ASYMPTOTICS FOR OPERATIONAL RISK QUANTIFIED WITH EXPECTED SHORTFALL

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ABSTRACT

In this paper we estimate operational risk by using the convex risk measure Expected Shortfall (ES) and provide an approximation as the confidence level converges to 100% in the univariate case. Then we extend this approach to the multivariate case, where we represent the dependence structure by using a Lévy copula as in Böcker and Klüppelberg (2006) and Böcker and Klüppelberg, C. (2008). We compare our results to the ones obtained in Böcker and Klüppelberg (2006) and (2008) for Operational VaR and discuss their practical relevance.

KEYWORDS

Operational risk, Expected Shortfall, Lévy copula, regular variation.

1. INTRODUCTION

Within the framework of Basel II banks not only have to put aside equity reserves for market and credit risk but also for *operational risk*. In §664 of Basel Committee of Banking Supervision (2004) the Basel Committee defines: "Operational risk is the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events".

The particular difficulty in measuring this new risk type arises from the fact that partially the corresponding events are extremely rare with enormously high losses and at the same time there are comparatively few data.

Banks have to apply one of three methods in order to calculate the capital requirement: the Basis Indicator Approach, the Standardized Approach or the Advanced Measurement Approach (AMA). Within the first two methods, the capital charge is a percentage of the average annual gross income. According to the AMA, a bank is allowed to develop an internal operational risk model with individual distributional assumptions and dependence structures. Hence it is of great interest to develop suitable methods to estimate the capital reserve.

The most common way of estimating the amount of equity reserve for operational risk is by using the risk measure Value at Risk (VaR). In Böcker

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and Klüppelberg (2005) the so-called Operational Value at Risk (OpVaR) at level $\kappa \in (0,1)$ is defined as the κ-quantile of the aggregated loss process. Operational Value at Risk has been extensively studied both in the univariate and multivariate case respectively in Böcker (2006), Böcker and Klüppelberg (2005), (2006) and (2007) and (2008).

An essential disadvantage of this risk measure is that, in general, it is not coherent. In particular, it can happen that VaR attributes more risk to a loss portfolio than to the sum of the single loss positions. Moreover, VaR exclusively regards the probability of a loss whereas its size remains out of consideration.

The most popular alternative to VaR is the Expected Shortfall (ES), which is also known as Average VaR, Conditional VaR or Tail VaR. This risk measure is coherent and indicates the expected size of a loss provided that it exceeds the VaR. In particular, the ES seems to be the best convex alternative to the VaR, since it is the smallest law-invariant, convex risk measure continuous from below that dominates VaR (Theorem 4.61 of Föllmer and Schied (2004)). In addition, within the framework of *Solvency II* and *the Swiss Solvency Test*, insurers have to calculate their target capital by using the ES. The Federal Office of Private Insurance justifies this in chapter 2.4.1 of Federal Office of Private Insurance (2006) as follows:

> The risk measure Expected Shortfall is more conservative than the VaR at the same confidence level. Since it can be assumed that the actual loss profile exhibits several extremely high losses with a very low probability, the Expected Shortfall is the more appropriate risk measure, as, in contrast to the VaR, it regards the size of this extreme losses.

This argumentation is also suitable for operational risk, since it is very similar to the quoted actuarial risk. In Chavez-Demoulin and Embrechts (2004) and Moscadelli (2004) ES is then suggested as an alternative to VaR for quantifying operational risk. Hence, in this paper we evaluate operational risk by using the Expected Shortfall and derive asymptotic results in univariate and multivariate models.

The organization of the paper is the following. First we consider a one-dimensional Loss Distribution Approach (LDA) model. Since in §667, Basel Committee of Banking Supervision (2004), the Basel Committee sets the confidence level at 99,9%, it is reasonable to focus on the right distribution tail instead of estimating the whole distribution. Therefore we study the asymptotic behavior of the right distribution tail and, assuming that the severity distribution has a regularly varying tail with index $\alpha > 1$, we derive an asymptotic approximation of the Operational Expected Shortfall:

$$ES_t(\kappa) \sim \frac{\alpha}{\alpha - 1} VaR_t(\kappa), \ \kappa \to 1, \ \alpha > 1.$$

Then we consider a multivariate model, whose cells represent the different operational risk classes, since according to the AMA, operational risk shall be allocated to eight business lines (Basel Committee of Banking Supervision

(2004), §654) and seven loss types (Basel Committee of Banking Supervision (2004), appendix 7).

