STOCHASTIC MODELS FOR ACTUARIAL USE: THE EQUILIBRIUM MODELLING OF LOCAL MARKETS

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ABSTRACT

In this paper, a long-term equilibrium model of a local market is developed. Subject to minor qualifications, the model is arbitrage-free. The variables modelled are the prices of risk-free zero-coupon bonds – both index-linked and conventional – and of equities, as well as the inflation rate. The model is developed in discrete (nominally annual) time, but allowance is made for processes in continuous time subject to continuous rebalancing. It is based on a model of the market portfolio comprising all the above-mentioned asset categories. The risk-free asset is taken to be the one-year index-linked bond. It is assumed that, conditionally upon information at the beginning of a year, market participants have homogeneous expectations with regard to the forthcoming year and make their decisions in mean-variance space. For the purposes of illustration, a descriptive version of the model is developed with reference to UK data. The parameters produced by that process may be used to inform the determination of those required for the use of the model as a predictive model. Illustrative results of simulations of the model are given.

KEYWORDS

Stochastic investment models, actuarial models, equilibrium models, no-arbitrage models, United Kingdom.

1. INTRODUCTION

Numerous stochastic models have been developed in the actuarial literature. In these models, the issues of arbitrage and equilibrium are generally not addressed; the models tend to be based on ex-post estimates. This means that they are essentially developed as descriptive models. While descriptive models may have use in actuarial practice, predictive models are needed for the pricing of the liabilities of financial institutions advised by actuaries and for the determination of capital adequacy. If a descriptive model is used for predictive purposes, however, it may produce risk-adjusted expected returns that exceed

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those of the market for some asset categories and understate those of the market for others.

If a model is to be used to indicate under- or over-priced asset categories, then it should not assume no-arbitrage or equilibrium conditions. For many actuarial applications, however, a model is required that will reflect market expectations. These applications include the estimation of fair-value prices of liabilities and the determination of benchmarks for the mandating of investment management and the measurement of investment performance. For the purposes of such applications, a model should be arbitrage-free. It should also arguably be an equilibrium model; that is, it should assume that, at any time, all market participants (including the financial institution concerned) are satisfied with their current exposures to the respective asset categories at current market prices after any adjustments at that time to their exposures and to those prices. Otherwise it cannot be held that the model reflects fair value.

While actuarial practice may relate to short-term liabilities, this paper focuses on long-term modelling typically required for life offices and retirement funds. A long-term equilibrium model of a local market is developed. Subject to minor qualifications, the model is arbitrage-free. The variables modelled are the prices of risk-free zero-coupon bonds – both index-linked and conventional – and of equities, as well as the inflation rate. These variables have been chosen as they comprise major constituent variables of the assets and liabilities of life offices and retirement funds. The model is developed in discrete (nominally annual) time, but allowance is made for processes in continuous time subject to continuous rebalancing. It is based on a model of the market portfolio comprising all the above-mentioned asset categories combined. That model is used as the basis of development of the arbitrage-free equilibrium model of its constituent asset categories. The risk-free asset is taken to be the one-year index-linked bond. It is assumed that, conditionally upon information at the beginning of a year, market participants have homogeneous expectations with regard to the forthcoming year and make their decisions in mean-variance space.

The distinction between a descriptive model and a predictive model is drawn in Thomson (2006). In this paper, that distinction is used to distinguish between the development and parameterisation of the proposed model for descriptive purposes and its parameterisation for predictive purposes. For the purposes of illustration, a descriptive version of the model is developed with reference to United Kingdom data. The parameters produced by that process may be used to inform the determination of those required for the use of the model for prediction purposes. Illustrative results of simulations of the model are given.

Relevant literature is reviewed in section 2. The theory of the equilibrium model is developed in section 3. The theory of the market model – that is, the model of the market portfolio – is developed in section 4. In section 5, a descriptive version of the model is estimated with reference to UK data. Illustrative results of simulations of the predictive model are presented in section 6. Conclusions are drawn, and some suggestions for further research are given, in section 7.

2. Literature Review

There have been a number of publications on the topic of stochastic models of investment returns. The Wilkie (1986) model was the first published stochastic investment return model for actuarial use. The model was extended in Wilkie (1995).

Thomson (1996) proposed a stochastic model of investment returns specifically for South Africa.

Whitten & Thomas (1999) suggested a threshold autoregressive system (Tong, 1990) as an improvement to the Wilkie (1995) model.

A model referred to by its authors as the "TY model" was proposed by Yakoubov, Teeger & Duval (1999).

All the above models include variously defined short-term and long-term interest rates. In effect, this implies a two-factor model for the term structure of interest rates. A model of the rest of the yield curve can be derived from the realisation of the short-term and long-term interest rates using the technique of principal-components analysis. (See Maitland (2002) for an application of this technique to the interpolation of the South African yield curve.)

Hibbert, Mowbray & Turnbull (unpublished) proposed a model that generates consistent values for the term structure of real and nominal interest rates, inflation rates, equity capital returns and dividend yields. They used a two-factor Hull-White (1990) model for the real interest rates.

Affine models are models in which the short-term interest rate can be expressed as (Dai & Singleton, 2000):

$$r(t) = \beta_0 + \sum_{i=1}^{M} \beta_i X_i(t);$$

where:

$X_i(t)$ are random state variables; and
M is the number of random factors driving the interest-rate model.

It has been shown (e.g. Duffie & Kahn, 1996) that affine models yield convenient closed-form expressions for the prices of zero-coupon bonds. For example, the price at time t of a zero coupon maturing at time T is expressed as:

$$P(t,T) = \exp\left(A(t,T) + \sum_{i=1}^{M} B_i(t,T)\, X_i(t)\right); \qquad (1)$$

where $A(t,T)$ and $B_i(t,T)$ are parameters expressed as functions of t and T whose forms need not concern us here. In other words, the exponent of the price formula is itself an affine function of the state variable. The model used by Hibbert, Mowbray & Turnbull (*op. cit.*) belongs to the affine class.

None of the above papers investigates the modelling of the market at equilibrium. While some of them may be arbitrage-free, the requirements of equilibrium are more exacting: under arbitrage-freedom a market participant cannot make a risk-free profit, whereas under equilibrium a market participant cannot even make a risk-adjusted profit.

3. The Equilibrium Model

In this section the theory of the arbitrage-free equilibrium model is developed. For the purposes of this section it is assumed that a model of the return during year t on the market portfolio has been developed, which may be expressed in the form:

$$\delta_{Mt} = \mu_{Mt} + \sigma_{Mt}\varepsilon_{Mt};$$

where:

μ_{Mt} is the expected return during that year, conditional on information at the start of that year;
σ_{Mt} is the standard deviation of the return during that year, conditional on information at the start of that year; and
$\varepsilon_{Mt} \sim \mathrm{N}(0,1)$ is such that $\mathrm{cov}(\varepsilon_{Mt}, \varepsilon_{Ms}) = 0$ for $t \neq s$.

The development of this model is deferred to section 4.

3.1. Assumptions

We assume that a local market comprises default-free index-linked and conventional government-issued zero-coupon bonds and equities. In principle, all risky capital assets (e.g. corporate loan stock, warrants and fixed property) should be included. Foreign assets should also be included to the extent to which local investors (i.e. investors with liabilities in the local currency) invest in such assets. For the purposes of this paper, such assets were excluded.

On the other hand the market is limited to capital assets in which equilibrium pricing may reasonably be supposed to be taking place. It therefore excludes unmarketable assets. In principle, also, assets held by foreign investors in local capital may be also excluded. This was not done in this paper. Derivative instruments and products issued by financial institutions should not be included. Only capital assets issued in the primary market to cover real investments in the economy should be included. For all other assets there are equal and opposite counterparties, whose holdings offset each other. The model may be used to price such instruments as described, for example, in Thomson (2005), but that is beyond the scope of this paper.

We further assume that market participants have homogeneous expectations and are able to borrow or lend unlimited amounts at the same risk-free return, and that the market is frictionless. Homogeneous expectations are required because, as indicated in section 1, a model is required that will reflect market expectations. The extension of the model to allow for differences between borrowing and lending rates and for friction is left for further research. At the end of a year, before decisions are made for the following year, the means and variances of factors affecting the average returns on each asset during the forthcoming year are known. (The choice of one-year intervals is arbitrary). For this purpose, we define the return on an asset during a year as the average instantaneous real rate of return over the year. At the beginning of the year, portfolios are selected by optimisation in mean-variance space so that the market is in equilibrium (i.e. so that all participants are satisfied with their positions in every asset at current prices). The extension of the model to higher moments than mean and variance is also a matter for further research. Real returns are used because, in the final analysis, equilibrium must relate to goods and services, not to currencies. Here the mean and variance are those of the returns during the forthcoming year.

