E[V_0(s-U)]}{c}$$

(see [1]. p. 4) and from (4) we get for $n = 0$:

$$V_1'(s) = \inf_{b>0} \lambda \left\{ \frac{V_0(s) - E[V_0(s - \min\{U, b\})]}{c - \rho E[(U-b)^+]} \right\}.$$

Thus we have $V_1'(s) \leq V_0'(s)$ for all $s \geq 0$. Now let $n \geq 1$ and s be fixed. For all b we have

$$
\begin{aligned}
V'_{n+1}(s)(c-\rho E[(U-b)^{+}]) &\leq \lambda V_n(s) - \lambda E[Vn(s-\min\{U,b\})] \\
&= \lambda E\left[\int_{s-\min\{U,b\}}^{s} V'_n(u)du\right] \\
&\leq \lambda E\left[\int_{s-\min\{U,b\}}^{s} V'_{n-1}(u)du\right] \\
&= \lambda V_{n-1}(s) - \lambda E\left[V_{n-1}(s-\min\{U,b\})\right].
\end{aligned}
$$

Here we used the induction hypothesis $V'_n(s) \leq V'_{n-1}(s)$ for all $s \geq 0$. Since b was arbitrary, we can switch to the infimum which gives us the required result

$$V'_{n+1}(s) \leq V'_n(s).$$

So $V'_n(s)$ is a decreasing sequence of continuous functions, and since $V'_n(s) > 0$ the sequence $V'_n(s)$ converges to a function $g(s)$, and with

$$V(s) = 1 - \int_s^{\infty} g(u)du$$

we have a nondecreasing continuous function $V(s)$ satisfying

$$g(s) = \inf_{b \geq \underline{b}} \left\{ \lambda \frac{V(s) - E[V(s-\min\{U,b\})]}{c - \rho E\left[(U-b)^{+}\right]} \right\}.$$

What is left is a proof for continuity of $g(s)$: then

$$V'(s) = g(s)$$

is continuous, and $V(s)$ satisfies equation (2). We first show that $g(s) > 0$ for all $s \geq 0$. The function $g(s)$ is the limit of the functions $V'_n(s)$. If the infimum in (4) is not attained in $[\underline{b}, s]$ then it is attained at $b = \infty$, and hence $V'_n(s)$ and $V'_0(s)$ are proportional for small s, i.e.

$$V'_n(s) \propto V'_0(s), 0 \leq s < \underline{b}.$$

Furthermore,

$$g(0) = \frac{\lambda V(0)}{c} > 0,$$

which implies $g(s) > 0$ for $0 \leq s < \underline{b}$. Assume that

$$s_0 = \inf\{s: g(s) = 0\} < \infty.$$

Then $s_0 \geq \underline{b}$, and there exists $s_0 \leq s < s_0 + \underline{b}$ for which $g(s) = 0$ or

$$\inf_{b \geq \underline{b}} \{V(s) - E[V(s - \min\{U, b\}]\} =$$
$$V(s) - E[V(s - \min\{U, \underline{b}\})] = 0,$$

i.e. $V(s) = V(s - \underline{b})$ (notice that $P(U > \underline{b}) > 0$). Then

$$0 = \int_{s-\underline{b}}^{s} g(u)du \geq \int_{s-\underline{b}}^{s_0} g(u)du$$

which contradicts the choice of s_0.

We next show that in the definition of the functions $V_n(s)$, $s \leq K$, the infimum can be restricted to the region $[b_1, \infty]$, where $b_1 > \underline{b}$. Assume the contrary, i.e. there exists a sequence $0 \leq s_n \leq K$ and $b_n \to \underline{b}$ such that

$$V'_{n+1}(s_n) \geq \lambda \frac{V_n(s_n) - E[V_n(s_n - \min\{U, b_n\})]}{c - \rho E[(U - b_n)^+]} - \frac{1}{n} \geq V'_{n+1}(s_n) - \frac{1}{n}.$$

Since $0 \leq V'_n(s) \leq V'_0(s)$ and $c - \rho E[(U - b_n)^+] \to 0$, we obtain

$$V_n(s_n) - E[V_n(s_n - \min\{U, b_n\})] \to 0,$$

and therefore for each accumulation point s_0 of the sequence s_n

$$V(s_0) - E\left[V\left(s_0 - \min\{U, \underline{b}\}\right)\right] = 0 = g(s_0),$$

a contradiction.