In the literature, the single risk classes are prevalently modelled by a compound Poisson process, i.e. the loss in one risk category i at time $t \geq 0$ is represented by the random sum

$$S_i(t) = \sum_{k=1}^{N_i(t)} X_k^i,$$

where $(X_k^i)_{k \in \mathbb{N}}$ is an independent and identically distributed (iid) severity process and $(N_i(t))_{t \geq 0}$ is a Poisson process independent of $(X_k^i)_{k \in \mathbb{N}}$. The total operational risk is the sum

$$S^+(t) = S_1(t) + \cdots + S_d(t).$$

However it is not realistic to assume that risk classes are independent. Hence in order to describe the dependencies between the $S_i(t)$, $1 \leq i \leq d$, we follow the approach of Böcker and Klüppelberg (2006) and use a *Lévy copula*. This yields a relatively simple model with comparatively few parameters as the dependencies between severities and frequencies are modelled simultaneously.

In this setting, we derive asymptotic conclusions for the OpES in various scenarios. For further details, we also refer to Ulmer (2007).

Finally we examine the practical relevance of our results.

2. Approximation of the OpES in a one-dimensional model

We suppose that operational risk follows an LDA model.

Definition 2.1. Loss Distribution Approach (LDA) model)

1. *The severity process: The severities are modelled by a sequence of positive iid random variables* $(X_k)_{k \in \mathbb{N}}$. *Let* F *be the distribution function (in short, df) of the* X_k.
2. *The frequency process: The random number* $N(t)$ *of losses in the time interval* $[0, t]$ *is a counting process, i.e. for* $t \geq 0$

$$N(t) := \sup\{n \geq 1 \colon T_n \leq t\}$$

   is generated by a sequence of random points in time $(T_n)_{n \in \mathbb{N}}$, *which satisfy* $0 \leq T_1 \leq T_2 \leq \ldots$ *a.s.*
3. *The severity process and the frequency process are assumed to be independent.*
4. *The aggregated loss process is defined as* $S(t) := \sum_{k=1}^{N(t)} X_k$.

In order to measure operational risk, we introduce the Operational Value at Risk (OpVaR) and the Operational Expected Shortfall (OpES). In this paper we will then focus on the OpES.

Definition 2.2. (OpVaR, OpES) *Let G_t be the df of the aggregated loss process $(S_t)_{t\geq 0}$ of an LDA model. The Operational Value at Risk until time t at level $\kappa \in (0,1)$ is the generalized inverse $G_t^{\leftarrow}$ of G_t*

$$VaR_t(\kappa) := G_t^{\leftarrow}(\kappa) = \inf\{x \in \mathbb{R} : G_t(x) \geq \kappa\}.$$

The Operational Expected Shortfall until time t at level $\kappa \in [0,1)$ is defined as

$$ES_t(\kappa) := \frac{1}{1-\kappa}\int_\kappa^1 VaR_t(u)\,du.$$

In order to compute these risk measures, we need to know the df G_t of $S(t)$. Because of the independence assumptions we know

$$G_t(x) = \mathbb{P}(S(t) \leq x) = \sum_{n=0}^{\infty} F^{n^*}(x)\,\mathbb{P}(N(t) = n), \tag{1}$$

where F^{n^*} is the n-th convolution of F and $F^{1^*} = F$ and $F^{0^*} = \mathbf{1}_{[0,\infty)}$.

We study now the asymptotic behavior of $\bar{G}_t(x) = \mathbb{P}(S(t) > x)$ for $x \to \infty$ and derive asymptotic results in univariate and multivariate models.

We say two real functions F, G are *asymptotically equal* for $x \to \infty$ ($F(x) \sim G(x)$, $x \to \infty$) if

$$\lim_{x\to\infty} \frac{F(x)}{G(x)} = 1.$$

Remark 2.3. *From the asymptotic equality of the summands we can infer the asymptotic equality of the sum. The same holds for the integrand and the integral:*

a) Let F_i, G_i, $i = 1, \ldots, d$, be positive real functions with

$$F_i(x) \sim G_i(x), \; x \to \infty. \tag{2}$$

Then

$$F_1(x) + \cdots + F_d(x) \sim G_1(x) + \cdots + G_d(x), \; x \to \infty.$$

b) Let $\varphi, \psi : [0,1] \to [0,\infty)$ with $\varphi(\kappa) \sim \psi(\kappa)$, $\kappa \to 1$, and suppose there exists a $\tau \in [0,1)$ such that $\int_\tau^1 \varphi(t)\,dt < \infty$ and $\int_\tau^1 \psi(t)\,dt < \infty$. Then

$$\int_\kappa^1 \varphi(t)\,dt \sim \int_\kappa^1 \psi(t)\,dt, \quad \kappa \to 1.$$