Arising from these assumptions, the capital-asset pricing model (CAPM) applies to the local market for a particular year, conditionally upon information and expectations at the end of the previous year. This follows from the assumptions of the CAPM (Elton & Gruber, 1995: 295). This model ensures, inter alia, that the equilibrium condition is satisfied.

In this paper, a model of the form of equation (1) is used for zero-coupon bonds. Instead of adopting the usual approach of deriving the pricing formula from the process for the short-term interest rate, the reverse is done here. In this way one can allow greater generality in the modelling of the term structure, as well as using the current yield curve as the starting point for simulations.

In this paper, a two-factor term-structure model is proposed. Although a two-factor model may adequately capture the volatility of the yield curve (e.g. Maitland, *op. cit.*), it suffers from the problem of not being able to mimic the correlation between the forward rates of different maturities. In particular, a two-factor model will over-estimate the correlation between forward rates for neighbouring maturities and under-estimate the correlation between forward rates with maturity dates far apart (Rebonato, 1998). This matter is further discussed in section 7.

3.2. Index-linked bonds

Let $P_{It}(s)$ denote the price at time $t = 0, \ldots, T$ of an index-linked bond maturing at time $t + s$, where T is the time horizon to which projections will be required. For $i, k = 1, \ldots, 6$ let:

$$\varepsilon_{it} \sim \mathrm{N}(0,1); \text{ and} \tag{2}$$

$$\operatorname{cov}(\varepsilon_{it}, \varepsilon_{kt}) = 0 \text{ for } i \neq k. \tag{3}$$

Let:

$$Y_{I;t}(s) = -\ln\{P_{It}(s)\} = f_{It}(s)\{1 + b_{I1}(s)\eta_{1t} + b_{I2}(s)\eta_{2t}\}; \tag{4}$$

where:

for $j = 1,2$:

$$\eta_{jt} = \sum_{i=1}^{6} a_{ij}\varepsilon_{it}; \text{ and} \tag{5}$$

$$\sum_{i=1}^{6} a_{ij} = \sqrt{6}. \tag{6}$$

The reason for the six dimensions referred to in equations (5) and (6) becomes apparent below. In equation (6), the value is an arbitrary scaling factor; its value is set so as to simplify equation (26) below. The dependence of the parameter $f_{It}(s)$ on t is explained below. From (4) it follows that the return on that bond during year t – i.e. the interval $(t-1, t]$ – is:

$$\delta_{It}(s) = \ln\frac{P_{It}(s)}{P_{I,t-1}(s+1)} = Y_{I,t-1}(s+1) - Y_{It}(s). \tag{7}$$

The expected return is:

$$\mu_{It}(s) = Y_{I,t-1}(s+1) - f_{It}(s). \tag{8}$$

Thus, from (4):

$$\delta_{It}(s) = \mu_{It}(s) - f_{It}(s)\{b_{I1}(s)\eta_{1t} + b_{I2}(s)\eta_{2t}\}. \tag{9}$$

Without loss of generality:

$$b_{I1}(0) = b_{I2}(0) = 0.$$

Therefore, from (7), since $P_{It}(0) \equiv 1$, the risk-free return for year t is:

$$\delta_{It}(0) = Y_{I,t-1}(1). \tag{10}$$

Without loss of generality, η_{1t} and η_{2t} may be taken as the drivers of the short rate ($s = 1$) and a suitable long rate (say $s = \tau$) respectively so that:

$$\tfrac{1}{\tau} b_{I1}(\tau) = 0; \text{ and}$$

$$b_{I2}(1) = 0. \tag{11}$$

3.3. Inflation

The average instantaneous rate of inflation during year t is modelled as:

$$\gamma_t = \mu_{\gamma t} + b_\gamma \eta_{3t}; \tag{12}$$

where:

$$\eta_{3t} = \sum_{i=1}^{6} a_{i3}\, \varepsilon_{it}; \text{ and} \tag{13}$$

$$\sum_{i=1}^{6} a_{i3} = \sqrt{6}. \tag{14}$$

The determination of $\mu_{\gamma t}$ is explained in section 3.4 below.

3.4. Conventional bonds

Let $P_{Ct}(s)$ denote the price at time t of a conventional bond maturing at time $t+s$. Let:

$$Y_{Ct}(s) = \ln\{P_{Ct}(s)\} = f_{Ct}(s)\{1 + b_{C1}(s)\eta_{4t} + b_{C2}(s)\eta_{5t}\}; \tag{15}$$

where:

for $j = 4, 5$:

$$\eta_{jt} = \sum_{i=1}^{6} a_{ij}\, \varepsilon_{it}; \text{ and} \tag{16}$$

$$\sum_{i=1}^{6} a_{ij} = \sqrt{6}. \tag{17}$$

Then the return on that bond during year t is:

$$\delta_{Ct}(s) = Y_{C,t-1}(s+1) - Y_{Ct}(s) - \gamma_t. \tag{18}$$

and the expected return is:

$$\mu_{Ct}(s) = Y_{C,t-1}(s+1) - f_{Ct}(s) - \mu_{\gamma,t}. \tag{19}$$

Thus, from (12) and (15):

$$\delta_{Ct}(s) = \mu_{Ct}(s) - b_\gamma \eta_{3t} - f_{Ct}(s)\{b_{C1}(s)\eta_{4t} + b_{C2}(s)\eta_{5t}\}. \tag{20}$$

As for index-linked bonds (section 3.2):

$$b_{C1}(0) = b_{C2}(0) = b_{C2}(1) = 0.$$

Suppose that the inflation risk premium

$$\phi_t = \mu_{Ct}(0) - \mu_{It}(0)$$

is constant, so that:

$$\phi_t = \phi \text{ for all } t.$$

Hence, from (10) and (19):

$$\mu_{\gamma t} = Y_{C,t-1}(1) - Y_{I,t-1}(1) - \phi. \tag{21}$$

3.5. Equities

Let P_{Et} denote the price of equities at time t, including reinvested dividends. Then the return on equities during year t is:

$$\delta_{Et} = \frac{\ln P_{Et}}{\ln P_{E,t-1}} = \mu_{Et} + b_{E1}\eta_{6t}; \tag{22}$$

where:

μ_{Et} is the expected return;

$$\eta_{6t} = \sum_{i=1}^{6} a_{i6}\,\varepsilon_{it}; \text{ and} \tag{23}$$

$$\sum_{i=1}^{6} a_{i6} = \sqrt{6}. \tag{24}$$

3.6. Notional risky assets

If there are 6 risky assets in a market and an investor maintains constant exposure w_i (at market prices) to asset i during a year then, if all income is reinvested when paid, the total return is:

$$\sum_{i=1}^{6} w_i\,\delta_i\,;$$

where δ_i is the average return on asset i during that year.

Consider a set of 6 notional risky assets, whose return during year t is:

$$\delta_{it} = c + d\varepsilon_{it} \text{ for } i = 1, \ldots, 6;$$

where ε_{it} is as defined in section 3.2.

Now η_{jt} being a linear function of ε_{it}, and $\delta_{It}(s)$, $\delta_{Ct}(s)$ and δ_{Et} being linear functions of η_{jt}, it follows from the preceding paragraph that any portfolio of index-linked bonds, conventional bonds and equities may be constructed from a portfolio of notional risky assets, and vice versa, with constant exposure to the constituents of the respective portfolios during year t. Furthermore, the decomposition of any portfolio of actual assets into the corresponding portfolio of notional assets constitutes a no-arbitrage hedging strategy, since the returns on the corresponding portfolio will be identical. This means that, as between the asset categories modelled, the model developed in this paper is arbitrage-free.