Finally, the relation

$$|g(x) - g(y)|$$
$$\leq \sup_{b \geq b_1} \lambda \frac{V(x) - E[V(x - \min\{U, b\})]}{c - \rho E[(U - b)^+]} - \lambda \frac{V(y) - E[V(y - \min\{U, b\})]}{c - \rho E[(U - b)^+]}$$

for $x, y \geq 0$ implies continuity of $g(s)$. ∎

Remark 1 Let

$$V(s, b) = \lambda \frac{V(s) - E[V(s - \min\{U, b\})]}{c - \rho E[(U - b)^+]}$$

where $V(s)$ is a smooth solution of the Bellman equation (3) with the properties of Theorem 1. Then the infimum over $b \geq \underline{b}$ is either

$$\frac{\lambda}{c}(V(s) - E[V(s - U)])$$

(no reinsurance or $b = \infty$) or

$$\lambda \frac{V(s) - E[V(s-U)] - V(0)P\{U \geq s\}}{c - \rho E[(U-s)^+]}$$

$(b = s)$ or

$$\inf_{\underline{b} < b < s} \lambda \left\{ \frac{V(s) - E[V(s - \min\{U, b\})]}{c - \rho E[(U-b)^+]} \right\}.$$

Since $V(s,b)$ has a continuous derivative w.r.t. b in $(\underline{b}, s)$ this last infimum – if attained in this interval – is attained at the point b for which this derivative is zero or

$$\lambda V'(s-b) = \rho V'(s).$$

So in each case there is (possibly more than) one point $b \in (\underline{b}, \infty]$ at which the infimum is attained, and a measurable selection of these points yields a measurable function $b(s)$. The corresponding strategy b_t^* is admissible, it can be represented by

$$\pi_n(T_1, \ldots, T_n, U_1, \ldots, U_n, t) = b(R(T_n) + B(t)),$$

where $R(T_n)$ is a measurable function of $T_1, \ldots, T_n, U_1, \ldots, U_n$ and

$$B(t) = ct - \rho \int_0^t E[(U - b_x)^+] dx.$$

The retention $b = \infty$ (no reinsurance) will be optimal for small values of s: For $s \leq \underline{b} < b$ we have

$$V(s,b) = \lambda \frac{V(s) - E[V(s-U)]}{c - \rho E[(U-b)^+]}$$

which is maximal for $b = \infty$.

Remark 2. The function $V(s)$ will not be concave in general: notice that the survival probability $\delta_0(s)$ will not be concave in general, and $V'(s)$ will be proportional to $\delta_0'(s)$ for $0 \leq s \leq \underline{b}$. However, if $\delta_0(s)$ is concave, the function $V(s)$ constructed above will be concave, too. To see this we have to show that all the above functions $V_n(s)$ are concave which is done by induction. If $V_n(s)$ is concave, then for $s \geq 0$ and $h > 0$ we have for arbitrary b

$$\begin{aligned}
&V_n(s+h) - E[V_n(s+h-\min\{U,b\})] \\
&= E \int_{s+h-\min\{U,b\}}^{s+h} V_n'(u)du \\
&\leq E \int_{s-\min\{U,b\}}^{s} V_n'(u)du \\
&= V_n(s) - E\left[V_n(s - \min\{U,b\})\right]
\end{aligned}$$

and hence

$$V'_{n+1}(s+h) \leq V'_{n+1}(s).$$

4. Verification theorem

In this section we will show that the strategy b_t^* derived from the maximizer $b^*(s)$ in (2) maximizes the survival probability. This is done through the following verification theorem. Notice that this theorem also implies uniqueness of the solution.

Theorem 2 *The strategy b_t^* maximizes survival probability: For any* s > 0 *and arbitrary predictable strategy b_t with survival probability $\delta(s)$ we have*

$$V(s) \geq \delta(s),$$

with equality for $b_t = b_t^*$.