Furthermore, by §667, Basel Committee of Banking Supervision (2004), operational risk usually presents a *heavy-tailed* distribution. We take this into account by admitting only *regularly varying distribution tails*.

Definition 2.4. *A positive measurable function U on* $(0,\infty)$ *is called* regularly varying in ∞ with index $\rho \in \mathbb{R}$ $(U \in \mathcal{R}_\rho)$ *if*

$$\lim_{x\to\infty} \frac{U(xt)}{U(x)} = t^\rho, \quad t>0.$$

A positive measurable function L on $(0,\infty)$ *is called* slowly varying *in* ∞ $(L \in \mathcal{R}_0)$ *if*

$$\lim_{x\to\infty} \frac{L(xt)}{L(x)} = 1, \quad t>0.$$

From now on we will consider dfs with regularly varying tails $\bar{F} \in \mathcal{R}_{-\alpha}$ for $\alpha \geq 0$. Note that F becomes more heavy-tailed for α smaller. Examples for this kind of dfs are the Pareto and the Burr distribution (see Examples 2.6 and 2.7).

Examples for slowly varying functions are the logarithm and functions that converge to a positive constant. For $U \in \mathcal{R}_\rho$, $L(x) := \frac{U(x)}{x^\rho} \in \mathcal{R}_0$. Thus, for every $U \in \mathcal{R}_\rho$ there exists an $L \in \mathcal{R}_0$ with $U(x) = x^\rho L(x)$.

By Theorem 2.13 of Böcker and Klüppelberg (2006) we obtain that given an LDA model for a fixed time $t>0$ with a severity distribution tail $\bar{F} \in \mathcal{R}_{-\alpha}$, $\alpha > 0$, the following asymptotic equality for the OpVaR holds:

$$VaR_t(\kappa) \sim F^{\leftarrow}\left(1 - \frac{1-\kappa}{\mathbb{E}[N(t)]}\right), \quad \kappa \to 1, \tag{3}$$

if there exists an $\varepsilon > 0$ such that

$$\sum_{n=0}^{\infty} (1+\varepsilon)^n \, \mathbb{P}(N(t) = n) < \infty. \tag{4}$$

For further details about (4), we refer to Theorem 1.3.9 of Embrechts, Klüppelberg and Mikosch (1997).

Both economically relevant frequency processes, the Poisson process and the negative binomial process (see Embrechts, Klüppelberg and Mikosch (1997), Example 1.3.11), satisfy condition (4). To derive a similar representation of the OpES as in (3) we need several properties of regularly varying distribution tails (see Appendix). We now prove our main result.

Theorem 2.5. (Analytic OpES) *Consider an LDA model at a fixed time $t>0$, where the severities have a distribution function F such that the distribution tail $\bar{F} \in \mathcal{R}_{-\alpha}$ for $\alpha > 1$. Assume that there exists an $\varepsilon > 0$ such that*

$$\sum_{n=0}^{\infty} (1+\varepsilon)^n \, \mathbb{P}(N(t) = n) < \infty.$$

Then we have the following asymptotic equality for the OpES:

$$ES_t(\kappa) \sim \frac{\alpha}{\alpha-1} F^{\leftarrow}\left(1 - \frac{1-\kappa}{\mathbb{E}[N(t)]}\right) \sim \frac{\alpha}{\alpha-1} VaR_t(\kappa), \quad \kappa \to 1. \tag{5}$$

PROOF. Put $q_\kappa := VaR_t(\kappa)$. By Corollary 4.49 of Föllmer and Schied (2004) the Expected Shortfall is given by

$$\begin{aligned} ES_t(\kappa) &= \mathbb{E}[S(t) | S(t) > VaR_t(\kappa)] = \frac{\mathbb{E}\left[S(t)\mathbf{1}_{\{S(t)>q_k\}}\right]}{\mathbb{P}(S(t) > q_k)} \\ &= \frac{1}{1-\kappa} \int_{q_k}^{\infty} x dG_t(x) \\ &= \frac{1}{1-\kappa}\left(q_k \bar{G}_t(q_k) + \int_{q_k}^{\infty} \bar{G}_t(x)\, dx\right) \end{aligned}$$