It may be shown that, in mean-variance space, the equilibrium market portfolio will reduce to equal exposure to each of these assets. The return on the market portfolio is thus:

$$\delta_{Mt} = \frac{1}{6}\sum_{i=1}^{6}\delta_{it};$$

and hence:

$$c = \mu_{Mt};$$

i.e. the expected return on the market portfolio. It may be shown that, since $\operatorname{cov}\{\varepsilon_{it}, \varepsilon_{jt}\} = 0$ for $i \neq j$, the variance of the return on that portfolio is:

$$\sigma_{Mt}^2 = \frac{d^2}{6};$$

so that:

$$d = \sqrt{6}\,\sigma_{Mt}.$$

From the above it may be shown that:

$$\delta_{Mt} = \mu_{Mt} + \sigma_{Mt}\lambda_t;$$

where:

$$\lambda_t = \frac{1}{\sqrt{6}}\sum_{i=1}^{6}\varepsilon_{it}.$$

From (5), (13), (16) and (23):

$$\eta_{jt} = \sum_{i=1}^{6} a_{ij}\varepsilon_{it}; \tag{25}$$

where, from (6), (14), (17) and (24) the common scaling factor is:

$$\sum_{i=1}^{6} a_{ij} = \sqrt{6}.$$

From the above, and from (2) and (3), it may be shown that:

$$\sigma_{\eta_j Mt} = \mathrm{cov}_{t-1}\{\eta_{jt}, \delta_{Mt}\} = \sigma_{Mt}. \tag{26}$$

As explained in section 3.2, the simplicity of this equation is attributable to the choice of scaling factor.

3.7. Development of the equilibrium model

In order for an asset $X \in \{(I; t, s), (C; t, s), (E; t)\}$ to satisfy the CAPM during year t, and therefore to ensure that the model is an equilibrium model, we require that:

$$\mu_X = \delta_{It}(0) + k_t \sigma_{XM}; \tag{27}$$

where:

$$k_t = \frac{\mu_{Mt} - \delta_{It}(0)}{\sigma_{Mt}^2}. \tag{28}$$

This is referred to by some authors as the 'market price of risk', though Elton & Gruber (*op. cit.*: 302-3) criticise that usage, preferring:

$$k_t = \frac{\mu_{Mt} - \delta_{It}(0)}{\sigma_{Mt}}.$$

For a given model of the return on the market portfolio in year t, (28) may be used to determine k_t. For each asset category, given the covariance of its return with that of the market, (27) may then be used to determine its expected return.

In particular, for each index-linked bond:

$$\mu_{It}(s) = \delta_{It}(0) + k_t \sigma_{IMt}(s); \tag{29}$$

where, from (9) and (26):

$$\sigma_{IMt}(s) = \mathrm{cov}_{t-1}[\delta_{It}(s), \delta_{Mt}] = -\sigma_{Mt} f_{it}(s)\{b_{I1}(s) + b_{I2}(s)\}. \tag{30}$$

Making $f_{It}(s)$ the subject of equation (8), we have, for $s < \tau$:

$$f_{It}(s) = Y_{I,t-1}(s+1) - \mu_{It}(s). \tag{31}$$

From the above it may be shown that, for $s < \tau$:

$$f_{It}(s) = \frac{Y_{I,t-1}(s+1) - \delta_{It}(0)}{1 - k_t \sigma_{Mt}\{b_{I1}(s) + b_{I2}(s)\}}. \tag{32}$$

In order to obtain the full yield curve, equation (32) will need to be evaluated for all values of s. A problem arises in the determination of $f_{It}(s)$ for the last point of the yield curve ($s = \tau$), where $b_{I1}(\tau + 1)$ and $b_{I2}(\tau + 1)$ are not defined. An assumption is required about the behaviour of the yield curve beyond τ. For the sake of simplicity it is assumed that, at any time t, the one-year forward rate for maturity at time $t + \tau$ is equal to the equivalent forward rate for maturity at time $t + \tau - 1$; i.e.:

$$\frac{P_{I,t-1}(\tau)}{P_{I,t-1}(\tau+1)} = \frac{P_{I,t-1}(\tau-1)}{P_{I,t-1}(\tau)};$$

whence:

$$Y_{I,t-1}(\tau+1) = 2Y_{I,t-1}(\tau) - Y_{I,t-1}(\tau-1). \tag{33}$$

From (7) and (33), since $b_{I1}(\tau) = 0$ it may be shown that:

$$f_{It}(\tau) = \frac{2Y_{I,t-1}(\tau) - Y_{I,t-1}(\tau-1) - \delta_{It}(0)}{1 - k_t \sigma_{Mt}\, b_{I2}(\tau)}. \tag{34}$$

Similarly we require that, for each conventional bond:

$$\mu_{Ct}(s) = \delta_{It}(0) + k_t \sigma_{CMt}(s); \tag{35}$$

where, from (20) and (26):

$$\sigma_{CMt}(s) = \mathrm{cov}_{t-1}\{\delta_{Ct}(s), \delta_{Mt}\} = -\sigma_{Mt}\left[b_\gamma + f_{Ct}(s)\{b_{C1}(s) + b_{C2}(s)\}\right]. \tag{36}$$

Making $f_{Ct}(s)$ the subject of equation (19), we have, for $s < \tau$:

$$f_{Ct}(s) = Y_{C,t-1}(s+1) - \mu_{\gamma t} - \mu_{Ct}(s).$$

From the above it may be shown that, for $s < \tau$:

$$f_{Ct}(s) = \frac{Y_{C,t-1}(s+1) - \mu_{\gamma t} - \delta_{It}(0) + k_t \sigma_{Mt}\, b_\gamma}{1 - k_t \sigma_{Mt}\{b_{C1}(s) + b_{C2}(s)\}}. \tag{37}$$

As for index-linked bonds:

$$f_{Ct}(\tau) = \frac{2Y_{C,t-1}(\tau) - Y_{C,t-1}(\tau-1) - \mu_{\gamma t} - \delta_{It}(0) + k_t \sigma_{Mt} b_\gamma}{1 - k_t \sigma_{Mt} b_{C2}(\tau)}. \tag{38}$$

For inflation, from (12) and (26):

$$\sigma_{\gamma Mt} = \mathrm{cov}_{t-1}\{\gamma_t, \delta_{Mt}\} = b_\gamma \sigma_{Mt}. \tag{39}$$

Finally, for equities:

$$\mu_{Et} = \delta_{It}(0) + k_t \sigma_{EMt}; \tag{40}$$

where, from (22) and (26):

$$\sigma_{EMt} = \mathrm{cov}_{t-1}\{\delta_{Et}, \delta_{Mt}\} = b_{E1}\sigma_{Mt}. \tag{41}$$

3.8. Summary of the equilibrium model

The model may be implemented as follows.

STEP 1:

The parameters required are as follows:

- for $s = 1, \ldots, \tau$:
  - $Y_{I0}(s)$ and $Y_{C0}(s)$; and
  - $b_{Ij}(s)$ and $b_{Cj}(s)$ for $j = 1, 2$; and
- b_γ;
- b_{E1};
- ϕ; and
- a_{ij} for $i, j = 1, \ldots, 6$.

STEP 2:

For $t = 1$ we then determine the variables μ_{Mt} and σ_{Mt}, using the market model. Also:

$$\delta_{It}(0) = Y_{I,t-1}(1) \text{ (equation (10))}.$$

STEP 3:

Using Monte Carlo methods we then simulate pseudorandom standard normal variables:

$$\varepsilon_{it} \text{ for } i = 1, \ldots, 6.$$

STEP 4:

From the above values we calculate:

$$k_t = \frac{\mu_{Mt} - \delta_{It}(0)}{\sigma_{Mt}^2} \text{ (equation (28));}$$

for $j = 1, \ldots, 6$:

$$\eta_{jt} = \sum_{i=1}^{6} a_{ij}\varepsilon_{it} \text{ (equation (25));}$$

$$\mu_{\gamma t} = Y_{C,t-1}(1) - Y_{I,t-1}(1) - \phi \text{ (equation (21));}$$

$$\gamma_t = \mu_{\gamma t} + b_\gamma \eta_{3t} \text{ (equation (12));}$$

for $s = 1, \ldots, \tau - 1$:

$$f_{It}(s) = \frac{Y_{I,t-1}(s+1) - \delta_{I;t}(0)}{1 - k_t \sigma_{Mt}\{b_{I1}(s) + b_{I2}(s)\}} \text{ (equation (32));}$$

$$f_{Ct}(s) = \frac{Y_{C,t-1}(s+1) - \mu_{\gamma t} - \delta_{It}(0) + k_t \sigma_{Mt} b_\gamma}{1 - k_t \sigma_{Mt}\{b_{C1}(s) + b_{C2}(s)\}} \text{ (equation (37));}$$

$$f_{It}(\tau) = \frac{2Y_{I,t-1}(\tau) - Y_{I,t-1}(\tau-1) - \delta_{It}(0)}{1 - k_t \sigma_{Mt} b_{I2}(\tau)} \text{ (equation (34));}$$

$$f_{Ct}(\tau) = \frac{2Y_{C,t-1}(\tau) - Y_{C,t-1}(\tau-1) - \mu_{\gamma t} - \delta_{It}(0) + k_t \sigma_{Mt} b_\gamma}{1 - k_t \sigma_{Mt} b_{C2}(\tau)} \text{ (equation (38));}$$