Proof. Let $V(s)$ be the smooth solution of (3) constructed in chapter 3, for which

$$0 \leq V(s) \leq 1$$

and

$$\lim_{s \to \infty} V(s) = 1.$$

We write $R(t)$ and $R^*(t)$ for the risk process of the insurance company with reinsurance strategy b_t and b_t^*, respectively, and initial capital s. Let τ and τ^* be the corresponding ruin times, X_t^*, X_t the stopped processes and W_t^*, W_t the stopped processes, transformed by $V(s)$, i.e.

$$W_t^* = V(X_t^*) = V\left(R^*\left(\min\{t, \tau^*\}\right)\right),$$
$$W_t = V(X_t) = V\left(R\left(\min\{t, \tau\}\right)\right).$$

Then, as in [5], p. 80, (2.16), we obtain

$$E[W_t] = V(s) + E\left[\int_0^t V'(X_s)\left(c - \rho E[(U - b_s)^+]\right) ds \right.$$
$$\left. + \lambda \int_0^t E[V(X_s - \min\{U, b_s\}) - V(X_s)] ds, \right]$$

and a corresponding formula for W_t^*. From the Hamilton-Jacobi-Bellman equation (2) we see that for all $t > 0$

$$E[W_t^*] = V(s) \geq E[W_t]. \tag{5}$$

Assume first that the predictable strategy b_t satisfies

$$b_t \geq B > 0 \text{ for all } t \geq 0, \tag{6}$$

where B satisfies $P\{U > B\} > 0$. We show that in this case the process $R(t)$ is unbounded on $\{\tau = \infty\}$. For this we prove

$$P\{R(t) \leq M \text{ for all } t \geq 0 \text{ and } \tau = \infty\} = 0 \tag{7}$$

for all $M > 0$. With $n > (M + c)/B$ the probability of more than n claims of size larger than B in an interval of length 1 is positive. Since the claims process has stationary and independent increments, with probability 1 there are more than n such claims in an interval $[t, t+1]$. For $R(t) \leq M$ we have

$$R(t+1) \leq M + c - nB < 0,$$

i.e. $\tau < \infty$. This proves (7).

For arbitrary ε we now construct a strategy b_t^+ with risk process $R^+(t)$ and ruin time τ^+ such that $P\{\tau = \infty \text{ and } \tau^+ < \infty\} < \varepsilon$ and $R^+(t) \to \infty$ on $\{\tau = \infty$ and $\tau^+ = \infty\}$. Let $M > s$ be sufficiently large such that $1 - \delta_0(M) < \varepsilon$ let $T = \inf\{t : R(t) = M\}$ which is finite almost everywhere on $\{\tau = \infty\}$, and define

$$b_t^+ = \begin{cases} b_t & \text{if } t \leq T \\ \infty & \text{if } t > T. \end{cases}$$

The strategy b_t^+ is predictable, and

$$P\{\tau = \infty, \tau^+ < \infty\} \leq 1 - \delta_0(M) < \varepsilon.$$

Furthermore, $T < \infty$ implies $R^+(t) \to \infty$.

Now repeat the above reasoning leading to (5) for $R^+(t)$ instead of $R(t)$. We obtain

$$E\left[V\left(R^*\left(\min\{t, \tau^*\}\right)\right)\right] = V(s) \geq E\left[V\left(R^+\left(\min\{t, \tau^+\}\right)\right)\right],$$

and with $t \to \infty$ we arrive with $V(R^*(\tau^*)) = 0$ and $V(R^+(\tau^+)) = 0$ at

$$\begin{aligned} P\{\tau^* = \infty\} &\geq V(s) \geq P\{\tau = \infty \text{ and } \tau^+ = \infty\} \\ &\geq P\{\tau = \infty\} - \varepsilon. \end{aligned}$$

Since ε was arbitrary, this is our assertion for the special case of a strategy b_t with property (6). In particular, since any solution $V(s)$ of (3) in the sense of

Theorem 1 will produce a strategy satisfying (6), we have uniqueness of the solution and $V(s) = P\{\tau^* = \infty\}$.

Next we show that for premium intensities $c, \bar{c}$ with $\bar{c} > c$ we have

$$W'(s) \geq \overline{W}'(s) \text{ for all } s \geq 0,$$

where $W(s)$ and $\overline{W}(s)$ are solutions to (3) with c and $\bar{c}$, respectively and $W(0) = \overline{W}(0) = \alpha$. Notice, that $W(s)$ and $\overline{W}(s)$ do not solve (3) in the sense of Theorem 1, the conditions $W(s) \to 1$ for $s \to \infty$ and $\overline{W}(s) \to 1$ for $s \to \infty$ will not hold. Let $W_n(s), \overline{W}_n(s)$ be the sequences constructed in the proof of Theorem 1 converging to $W(s)$ and $\overline{W}(s)$ with $W(s)$, respectively $\overline{W}(s)$ defined by