Since condition (4) is satisfied and the df F is *subexponential*[1] due to Proposition A.1 b), by Theorem 1.3.9 of Embrechts, Klüppelberg and Mikosch (1997) we have that

$$\bar{G}_t(x) \sim \mathbb{E}[N(t)]\, \bar{F}(x), \quad x \to \infty.$$

Hence

$$\begin{aligned} ES_t(\kappa) &\overset{(Rem. 2.3b)}{\sim} \frac{\mathbb{E}[N(t)]}{1-\kappa}\left(q_k \bar{F}(q_k) + \int_{q_k}^{\infty} \bar{F}(x)\, dx\right) \\ &= \frac{\mathbb{E}[N(t)]}{1-\kappa} q_k \bar{F}(q_k) \left(1 + \frac{\int_{q_k}^{\infty} \bar{F}(x)\, dx}{q_k \bar{F}(q_k)}\right) \\ &\overset{(Prop. A.1 f)}{\sim} \frac{\mathbb{E}[N(t)]}{1-\kappa} \frac{\alpha}{\alpha-1} q_k \bar{F}(q_k). \end{aligned} \tag{6}$$

From Theorem 2.13 of Böcker and Klüppelberg (2006) we know:

$$q_k := VaR_t(\kappa) \overset{(3)}{\sim} F^{\leftarrow}\left(1 - \frac{1-\kappa}{\mathbb{E}[N(t)]}\right), \quad \kappa \to 1. \tag{7}$$

[1] Let $X_k, k \in \mathbb{N}$, be positive iid random variables with df F. The df F (or $\bar{F}$) is called subexponential, if $\bar{F}(x) = 0$ for all $x \in \mathbb{R}$, and if for all $n \geq 2$:

$$\lim_{x \to \infty} \frac{\mathbb{P}(X_1 + \cdots + X_n > x)}{\mathbb{P}(\max(X_1, \ldots, X_n) > x)} = 1.$$

Since $\bar{F} \in \mathcal{R}_{-\alpha}$ and by (7), from Proposition A.1c) with $c = 1$ we have that

$$\bar{F}(q_k) \sim \bar{F}\left(F^{\leftarrow}\left(1 - \frac{1-\kappa}{\mathbb{E}[N(t)]}\right)\right), \quad \kappa \to 1. \tag{8}$$

Moreover, since $\bar{F} \in \mathcal{R}_{-\alpha}$, by Resnick (1987) page 15 we have that

$$\bar{F}(\bar{F}^{\leftarrow}(x)) \sim x, \;\; x > 0. \tag{9}$$

Putting everything together we obtain:

$$\begin{aligned} ES_t(\kappa) &\overset{(8)}{\sim} \frac{\mathbb{E}[N(t)]}{1-\kappa} \frac{\alpha}{\alpha-1} F^{\leftarrow}\left(1 - \frac{1-\kappa}{\mathbb{E}[N(t)]}\right) \bar{F}\left(F^{\leftarrow}\left(1 - \frac{1-\kappa}{\mathbb{E}[N(t)]}\right)\right) \\ &\overset{(9)}{\sim} \frac{\alpha}{\alpha-1} F^{\leftarrow}\left(1 - \frac{1-\kappa}{\mathbb{E}[N(t)]}\right) \\ &\overset{(7)}{\sim} \frac{\alpha}{\alpha-1} VaR_t(\kappa), \quad \kappa \to 1, \end{aligned}$$

that proves (5). □

Example 2.6 (Pareto distribution) *If the severities are Pareto distributed, i.e. with distribution function*

$$F(x) = 1 - \left(1 + \frac{x}{\theta}\right)^{-\alpha}, \; \alpha, \theta, x > 0,$$

then $\bar{F} \in \mathcal{R}_{-\alpha}$*. By* (5) *we obtain*

$$\begin{aligned} ES_t(\kappa) &\sim \frac{\alpha}{\alpha-1} F^{\leftarrow}\left(1 - \frac{1-\kappa}{\mathbb{E}[N(t)]}\right) \\ &\sim \frac{\alpha}{\alpha-1} \theta \left(\frac{\mathbb{E}[N(t)]}{1-\kappa}\right)^{\frac{1}{\alpha}}, \quad \kappa \to 1. \end{aligned} \tag{10}$$