for $s = 1, \ldots, \tau$:

$$\sigma_{IMt}(s) = -\sigma_{Mt} f_{it}(s)\{b_{I1}(s) + b_{I2}(s)\} \text{ (equation (30));}$$

$$\sigma_{CMt}(s) = -\sigma_{Mt}\left[b_\gamma + f_{Ct}(s)\{b_{C1}(s) + b_{C2}(s)\}\right] \text{ (equation (36));}$$

$$\mu_{It}(s) = \delta_{It}(0) + k_t \sigma_{IMt}(s) \text{ (equation (29));}$$

$$\mu_{Ct}(s) = \delta_{It}(0) + k_t \sigma_{CMt}(s) \text{ (equation (35));}$$

$$\delta_{It}(s) = \mu_{It}(s) - f_{It}(s)\{b_{I1}(s)\eta_{1t} + b_{I2}(s)\eta_{2t}\} \text{ (equation (9));}$$

$$\delta_{Ct}(s) = \mu_{Ct}(s) - b_\gamma \eta_{3t} - f_{Ct}(s)\{b_{C1}(s)\eta_{4t} + b_{C2}(s)\eta_{5t}\} \text{ (equation (20));}$$

$$\sigma_{EMt} = b_{E1}\sigma_{Mt} \text{ (equation (41));}$$

$$\mu_{Et} = \delta_{It}(0) + k_t \sigma_{EMt} \text{ (equation (40));}$$

$$\delta_{Et} = \mu_{Et} + b_{E1}\eta_{6t} \text{ (equation (22)).}$$

STEP 5:

For $t < T$, we calculate:

for $s = 1, \ldots, \tau$:

$$Y_{It}(s) = Y_{I,t-1}(s+1) - \delta_{It}(s) \text{ (from equation (7)); and}$$

$$Y_{Ct}(s) = Y_{C,t-1}(s+1) - \delta_{Ct}(s) - \gamma_t \text{ (from equation (18)).}$$

Steps 2 to 5 are repeated for $t = 2, \ldots, T$.

4. MARKET MODELS

In the equilibrium model, no consideration is given to the processes governing the variables μ_{Mt} and σ_{Mt}.

Depending on the local market, these variables can be treated as constants, or they can be modelled using a model such as a regime-switching or ARCH model.

It is not possible to assume that μ_{Mt} is constant, otherwise whenever $\delta_{It}(0) > \mu_M$, we have $k_t < 0$, which means that the market price of risk is negative. In order to address this problem, the expected return on the market portfolio is expressed as a function of the risk-free return as follows:

$$\begin{aligned} \mu_{Mt} &= g\delta_{It}(0) + h \ \text{ for } \ \delta_{It}(0) > 0 \\ &= \delta_{It}(0) \text{ otherwise.} \end{aligned} \tag{42}$$

For $\delta_{It}(0) > 0$, equation (42) is justified on the grounds that the risk premium

$$\pi_t = \frac{\mu_{Mt} - \delta_{It}(0)}{\sigma_{Mt}}.$$

is positive, though it may vary according to the level of $\delta_{It}(0)$. In general, since the sensitivity of the volatility of the return on the market to the risk-free return may be expected to be positive, it may be expected that $g > 1$. For $\delta_{It}(0) \leq 0$, this does not apply; under such circumstances it is effectively being assumed that the risk premium is zero. While this is an arbitrary assumption, it is unlikely to apply frequently.

The exploration of alternative market models is left for further research.

5. DESCRIPTIVE ESTIMATION OF THE MODEL

The method of determination of the model parameters is explained in the Appendix for the purpose of the estimation of a descriptive version of the model. In this section the results of the descriptive estimation of the parameters are presented.

The historical data required include, for each year:

– the zero-coupon yield (conventional and index-linked) for each maturity modelled (in this case from 1 to 30 years at yearly intervals);
– the total return on equities;
– the inflation rate; and
– the composition of the market portfolio.

The composition of the market portfolio is represented by the split of the total investment market capitalisation into equity and conventional and index-linked bonds. For this purpose, the market capitalisation of the bonds was split by term to maturity. Since the bonds being modelled are zero-coupon bonds, each traded bond was decomposed into a series of zero-coupon bonds, which were aggregated by maturity date into annual buckets.

For the purposes of the illustrative estimation of the descriptive model, the total market capitalisation of the FTSE All-Share Index was taken as a proxy for the market capitalisation of equities.

The yields on conventional bonds were obtained from the zero-coupon yield curves published by the UK Debt Management Office (DMO). These are denoted as $CONV01, \ldots, CONV30$. The history of these yields was obtained from 31 December 1979 to 31 December 2006 at yearly intervals.

The yields on index-linked bonds were likewise obtained from zero-coupon yield curves published by DMO. The index-linked zero-coupon bond yields for maturities $1, \ldots, 30$ years are denoted as $ILB01, \ldots, ILB30$. These were obtained at yearly intervals from 31 December 1985 to 31 December 2006.

Figure 1 shows the yield curves of conventional and index-linked bonds as at 31 December 2006.

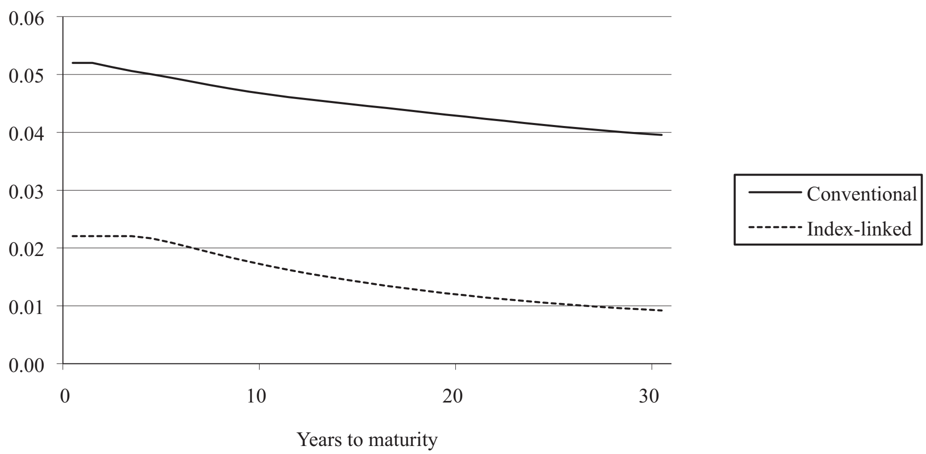

FIGURE 1. Yield curves of conventional and index-linked bonds as at 31 December 2006.

Historical inflation figures were derived as:

$$\mathring{\gamma}_t = \ln \frac{R_t}{R_{t-1}}$$

where R_t is the value of the UK retail prices index at the end of year t[1].

Historical equity returns were derived from the FTSE All-Share total-return index as follows:

$$\mathring{\delta}_{Et} = \ln \frac{T_t}{T_{t-1}};$$

where T_t is the value of the relevant equity index at the end of year t[2].

Market capitalisations for bond markets were available only from 31 December 1998 onwards[3]. It was assumed that the split of total market capitalisation between equities and bonds prior to 1998 was the same as at 31 December 1998.

Since the yield curves for index-linked bonds were available only since 31 December 1985, it was assumed that the market capitalisation of those bonds was zero before that date. Since the market capitalisation of index-linked bonds is small compared with that of conventional bonds and equities, this is not expected to skew results significantly.

For the purposes of estimating μ_{Mt} in terms of equation (42), a linear regression was carried out; for this purpose, the risk-free return for the years prior to 1986 was calculated using the simplifying assumption that:

$$\delta_{It}(0) = \delta_{Ct}(0) - \phi.$$

It was found that the intercept constant h was not significant at the 95% level. Fixing the intercept at zero, we obtain an estimate of $g = 1.833$ (p-value = 0.008).

The estimated parameters of the yields on zero-coupon conventional and index-linked bonds are shown in Table 1. The other model parameters were estimated as follows:

$$\begin{aligned} b_\gamma &= -0.0083; \\ b_{E1} &= 0.0685; \text{ and} \\ \sigma_M &= 0.12026. \end{aligned}$$

The inflation risk premium (ϕ) was fixed at the arbitrary value of 0.3% per annum. Further research is required on the reliable estimation of the inflation risk premium.

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1 Data supplied by Professor A.D. Wilkie, InQA Limited.