$$W(s) = \alpha + \int_0^s g(u)du \text{ and } \overline{W}(s) = \alpha + \int_0^s \bar{g}(u)du,$$

where $g(s)$, $\bar{g}(s)$ are the limits of the sequences $W_n'(s)$, $\bar{W}_n'(s)$. We prove by induction that

$$W_n'(s) \geq \overline{W}_n'(s), \quad n = 0, 1, 2, \ldots. \tag{8}$$

For $n = 0$ we have

$$W_0'(s) = \frac{\lambda}{c}(W_0(s) - E[W_0(s-U)]),$$
$$\overline{W_0'}(s) = \frac{\lambda}{\bar{c}}(\overline{W}_0(s) - E[\overline{W}_0(s-U)]).$$

At $s = 0$ we have $W_0'(s) > \bar{W}_0'(s)$. Assume now that

$$s_0 = \inf\{s: W_0'(s) \leq \overline{W}_0'(s)\} < \infty.$$

By continuity, $s_0 > 0$. Then

$$\overline{W}'_0(s_0) = \frac{\lambda}{\bar{c}} E\left[\int_{s_0-U}^{s_0} \overline{W}'_0(u)du\right] \leq \frac{\lambda}{\bar{c}} E\left[\int_{s_0-U}^{s_0} W_0'(u)du\right] < W_0'(s_0),$$

a contradiction. Assume now that (8) holds for n. Then for all $b > 0$

$$W_n(s) - E[W_n(s - \min\{U,b\})] = E\left[\int_{s-\min\{U,b\}}^{s} W_n'(u)du\right] \geq$$
$$E\left[\int_{s-\min\{U,b\}}^{s} \overline{W_n'}(u)du\right] = \overline{W}_n(s) - E\left[\overline{W}_n(s - \min\{U,b\})\right]$$

which implies $W_{n+1}'(s) \geq \overline{W}_{n+1}'(s)$ and finally the desired result $W(s) \geq \overline{W}(s)$ for all s.

Now let c_n converge monotonically to c from above, and $W_n(s)$ the corresponding solutions of (3) with c_n instead of c. Then the sequence of functions $W_n'(s)$ is monotone and bounded by $W_0'(s)$, the function corresponding to c. Let $g(s)$ and $W(s)$ be the limits of $W_n'(s)$ and $W_n(s)$, respectively. As in the proof of Theorem 1 we obtain continuity of $g(s)$, and so $W'(s) = g(s)$ and $W(s)$ is a solution of (3) with c. Uniqueness of the solution for every α implies $W(s) = W_0(s)$.

To obtain a solution $V(s)$ of (3) satisfying $V(s) \to 1$ for $s \to \infty$ let $V_n(s)$ be a sequence of functions with

$$V_n(s) = \frac{V_n(0)}{\alpha} W_n(s),$$

then for $n \to \infty$ we have

$$V_n(s) \to \frac{\gamma}{\alpha} W_0(s) \text{ or}$$
$$V_n(s) \to \gamma' V(s).$$

For $s \to \infty$ we have

$$1 = \frac{\gamma}{\alpha} W_0(s) = \gamma' V(s)$$

and therewith $\gamma' = 1$. The same argumentation with $\overline{V}_n(s)$ instead of $V_n(s)$ leads us to $\overline{V}(s)$ with $\overline{V}(s) \geq V(s)$ for $\bar{c} > c$. For fixed s and arbitrary small $\varepsilon > 0$ we can find $\bar{c} > c$ for which

$$\overline{V}(s) < V(s) + \varepsilon.$$

Let $\overline{R}(t)$ be the risk process with strategy b_t, premium intensity $\bar{c}$ and $\bar{\tau}$ its ruin time. Then on $\{\tau = \infty\}$ we have $\overline{R}(t) \to \infty$ and hence, with $\bar{\tau} \geq \tau$

$$\begin{aligned} P\{\tau = \infty\} &= \lim_{t \to \infty} E\left[\overline{V}(\overline{R}(\min\{t, \bar{\tau}\}))1_{\{\tau = \infty\}}\right] \\ &\leq \lim_{t \to \infty} E\left[\overline{V}(\overline{R}(\min\{t, \bar{\tau}\}))\right] \leq \overline{V}(s) < V(s) + \varepsilon. \end{aligned}$$

So with $\varepsilon \to 0$

$$P\{\tau^* = \infty\} \geq V(s) \geq P\{\tau = \infty\}$$

which proves the verification theorem. ■

5. Numerical Examples

Here we present numerical computations for three different claim size distributions. Our first example has exponential claim sizes with mean $1/m$. Even in this simple case it seems to be impossible to find an analytical solution of (2).