Example 2.7 (Burr distribution) *Let*

$$F(x) = 1 - \left(1 + \frac{x^{\tau}}{\theta}\right)^{-\alpha}, \; \tau, \alpha, \theta, x > 0$$

be the Burr df. Then $\bar{F} \in \mathcal{R}_{-\alpha\tau}$ *since*

$$\lim_{x \to \infty} \frac{\bar{F}(xt)}{\bar{F}(x)} = \lim_{x \to \infty} \left(\frac{\theta + (xt)^{\tau}}{\theta + x^{\tau}}\right)^{-\alpha} = t^{-\alpha\tau}, \quad t > 0.$$

Thus, the Burr distribution satisfies the conditions of Theorem 2.5 if $\alpha\tau > 1$, *and we have*

$$ES_t(\kappa) \sim \frac{\alpha}{\alpha-1}\left[\theta\left(\left(\frac{\mathbb{E}[N(t)]}{1-\kappa}\right)^{\frac{1}{\alpha}}-1\right)\right]^{\frac{1}{\tau}}, \quad \kappa \to 1. \tag{11}$$

For a further example, we also refer to Section 2 of Böcker (2006), where an analytical expression for the ES of operational risk has been computed for high-severity losses following a generalized Pareto distribution.

Comparing our result with the ones of Böcker and Klüppelberg (2006), we have

$$\lim_{\kappa \to 1} \frac{ES_t(\kappa)}{VaR_t(\kappa)} > 1,$$

and the closer α is to 1, the higher is the difference between Expected Shortfall and Value at Risk. For instance if

$$\begin{aligned} \alpha = 1{,}1 \quad & ES_t(\kappa) \sim 11 \cdot VaR_t(\kappa),\ \kappa \to 1, \\ \alpha = 2 \quad & ES_t(\kappa) \sim 2 \cdot VaR_t(\kappa),\ \kappa \to 1. \end{aligned}$$

Hence using OpVaR and its asymptotic estimation, we obtain an *underestimation of the capital reserve* that becomes bigger for α smaller.

3. Total OpES in the multivariate model

As mentioned before, the banks using AMA are required to divide their operational risk into several risk classes. Therefore, we investigate now a higher dimensional model, in which the single risk cells may be dependent.

Following the approach of Böcker and Klüppelberg (2006) we model the dependence structure with a Lévy copula. From now on we assume that the frequency process is a Poisson process. Since operational risks are always losses, we concentrate on Lévy processes admitting only positive jumps in every component, hereafter called *spectrally positive Lévy processes.*

Being interested in very high losses we introduce the notion of tail integral.

Definition 3.1. (tail integral) *Let L be a spectrally positive Lévy process on* $\mathbb{R}^d$ *with Lévy measure* Π. *The tail integral of L is the function* $\overline{\Pi} : [0,\infty]^d \to [0,\infty]$ *with the following properties:*

1. $\overline{\Pi}(x) = \Pi([x_1,\infty) \times \cdots \times [x_d,\infty)),\ x \in [0,\infty)^d$,
   where $\overline{\Pi}(0) = \lim_{x_1 \downarrow 0, \ldots, x_d \downarrow 0} \Pi([x_1,\infty) \times \cdots \times [x_d,\infty))$.
2. $\overline{\Pi}(x) = 0$ *if for any* $i \in \{1, \ldots, d\}$ $x_i = \infty$.

3. *$\overline{\Pi}(0, \ldots, 0, x_i, 0, \ldots, 0) = \overline{\Pi}_i(x_i)$, $x_i \in [0,\infty)$, $i = 1, \ldots, d$, where $\overline{\Pi}_i(x) = \Pi_i([x_1,\infty))$ is the tail integral of the i-th component.*

For a one-dimensional compound Poisson process with any jump size df F, we have that $\overline{\Pi}(x) = \lambda \overline{F}(x)$.