2 Source: Communication from info@ftse.com

3 Source: www.dmo.gov.uk

TABLE 1

ESTIMATED PARAMETERS OF THE MODEL FOR CONVENTIONAL AND INDEX-LINKED BONDS

s	$Y_{I,0}(s)$	$b_{I,1}(s)$	$b_{I,2}(s)$	$Y_{C,0}(s)$	$b_{C,1}(s)$	$b_{C,2}(s)$
1	0.0220	0.1302	0.0000	0.0520	–0.1208	0.0000
2	0.0441	0.1300	–0.0001	0.1026	–0.1140	–0.0236
3	0.0661	0.1226	0.0192	0.1518	–0.1072	–0.0357
4	0.0868	0.0834	0.0253	0.2001	–0.0861	–0.0422
5	0.1047	0.0862	0.0502	0.2472	–0.0890	–0.0484
6	0.1205	0.0733	0.0571	0.2929	–0.0799	–0.0525
7	0.1346	0.0633	0.0609	0.3373	–0.0712	–0.0560
8	0.1472	0.0554	0.0628	0.3807	–0.0631	–0.0593
9	0.1587	0.0491	0.0636	0.4234	–0.0555	–0.0624
10	0.1691	0.0439	0.0637	0.4655	–0.0486	–0.0654
11	0.1785	0.0395	0.0635	0.5071	–0.0421	–0.0683
12	0.1872	0.0357	0.0632	0.5483	–0.0361	–0.0711
13	0.1951	0.0324	0.0629	0.5889	–0.0307	–0.0738
14	0.2024	0.0294	0.0627	0.6290	–0.0257	–0.0764
15	0.2092	0.0266	0.0627	0.6683	–0.0213	–0.0789
16	0.2154	0.0241	0.0627	0.7070	–0.0174	–0.0813
17	0.2211	0.0218	0.0630	0.7449	–0.0140	–0.0835
18	0.2265	0.0196	0.0634	0.7821	–0.0112	–0.0856
19	0.2315	0.0175	0.0639	0.8185	–0.0089	–0.0875
20	0.2362	0.0155	0.0646	0.8542	–0.0070	–0.0892
21	0.2407	0.0136	0.0655	0.8892	–0.0054	–0.0908
22	0.2450	0.0118	0.0665	0.9236	–0.0041	–0.0923
23	0.2491	0.0101	0.0677	0.9574	–0.0031	–0.0936
24	0.2530	0.0084	0.0690	0.9908	–0.0024	–0.0949
25	0.2569	0.0068	0.0702	1.0236	–0.0018	–0.0960
26	0.2607	0.0053	0.0714	1.0562	–0.0014	–0.0970
27	0.2645	0.0038	0.0725	1.0888	–0.0010	–0.0980
28	0.2684	0.0025	0.0736	1.1214	–0.0007	–0.0988
29	0.2722	0.0012	0.0746	1.1540	–0.0003	–0.0997
30	0.2760	0.0000	0.0756	1.1866	0.0000	–0.1005

TABLE 2

COVARIANCES OF η_{jt}

	η_{1t}	η_{2t}	η_{3t}	η_{4t}	η_{5t}	η_{6t}
η_{1t}	6.00	0.11	2.04	3.49	0.84	0.02
η_{2t}	0.11	5.78	–0.36	0.28	2.62	3.13
η_{3t}	2.04	–0.36	3.90	1.03	0.02	1.00
η_{4t}	3.49	0.28	1.03	3.09	1.03	–0.34
η_{5t}	0.84	2.62	0.02	1.03	1.89	1.02
η_{6t}	0.02	3.13	1.00	–0.34	1.02	3.75

TABLE 3

COEFFICIENTS a_{ij}

$j \setminus i$	1	2	3	4	5	6
1	2.449	0.000	0.000	0.000	0.000	0.000
2	0.046	2.403	0.000	0.000	0.000	0.000
3	0.833	–0.166	1.782	0.000	0.000	0.000
4	1.427	0.087	–0.083	1.018	0.000	0.000
5	0.344	1.084	–0.048	0.435	0.634	0.000
6	0.010	1.300	0.678	–0.403	–0.297	1.161

The covariance matrix of η_{jt} and the coefficients a_{ij} (see Tables 2 and 3 respectively) were determined as described in the Appendix.

6. Illustrative Results of the Model

In this section, illustrative results of the predictive model are presented.

Figures 2 to 7 show the results of the projections based on the parameters estimated from historical data and shown in the previous section. The simulated variables include short-term interest rates (one-year zero-coupon yields, both conventional and index-linked), long-term interest rates (20-year zero-coupon yields, both conventional and index-linked), inflation rates and equity returns. For each variable the historical values are shown and the mean and 95% confidence intervals are shown for each of the next 20 years based on 10,000 simulations of the model.

Equity returns show fairly stable means and confidence limits, though it should be noted that the scale of Figure 2 is greater than those of the other figures. In fact, while the drift is gradual, mean returns drift downwards to very low levels, even though they remain greater than risk-free returns. There were two apparent historical breaches of the confidence limits, both on the downside. This is not unreasonable over a 27-year period.

In general, the other variables exhibited an initial widening of the funnel of doubt until the lower confidence limit reached low levels, after which the lower confidence interval contracted. The exception was the upper confidence limit of short-term index-linked bonds (Figure 7), which also contracted. This contraction may be largely explained by the theoretical lower bound of zero, coupled with the decreasing mean: although the distribution is skew, it is not sufficiently skew to produce indefinite expansion of the upper tail.

As shown in Figure 3, inflation rates breached the asymptotes of the confidence limits two or three times – again not particularly remarkable, especially since the asymptotes are driven by current yield curves, not by those in 1979.

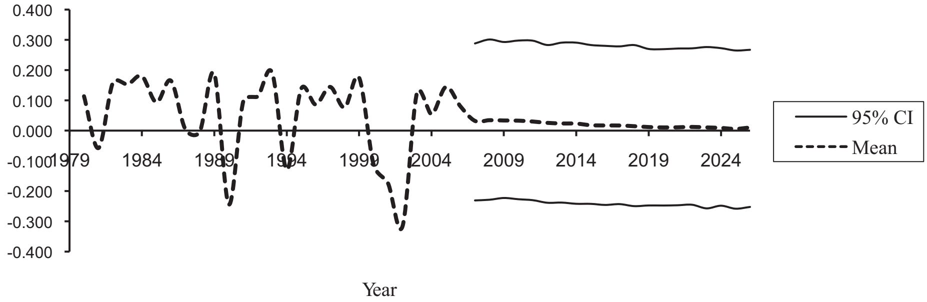

FIGURE 2. Mean and 95% confidence interval for equity returns.

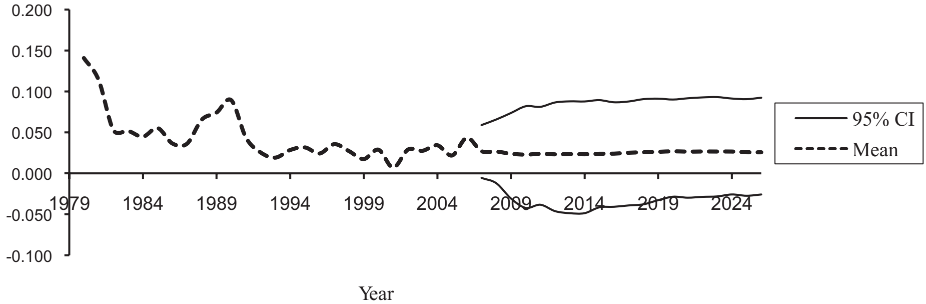

FIGURE 3. Mean and 95% confidence interval for inflation.

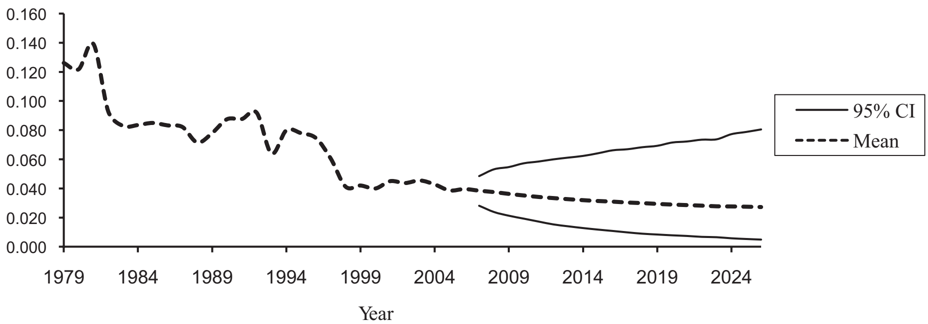

FIGURE 4. Mean and 95% confidence interval for yields on long-term conventional bonds.