The survival probability of an insurance company using no reinsurance, i.e. $b_t = \infty$ for all t, can be expressed explicitly by

$$\delta(s) = 1 - \frac{\lambda}{mc} \exp\left(-\left(m - \frac{\lambda}{c}\right)s\right) \tag{9}$$

(see [4]. p. 164). We will use the same parameters as in [6], i.e. $m = 1$, (which implies $\mu = 1$), $\lambda = 1$, premium rate $c = 1.5$ and $\rho = 1.7$. Since $V(0)$ is unknown we start with $V(0) = \delta(0)$ from (9) and norm the function $V(s)$ replacing $V(s)$ by $V(s)/V(s_1)$ where s_1 is sufficiently large. Figure 1 gives the survival probabilities for no reinsurance (lower graph) and for optimal excess of loss reinsurance (upper graph) for reserves $s \in [0, 15]$. We see that optimal excess of loss reinsurance gives a considerably higher survival probability. Figure 2 gives the optimal strategy $b^*(s)$ for values $s \in [0, 5]$; for $s \geq 5$ the optimal strategy is nearly constant. For small s the optimal strategy is, as expected, to keep the whole risk. At the point $s \approx 0.376$ the optimal strategy is $b(s) = s$, which means that independent of the following claim size the reserve remains nonnegative immediately after the claim. For $s \gtrsim 0.797$ we have to choose strategies $b(s) < s$ and the optimal strategy tends to be constant. Figure 3 is used to explain the optimal strategy presented in Figure 2. For each curve we fixed s ($s = 0.4$ at upper graph, then $s = 0.59, s = 0.6, s = 0.61, s = 0.8$ and finally $s = 0.9$ at lowest graph) and calculated $V(s, b)$ (defined in Remark 1) for varying $b \in [0.15, 1]$. For s small $V(s, b)$ is minimized for $b = \infty$. For $s \in [0.376, 0.797]$ the minimum is achieved at the jump, which means $b = s$. For larger values of s, here for