We model the dependence structure of the d components with a Lévy copula. By Definition 3.1 of Kallsen and Tankov (2004) we have

Definition 3.2. (Lévy copula) *A d-dimensional Lévy copula of a spectrally positive Lévy process is a function $C : [0,\infty]^d \to [0,\infty]$ such that*

1. *$C(u_1, \cdots, u_d) \neq \infty$ for $(u_1, \cdots, u_d) \neq (\infty, \cdots, \infty)$,*
2. *$C(u_1, \cdots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \cdots, d\}$,*
3. *C is d-increasing,*
4. *the margins $C_i(u_i) := \lim_{u_j \to +\infty, \forall j \neq i} C(u_1, \cdots, u_i, \cdots, u_d) = u_i$ for any $i \in \{1, \cdots, d\}$.*

From now on we consider a special case of the LDA model and assume that the severity distribution satisfies all the prerequisites of Theorem 2.5 such that in this model the asymptotic approximation (5) for the OpES holds.

Definition 3.3. (RVCP model) *A regularly varying compound Poisson model consists of the following elements:*

1. *The severity process: The severities are modelled by a sequence of positive iid random variables $(X_k)_{k\in\mathbb{N}}$. Let the distribution tail $\overline{F}$ of the X_k be regularly varying with index $-\alpha$, $\alpha > 1$, and continuous.*
2. *The frequency process: The random number $N(t)$ of losses in the time interval $[0,t]$, $t \geq 0$, is a Poisson process with parameter $\lambda > 0$.*
3. *The severity process and the frequency process are assumed to be independent.*
4. *The aggregated loss process is defined as $S(t) := \sum_{k=1}^{N(t)} X_k$, $t \geq 0$.*

The severities X_k being positive, $(S_t)_{t\geq 0}$ is a compound Poisson process with positive jumps and tail integral given by

$$\overline{\Pi}(x) = \lambda \overline{F}(x), \; x \geq 0.$$

According to the AMA operational risk shall be divided into eight business lines and seven loss types. We describe every single risk cell with an RVCP model in order to be able to approximate the OpES as in Theorem 2.5. As in Böcker and Klüppelberg (2006) we model the dependence structure by a Lévy copula and focus on a multivariate RVCP model.

Definition 3.4. (Multivariate RVCP model)

1. *Let every single risk cell be an RVCP model with aggregated loss process* S_i, *severity distribution tail* $\bar{F}_i \in \mathcal{R}_{-\alpha_i}$, $\alpha_i = 1$, *and Poisson process* N_t^i *with parameter* λ_i, $1 \leq i \leq d$.
2. *The dependence between cells is modelled by a Lévy copula. More precisely, with the tail integral* $\bar{\Pi}_i(x) = \lambda_i \bar{F}_i(x)$ *of* S_i, $1 \leq i \leq d$, *and a Lévy copula* C *the tail integral of* $(S_1, \ldots, S_d)$ *is given by*

$$\bar{\Pi}(x_1, \ldots, x_d) = C(\bar{\Pi}_1(x_1), \ldots, \bar{\Pi}_d(x_d)), \ (x_1, \ldots, x_d) \in [0, \infty)^d.$$

3. *The total aggregated loss process is defined as*

$$S^+(t) := S_1(t) + \ldots + S_d(t), \ t > 0,$$

with tail integral

$$\bar{\Pi}^+(x) = \Pi(\{(y_1, \ldots, y_d) \in [0, \infty)^d : \sum_{i=1}^{d} y_i > x\}), \ x \geq 0.$$

We denote G_t^+ *the df of* $S^+(t)$.

Sklar's Theorem (see Theorem 3.6 in Kallsen and Tankov (2004)) yields that $(S_1, \ldots, S_d)$ is a d-dimensional spectrally positive Lévy process.

Definition 3.5. (total OpES, total OpVaR) *The total Operational Expected Shortfall until time* $t > 0$ *at level* $\kappa \in [0,1)$ *is defined as*

$$ES_t^+(\kappa) := \frac{1}{1-\kappa} \int_\kappa^1 VaR_t^+(u)\, du,$$

where $VaR_t^+(\kappa) := \inf\{x \in \mathbb{R} : G_t^+(x) \geq \kappa\}$ *is the total Operational Value at Risk until time* t *at level* κ.

In this setting we obtain the following results for the total OpEs as a consequence of Theorem 2.5.