As shown in Figures 4 to 7, yields on bonds – both conventional and index-linked – have reduced over the historical period. While the means and lower confidence limits continue this trend, the upper confidence limit allows for a possible reversal.

As shown in Figures 5 and 7, there remains a small probability of negative yields (both nominal and real), which, in principle, allows a possibility of arbitrage. This is discussed in section 7.

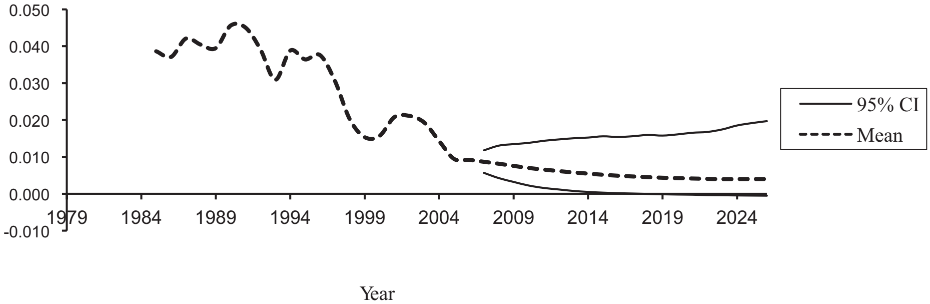

FIGURE 5. Mean and 95% confidence interval for yields on long-term index-linked bonds.

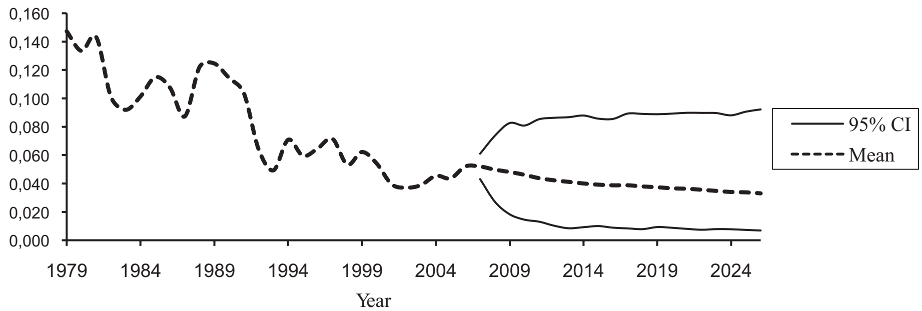

FIGURE 6. Mean and 95% confidence interval for yields on short-term conventional bonds.

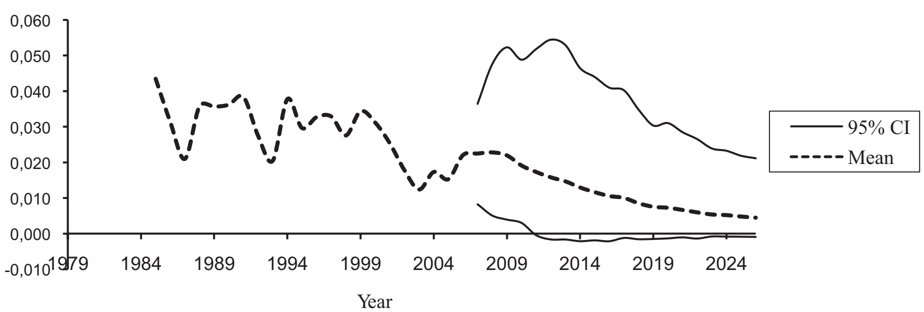

FIGURE 7. Mean and 95% confidence interval for yields on short-term index-linked bonds.

7. Conclusions

The model presented in this paper is a long-term model of a local market. It comprises a model of the market portfolio, which, subject to certain constraints, may be specified by the user, as well as an equilibrium model of equity, bonds and inflation. While no arbitrage is present as between the asset classes modelled, there remains a small probability of negative yields (both nominal and real), which, in principle, allows a possibility of arbitrage. That may be avoided by eliminating projections that produce negative yields, or by applying a lower limit of zero, but this will have the effect of distorting the distribution of the returns on the asset classes so that arbitrage may be possible between asset classes, or so that the equilibrium equations do not apply, or so that the fidelity of the predictive model to the descriptive model is compromised.

In practice, negative yields should be monitored. If the effects are negligible in relation to the purpose to which the model is being applied, they may be ignored or avoided at the discretion of the user. Otherwise the model should not be used. It should be recognised, however, that, while negative yields allow arbitrage in principle, it may in practice be difficult to achieve. Particularly in the case of real yields, it is impossible to hold the basket of goods and services comprising a retail prices index without considerable cross-hedge risk or holding costs. Even in the case of nominal yields, there are costs in holding cash.

As mentioned in section 3.1, a two-factor model of the term structure of interest rates overestimates the correlation between forward rates for neighbouring maturities and underestimates the correlation between forward rates with maturity dates far apart. Depending on the application for which the model is required, it may be necessary to consider a third factor for either or both of the models for conventional and index-linked bonds.

The following further research is required:

– the compilation of more historical data;
– the inclusion of corporate loan stock and fixed property;
– the inclusion of a third factor for either or both of the models for conventional and index-linked bonds;
– the comparison of results for various markets;
– the estimation of the inflation risk premium and the modelling of inflation;
– the development of, and comparisons between, alternative models of the return on the market portfolio;
– the advantages and disadvantages of including a third factor in the bond pricing models;
– an investigation of the problem of negative yields; and
– the use of the model for research on the pricing of liabilities.

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APPENDIX

A. Descriptive Estimation of the Model

A.1. Introduction

In this appendix, the method of determination of the model parameters is developed for the purpose of the estimation of a descriptive version of the model. The purpose of this process is to estimate values of the parameters required both for the equilibrium model and for a market model. While the mathematical specification of the equilibrium model does not require specification of the market model, the estimation of the former requires estimation of the latter. The latter is therefore dealt with first.

In determining the required parameters, we deliberately invoke the requirements of equilibrium, particularly through the use of the relationships between expected returns on the respective asset categories and the expected return on the market portfolio, as discussed in section 3.7. This means that the estimates of these expected returns are not necessarily unbiased estimates ex post. Equilibrium is essentially established ex ante, and it is therefore important that, so far as it is possible, ex-ante expected values be estimated. Under the rational expectations hypothesis, which is normally invoked in the estimation of stochastic investment models, it is assumed that ex-post estimates are unbiased estimates of ex-ante expectations. Where that hypothesis conflicts with the requirements of equilibrium modelling, it is not invoked in this paper.

A.2. Estimation of the Market Model

Consider a sample historical period $t = t_0 + 1, t_0 + 2, \ldots, t_0 + T$. Let $\mathring{\gamma}_t$ denote the continuously compound rate of inflation during year t. Let $\mathring{y}_{It}(s)$ and $\mathring{y}_{Ct}(s)$ denote the effective continuously compound spot yields at time t on zero-coupon bonds – index-linked and conventional respectively – maturing at time $t + s$. We assume that, for each t, these have been graduated (either parametrically or non-parametrically) using standard techniques for the fitting of yield curves (see for example Van Deventer *et. al.* (2004)). The corresponding one-year returns are:

$$\mathring{\delta}_{It}(s) = \mathring{Y}_{I,t-1}(s+1) - \mathring{Y}_{It}(s); \text{ and}$$
$$\mathring{\delta}_{Ct}(s) = \mathring{Y}_{C,t-1}(s+1) - \mathring{Y}_{Ct}(s) - \mathring{\gamma}_t;$$

where:

$$\mathring{Y}_{It}(s) = s\,\mathring{y}_{It}(s); \text{ and}$$
$$\mathring{Y}_{Ct}(s) = s\,\mathring{y}_{Ct}(s).$$

Let $\mathring{\delta}_{Et}$ be the continuously compound (real) return on equities during year t.

Let:

- $\mathring{w}_{Et}$ be the observed proportion of the market portfolio in equities, by market capitalisation, in year t; and
- $\mathring{w}_{It}(s)$ and $\mathring{w}_{Ct}(s)$ be the corresponding proportions in index-linked and conventional bonds respectively, with payment dates at time $t + s$.

For the purposes of calculation of the above proportions, coupon-paying bonds need to be notionally stripped into zero-coupon bonds. Bond payments need to be notionally apportioned between integral payment dates. Allowance needs to be made for lags in index-linking. Approximations may need to be made in order to avoid excessive data collection or to accommodate missing data, especially where the inclusion of such data would merely produce spurious accuracy.