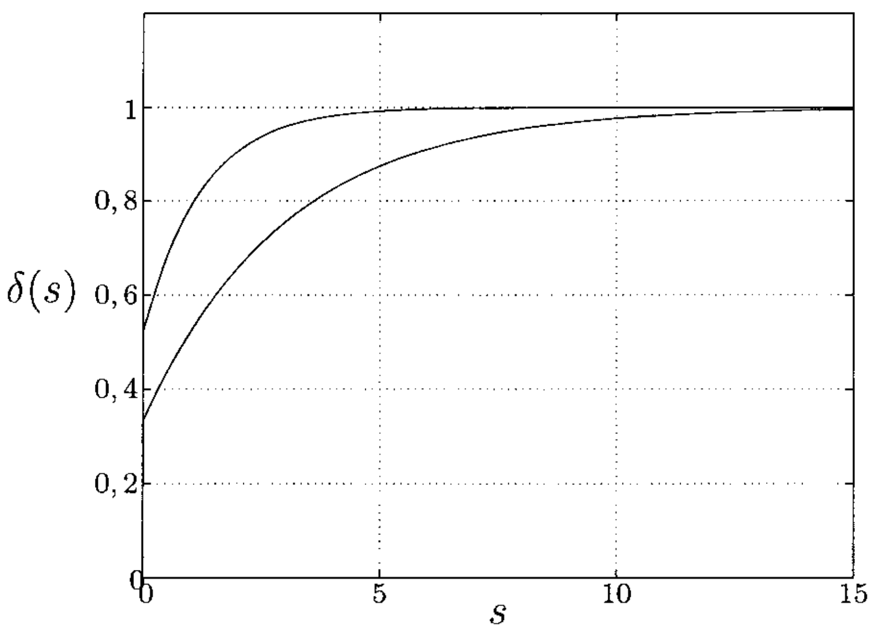

Figure 1: Survival probabilities

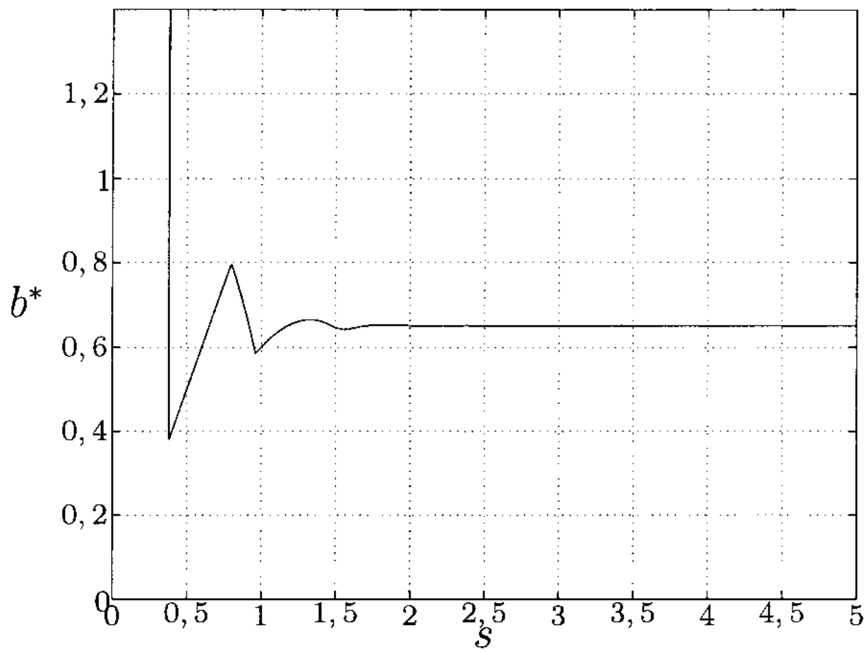

Figure 2: Optimal strategy for exponential distribution

example $s = 0.9$ the minimal $V(s, b)$ is achieved at a point before the jump. Looking at

$$E[\delta(s - \min\{U, b\})] = \int_0^b \delta(s - x) f(x) dx + \delta(s - b)(1 - F(b))$$

we can see why the jumps occur, if $b > s$ the term $\delta(s - b)(1 - F(b))$ equals zero. Figure 4 gives the optimal strategy $b^*(s), s \in [0, 15]$ in the case of a non concave solution $V(s)$. To achieve such a $V(s)$ we use a distribution with density

$$p(x) = m(\exp(-m(x - 1))), \quad x > 1$$

which is an exponential distribution shifted by 1, and solve the corresponding Hamilton-Jacobi-Bellman equation for parameters $m = 1$, $\lambda = 1$ and premium rates $c = 3$ and $\rho = 3.5$. Notice that in this case we have to choose $c > 2$ to keep the condition $c > \lambda E[U_i]$. In the last example we consider Pareto distributed claim sizes with parameter $a = 2$, i.e. claims with density

$$p(x) = 2(1 + x)^{-3}, x > 0.$$

Like in the first example we choose $\lambda = 1$ and the premium rates $c = 1.5$ and $\rho = 1.7$. Without reinsurance the survival probability at $s = 0$ is