1. **One dominating cell:** First we consider the case where one severity distribution is more heavy-tailed than the other severity distributions. Without loss of generality we assume that it is the first cell. By Theorem 2.5 and Theorem 3.4 of Böcker and Klüppelberg (2006) we obtain the following result for the case of OpES.

   Proposition 3.6. *Consider a multivariate RVCP model with* $1 < \alpha_1 < \alpha_i$, $2 \leq i \leq d$ *and jump size df* F^+ *of the compound Poisson process* S^+. *Then*

$$\bar{F}^+(x) \sim \frac{\lambda_1}{\lambda^+} \bar{F}_1(x), \ x \to \infty, \tag{12}$$

   and the total OpES is asymptotically equal to the OpES of the first cell

$$ES_t^+(\kappa) \sim ES_t^1(\kappa), \ \kappa \to 1.$$

We see that in this case *the total OpES is asymptotically equal to the OpES of the first cell independently of the general dependence structure*. Consequently, a huge operational loss occurs very likely because of one single loss in the first cell instead of several dependent losses in different risk cells.

2. **Completely dependent cells:** we now assume that in all risk cells losses occur simultaneously, i.e. that the compound Poisson processes $S_1, \ldots, S_d$ always jump together. With a slight abuse of language, we say that in this case the Lévy processes $S_i(t)$, $1 \leq i \leq d$, are completely dependent (see also Böcker and Klüppelberg (2006) and Böcker and Klüppelberg (2007)). By Theorem 3.5 of Böcker and Klüppelberg (2006) the total OpVaR is asymptotically equal to the sum of the OpVaR of the cell processes

$$VaR_t^+(\kappa) \sim \sum_{i=1}^{d} VaR_t^i(\kappa), \ \kappa \to 1. \tag{13}$$

We have that the same holds for the OpES by using Theorem 2.5.

Proposition 3.7. *Consider a multivariate RVCP model at fixed time $t > 0$. We assume that the aggregated loss processes $S_1, \ldots, S_d$ are completely dependent with strictly increasing severity dfs $F_1, \ldots, F_d$. Then the total OpES asymptotically equals the sum of the cell OpES*

$$ES_t^+(\kappa) \sim \sum_{i=1}^{d} ES_t^i(\kappa), \ \kappa \to 1. \tag{14}$$

In §669 d) of Basel Committee of Banking Supervision (2004), the Basel Committee indicates the sum over all the risk cells as the standard procedure to quantify the total risk. Therefore, it seems that the Basel Committee acts on the assumption that the completely dependent case is the worst case that can happen. If applying a coherent, convex or subadditive risk measure like Expected Shortfall, this assumption is true, since the ES of a loss portfolio is always less or equal than the sum of the ES of the single losses, in spite of the prevailing kind of dependence. It fails, however, if VaR is applied. Note that ES has for $\alpha > 1$ the same properties of VaR at least asymptotically.

3. **Dependent model with b dominating cells:** we now assume that the first $b \in \{1, \ldots, d\}$ risk cells are more heavy-tailed than the remaining risk cells. Also in this case the total OpES is asymptotically equivalent to the OpES of the dominating cells, as it also happens in the case of the OpVaR (see Proposition 3.7 of Böcker and Klüppelberg (2006)).

Proposition 3.8. *Consider a multivariate RVCP model at fixed time $t > 0$. We assume that the aggregated loss processes $S_1, \ldots, S_d$ are completely dependent with strictly increasing severity dfs $F_1, \ldots, F_d$. Let $b \in \{1, \ldots, d\}$ and*

$1 < \alpha_1 = \ldots = \alpha_b =: \alpha < \alpha_j,\ j = b+1, \ldots, d$ *and let* $c_i \in (0, \infty),\ i = 2, \ldots, b$ *such that*

$$\lim_{x \to \infty} \frac{\bar{F}_i(x)}{\bar{F}_1(x)} = c_i.$$

Then with $c_1 := 1$ *and* $C := \sum_{i=1}^{b} c_i^{1/\alpha}$

$$ES_t^+(\kappa) \sim C \cdot ES_t^1(\kappa) \sim \frac{\alpha}{\alpha - 1} F_1^{-1}\left(1 - \frac{1-\kappa}{\lambda t C^\alpha}\right), \quad \kappa \to 1. \tag{15}$$

4. **Independent cells:** we now turn to the case where the aggregated loss processes $S_1, \ldots, S_d$ are independent. This holds if and only if they almost surely never jump together. By Theorem 2.5 it follows that total OpES behaves asymptotically as in the one-dimensional case, analogously to the case of the OpVaR (see Theorem 3.10 of Böcker and Klüppelberg (2006)).