From the above values, the return on the market portfolio during year t may be calculated as:

$$\mathring{\delta}_{Mt} = \sum_{s=1}^{\infty} \left\{ \mathring{w}_{It}(s)\,\mathring{\delta}_{It}(s) + \mathring{w}_{Ct}(s)\,\mathring{\delta}_{Ct}(s) \right\} + \mathring{w}_{Et}\,\mathring{\delta}_{Et}.$$

From these values we may estimate σ_M as follows:

$$\hat{\sigma}_M = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(\mathring{\delta}_{Mt} - \hat{\mu}_M\right)^2}.$$

As explained in section 4, we may determine the ex-ante estimate of μ_{Mt} as:

$$\begin{aligned} \hat{\mu}_{Mt} &= g\delta_{It}(0) + h, \text{ for } \delta_{I,t}(0) > 0 \\ &= \delta_{It}(0) \text{ otherwise.} \end{aligned}$$

A.3. Estimation of the Equilibrium Model

A.3.1. Parameters required

As stated in section 3.8, the parameters required are as follows:

- for all required values of s:

  $Y_{I0}(s)$ and $Y_{C0}(s)$; and

  $b_{Ij}(s)$ and $b_{Cj}(s)$ for $j = 1, 2$; and

– b_γ;
– b_{E1}; and
– a_{ij} for $i, j = 1, \ldots, 6$.

A.3.1. Estimation of $Y_{I;0}(s)$ and $Y_{C;0}(s)$

From equation (4), $Y_{I0}(s)$ may be estimated as:

$$\mathring{Y}_{I0}(s) = -\ln\left(\mathring{P}_{I0}(s)\right);$$

where $\mathring{P}_{I0}(s)$ is the observed value of $P_{I0}(s)$. Similarly:

$$\mathring{Y}_{C0}(s) = -\ln\left(\mathring{P}_{C0}(s)\right).$$

A.3.2. Estimation of $b_{Ij}(s)$

The values of $b_{Ij}(s)$ may be estimated as follows. From (30) it is clear that, since σ_{Mt} is constant (say σ_M) for all t, the value of:

$$\chi_I(s) = \frac{\sigma_{IMt}(s)}{f_{It}(s)} \tag{43}$$

will also be constant. Since $f_{It}(s)$ is known ex ante at time $t-1$ (though it is unobservable ex post), and since $\chi_I(s)$ is defined ex ante, and since it is constant, we may write:

$$\chi_I(s) = \mathrm{cov}\left\{\frac{\delta_{It}(s)}{f_{It}(s)}, \delta_{Mt}\right\}.$$

After substitution of equation (31), the ex-post estimate of the ex-ante value of $\chi_I(s)$ is given by:

$$\hat{\chi}_I(s) = \frac{1}{T-1}\sum_{t=1}^{T}\frac{\mathring{\delta}_{It}(s) - \hat{\mu}_{It}(s)}{\mathring{Y}_{I,t-1}(s+1) - \hat{\mu}_{It}(s)}\left(\mathring{\delta}_{Mt} - \hat{\mu}_{Mt}\right). \tag{44}$$

From (28) we have:

$$\hat{k}_t = \frac{\hat{\mu}_{Mt} - \mathring{\delta}_{It}(0)}{\hat{\sigma}_M^2};$$

and from (29):

$$\hat{\mu}_{It}(s) = \mathring{\delta}_{It}(0) + \hat{k}_t\,\hat{\chi}_I(s)\hat{f}_{It}(s).$$

This linear constraint explains the division by $T-1$ in equation (44).

From (31) it may be shown that:

$$\hat{\mu}_{It}(s) = \frac{\mathring{\delta}_{It}(0) + \hat{k}_t \hat{\chi}_I(s) \mathring{Y}_{I,t-1}(s+1)}{1 + \hat{k}_t \hat{\chi}_I(s)}. \tag{45}$$

Substituting this into (44) we obtain, after some algebra:

$$\hat{\chi}_I(s) = \frac{\frac{1}{T-1} \sum_{t=1}^{T} (1 - \kappa_{It}(s)) \left(\mathring{\delta}_{Mt} - \hat{\mu}_{Mt} \right)}{1 + \frac{1}{T-1} \sum_{t=1}^{T} \hat{k}_t \kappa_{It}(s) \left(\mathring{\delta}_{Mt} - \hat{\mu}_{Mt} \right)}; \tag{46}$$

where

$$\kappa_{It}(s) = \frac{\mathring{Y}_{I,t-1}(s+1) - \mathring{\delta}_{It}(s)}{\mathring{Y}_{I,t-1}(s+1) - \mathring{\delta}_{It}(0)}.$$

On substituting the value of (46) into (45) we obtain ex-post estimates $\hat{\mu}_{It}(s)$ of the ex-ante expected returns. Ex post, these are clearly biased estimates of $\mu_{It}(s)$; that is a consequence of the need to estimate ex-ante expected values, which may be ex-post biased. As mentioned in section B.1, we are deliberately avoiding the rational-expectations hypothesis to the extent that it conflicts with the requirements of equilibrium modelling.

From (9) we may write:

$$\boldsymbol{x}_t = -(\hat{\boldsymbol{b}}_{I1} \eta_{1,t} + \hat{\boldsymbol{b}}_{I2} \eta_{2t}); \tag{47}$$

where:

$$\boldsymbol{x}_t = \begin{pmatrix} x_{t1} \\ \vdots \\ x_{t\tau^*} \end{pmatrix};$$

$$x_{ts} = \frac{\mathring{\delta}_{It}(s) - \mathring{\mu}_{It}(s)}{\mathring{f}_{It}(s)}$$

$$\hat{\boldsymbol{b}}_{Ij} = \begin{pmatrix} \hat{b}_{Ij}(1) \\ \vdots \\ \hat{b}_{Ij}(\tau) \end{pmatrix};$$

and $\hat{b}_{Ij}(s)$ is the estimate of $b_{Ij}(s)$ to be determined. This may be done by finding the first two principal components (e.g. Jackson, 2003), as follows.

First we estimate the covariance matrix of $\boldsymbol{x}_t$ as:

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{11} & \cdots & \hat{\sigma}_{1t^*} \\ \vdots & & \vdots \\ \hat{\sigma}_{\tau^*1} & \cdots & \hat{\sigma}_{\tau^*\tau^*} \end{pmatrix};$$

where:

$$\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T} x_{ti}x_{tj}.$$

Once again, we are working with an (ex-post biased) estimate of ex-ante expectations.

Next we determine the first two eigenvalues l_1 and l_2 and the eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ of $\hat{\Sigma}$, so that:

$$\boldsymbol{U}\hat{\boldsymbol{\Sigma}}\boldsymbol{U} = \boldsymbol{L};$$

where:

$$\boldsymbol{U} = (\boldsymbol{u}_1 | \boldsymbol{u}_2)$$

$$\boldsymbol{u}_1 = \begin{pmatrix} u_{1,1} \\ \vdots \\ u_{1,\tau^*} \end{pmatrix};$$

$$\boldsymbol{u}_2 = \begin{pmatrix} u_{2,1} \\ \vdots \\ u_{2,\tau^*} \end{pmatrix}; \text{ and}$$

$$\boldsymbol{L} = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}.$$

The eigenvalues and eigenvectors may be determined either by means of the power method (Jackson, *op. cit.*) or by means of more efficient techniques available in numerous computer packages (*ibid.*). The matrix $\boldsymbol{L}$ is the variance matrix of the principal components (their covariances being zero as they are uncorrelated). The principal-component scores corresponding to the observed values $\boldsymbol{x}_t$ are:

$$z_{1,t} = \boldsymbol{u}_1'\boldsymbol{x}_t \text{ and } z_{2,t} = \boldsymbol{u}_2'\boldsymbol{x}_t.$$

Now we need to determine $\hat{\boldsymbol{b}}_{I,1}$ and $\hat{\boldsymbol{b}}_{I,2}$. Assuming that the third and higher-order principal components may be ignored, we have (*ibid.*):

$$x_{ts} = u_{1s} z_{1t} + u_{2s} z_{2t}. \tag{48}$$

Let:

$$\eta_{1t} = c_{11} z_{1t} + c_{12} z_{2t}; \text{ and}$$
$$\eta_{2t} = c_{21} z_{1t} + c_{22} z_{2t}.$$

Then, from (47) and (48), for all values of t:

$$-\left\{\hat{b}_{I1}(s)\,(c_{11} z_{1t} + c_{12} z_{2t}) + \hat{b}_{I2}(s)\,(c_{21} z_{1t} + c_{22} z_{2t})\right\} = u_{1s} z_{1t} + u_{2s} z_{2t}.$$