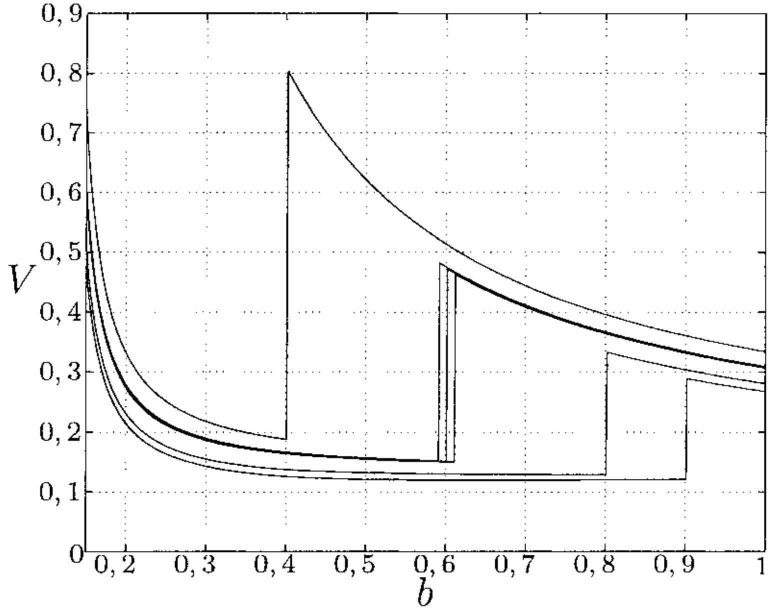

Figure 3: $V(s,b)$ for different values of s and varying b

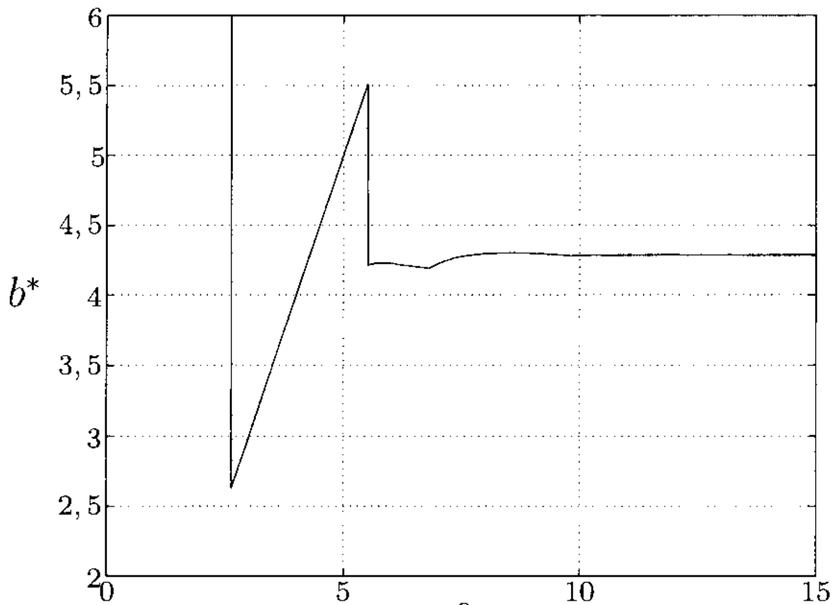

Figure 4: Optimal strategy for shifted exponential distribution

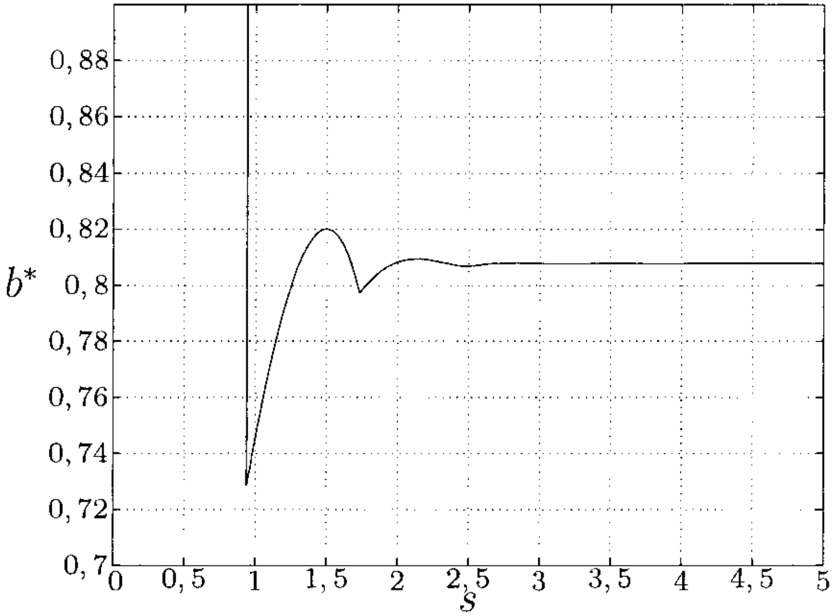

Figure 5: Optimal strategy for Pareto distribution

$$\delta(0) = 1 - \frac{\lambda}{c(a-1)}.$$

In Figure 5 we show the optimal reinsurance strategy for $s \in [0, 5]$ Contrary to the case of exponential distributed claim sizes there exists no interval in which we can choose $b^*(s) = s$. The optimal strategy for large values of s is constant, $b^*(s) \approx 0.8077$ for $s = 5$.

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