Equating the coefficients of z_{1t} and those of z_{2t}, we have, respectively:

$$-\left\{\hat{b}_{I1}(s)\,c_{11} + \hat{b}_{I2}(s)\,c_{21}\right\} = u_{1s}; \text{ and} \tag{49}$$

$$-\left\{\hat{b}_{I1}(s)\,c_{12} + \hat{b}_{I2}(s)\,c_{22}\right\} = u_{2s}. \tag{50}$$

In particular, for $s = 1$ and τ, we have, from (11):

$$\begin{aligned} c_{11}\hat{b}_{I1}(1) &= -u_{11}; \\ c_{21}\hat{b}_{I2}(\tau) &= -u_{1\tau}; \\ c_{12}\hat{b}_{I1}(1) &= -u_{21}; \text{ and} \\ c_{22}\hat{b}_{I2}(\tau) &= -u_{2\tau}; \end{aligned} \tag{51}$$

Now from (30) and (43):

$$\hat{b}_{I1}(1) = -\frac{\hat{\chi}_I(1)}{\hat{\sigma}_M}; \text{ and}$$

$$\hat{b}_{I2}(\tau) = -\frac{\hat{\chi}_I(\tau)}{\hat{\sigma}_M}.$$

Thus, from (51), we obtain:

$$c_{11} = -\frac{u_{11}}{\hat{b}_{I1}(1)};$$

$$c_{21} = -\frac{u_{1\tau}}{\hat{b}_{I2}(\tau)};$$

$$c_{12} = -\frac{u_{21}}{\hat{b}_{I1}(1)}; \text{ and}$$

$$c_{22} = -\frac{u_{2\tau}}{\hat{b}_{I2}(\tau)}.$$

Equations (49) and (50) may be represented as:

$$\hat{\boldsymbol{B}}_I \boldsymbol{C} = \boldsymbol{U};$$

where:

$$\hat{\boldsymbol{B}}_I = \left(\hat{\boldsymbol{b}}_{I1} \,\middle|\, \hat{\boldsymbol{b}}_{I2}\right); \text{ and}$$

$$\boldsymbol{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Thus:

$$\hat{\boldsymbol{B}}_I = \boldsymbol{U}\boldsymbol{C}^{-1}.$$

A.3.3. Estimation of b_γ

From equation (21) we may estimate $\mu_{\gamma t}$ as:

$$\hat{\mu}_{\gamma t} = \mathring{Y}_{C,t-1}(1) - \mathring{Y}_{I,t-1}(1) - \phi.$$

An ex-post estimate of the ex-ante value of $\sigma_{\gamma m}$ may be determined as:

$$\hat{\sigma}_{\gamma M} = \frac{1}{T-1}\sum_{t=1}^{T}\left(\mathring{\gamma}_t - \hat{\mu}_{\gamma t}\right)\left(\mathring{\delta}_{Mt} - \hat{\mu}_{Mt}\right).$$

From equation (39), b_γ may then be estimated as:

$$\hat{b}_\gamma = \frac{\hat{\sigma}_{\gamma M}}{\hat{\sigma}_M}.$$

From equation (12) we may also derive:

$$\hat{\eta}_{3t} = \frac{\mathring{\gamma}_t - \hat{\mu}_{\gamma t}}{\hat{b}_\gamma}.$$

A.3.4. Estimation of $b_{Cj}(s)$

For conventional bonds, as for index-linked bonds, we have:

$$\hat{\mu}_{Ct}(s) = \mathring{\delta}_{It}(0) + \hat{k}_t\hat{\chi}_C(s)\hat{f}_{Ct}(s);$$

where $\hat{\chi}_C(s)$ is the ex-post estimate of the ex-ante value of:

$$\hat{\chi}_C(s) = \frac{\sigma_{CMt}(s)}{f_{Ct}(s)}.$$

Also:

$$\hat{\mu}_{Ct}(s) = \frac{\mathring{\delta}_{It}(0) + \hat{k}_t\,\hat{\chi}_C(s)\left\{\mathring{Y}_{C,t-1}(s+1) - \hat{\mu}_{\gamma t}\right\}}{1 + \hat{k}_t\,\hat{\chi}_C(s)}.$$

As for index-linked bonds, we obtain:

$$\hat{\chi}_C(s) = \frac{\frac{1}{T-1}\sum_{t=1}^{T}(1-\kappa_{Ct}(s))\left(\mathring{\delta}_{Mt} - \hat{\mu}_{Mt}\right)}{1 + \frac{1}{T-1}\sum_{t=1}^{T}\hat{k}_t\,\kappa_{Ct}(s)\left(\mathring{\delta}_{Mt} - \hat{\mu}_{Mt}\right)};$$

where:

$$\kappa_{Ct}(s) = \frac{\mathring{Y}_{I,t-1}(s+1) - \hat{\mu}_{\gamma t} - \mathring{\delta}_{Ct}(s)}{\mathring{Y}_{I,t-1}(s+1) - \hat{\mu}_{\gamma t} - \mathring{\delta}_{It}(0)}.$$

As for index-linked bonds, $\hat{b}_{Cj}(s)$ may be determined by finding the first two principal components as in section B.3.2. In this case:

$$x_{ts} = \frac{\mathring{\delta}_{Ct}(s) - \mathring{\mu}_{Ct}(s) + \hat{b}_\gamma\,\hat{\eta}_{3t}}{\mathring{f}_{Ct}(s)}.$$

A.3.5. Estimation of b_{E1}

From (40):

$$\hat{\mu}_{Et} = \mathring{\delta}_{It}(0) + \hat{k}_t\,\hat{\sigma}_{EMt}.$$

From (41) it is clear that, since $\sigma_{Mt} = \sigma_M$ is constant, σ_{EMt} will also be constant (say σ_{EM}). Let:

$$\hat{\sigma}_{EM} = \frac{1}{T-1}\sum_{t=1}^{T}\left(\mathring{\delta}_{Et} - \hat{\mu}_{Et}\right)\left(\mathring{\delta}_{Mt} - \hat{\mu}_{Mt}\right). \tag{52}$$

From (40) we have:

$$\hat{\mu}_{Et} = \mathring{\delta}_{It}(0) + \hat{k}_t\,\hat{\sigma}_{EMt}. \tag{53}$$

Substituting (53) into (52), we have, after some rearrangement:

$$\hat{\sigma}_{EM} = \frac{\frac{1}{T-1}\sum_{t=1}^{T}\left(\mathring{\delta}_{Et} - \mathring{\delta}_{It}(0)\right)\left(\mathring{\delta}_{Mt} - \hat{\mu}_M\right)}{1 + \frac{1}{T-1}\sum_{t=1}^{T}\hat{k}_t\left(\mathring{\delta}_{Mt} - \hat{\mu}_M\right)}.$$

From (41):

$$b_{E1} = \frac{\hat{\sigma}_{EM}}{\hat{\sigma}_M}.$$

Also, from (22)

$$\hat{\eta}_{6t} = \frac{\delta_{Et} - \hat{\mu}_{Et}}{b_{E1}}.$$

A.3.6. Estimation of a_{ij}

The estimation of a_{ij} proceeds by Cholesky decomposition of the sample covariance matrix:

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{11} & \cdots & \hat{\sigma}_{16} \\ \vdots & & \vdots \\ \hat{\sigma}_{61} & \cdots & \hat{\sigma}_{66} \end{pmatrix}.$$

where:

$$\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}\eta_{it}\eta_{jt}.$$

First, using the values of η_{jt}, the residuals of the descriptive model, as determined in the estimation process above, we define:

$$\boldsymbol{\eta}_t = \begin{pmatrix} \eta_{1t} \\ \vdots \\ \eta_{6t} \end{pmatrix}.$$

Now we calculate the sample covariance matrix $\hat{\Sigma}$.

From equation (25):

$$\begin{aligned}\hat{\sigma}_{ij} &= \operatorname{cov}(\eta_i, \eta_j) \\ &= \sum_{k=1}^{6}\sum_{l=1}^{6} a_{ki}a_{lj}\operatorname{cov}(\varepsilon_k, \varepsilon_l) \\ &= \sum_{l=1}^{6} a_{ki}a_{kj}.\end{aligned}$$

We now require the matrix:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{16} \\ \vdots & & \vdots \\ a_{61} & \cdots & a_{66} \end{pmatrix}.$$

such that

$$\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{\Sigma}.$$

This may be found by Cholesky decomposition.

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