Theorem 3.9. *Consider a multivariate RVCP model at fixed time* $t > 0$ *with independent aggregated loss processes* $S_1, \ldots, S_d$.

a) Then S^+ *is a one-dimensional RVCP model with Poisson parameter*

$$\lambda^+ = \lambda_1 + \cdots + \lambda_d$$

and severity distribution tail

$$\bar{F}^+(x) = \frac{1}{\lambda^+}\left(\lambda_1 \bar{F}_1(x) + \cdots + \lambda_d \bar{F}_d(x)\right) \in \mathcal{R}_{-\alpha},\ \alpha := \min(\alpha_1, \ldots, \alpha_d).$$

The total OpES behaves asymptotically as in the one-dimensional case, i.e.

$$ES_t^+(\kappa) \sim \frac{\alpha}{\alpha - 1} F^{+\leftarrow}\left(1 - \frac{1-\kappa}{\lambda^+ t}\right), \quad \kappa \to 1. \tag{16}$$

b) Let $1 < \alpha_1 = \ldots = \alpha_b =: \alpha < \alpha_j, j = b+1, \ldots, d$ *for* $b \in \{1, \ldots, d\}$ *and consider for* $i = 2, \ldots, b$ $c_i \in (0, \infty)$ *with*

$$\lim_{x \to \infty} \frac{\bar{F}_i(x)}{\bar{F}_1(x)} = c_i.$$

Then the total OpES can be approximated in the following way:

$$ES_t^+(\kappa) \sim \frac{\alpha}{\alpha - 1} F_1^{\leftarrow}\left(1 - \frac{1-\kappa}{C_\lambda t}\right) \sim \frac{\alpha}{\alpha - 1} VaR_t^+(\kappa), \quad \kappa \to 1, \tag{17}$$

with $C_\lambda := \lambda_1 + c_2 \lambda_2 + \cdots + c_b \lambda_b$.

We conclude the Section with some examples.

Example 3.10. *Let F_i, $i = 1, \ldots, d$, the Pareto distributions with parameters α_i, $\theta_i > 0$ and suppose that for $b \in \{1, \ldots, d\}$ $1 < \alpha_1 = \ldots = \alpha_b =: \alpha < \alpha_j$, $j = b + 1, \ldots, d$ holds.*

If the aggregated loss processes $S_1, \ldots, S_d$ are completely dependent, i.e. $N_t^i = N_t$ $\forall i = 1, \ldots, d$, then for $i = 1, \ldots, b$ it follows that

$$\lim_{x\to\infty} \frac{\bar{F}_i(x)}{\bar{F}_1(x)} = \lim_{x\to\infty} \frac{\left(1 + \frac{x}{\theta_1}\right)^{-\alpha_i}}{\left(1 + \frac{x}{\theta_1}\right)^{-\alpha_1}} = \lim_{x\to\infty} \left(\frac{(\theta_1 + x)\,\theta_i}{(\theta_i + x)\,\theta_1}\right)^{\alpha} = \left(\frac{\theta_i}{\theta_1}\right)^{\alpha}.$$

Hence, we know that $c_i = \left(\frac{\theta_i}{\theta_1}\right)^{\alpha}$ in Proposition 3.8. For the severity distribution tail $\bar{F}^+$ of the compound Poisson process S^+ we have

$$\begin{aligned} \bar{F}^+(x) &\sim \left(\sum_{i=1}^{b} c_i^{1/\alpha}\right)^{\alpha} \bar{F}_1(x) = \left(\sum_{i=1}^{b} \frac{\theta_i}{\theta_1}\right)^{\alpha} \left(1 + \frac{x}{\theta_1}\right)^{-\alpha} \\ &= \left(\sum_{i=1}^{b} \theta_i\right)^{\alpha} (\theta_1 + x)^{-\alpha} \sim \left(\sum_{i=1}^{b} \theta_i\right)^{\alpha} x^{-\alpha}, \quad x \to \infty. \end{aligned}$$

For the total Operational Expected Shortfall we obtain

$$\begin{aligned} ES_t^+(\kappa) \sim C \cdot ES_t^1(\kappa) &\overset{(10)}{\sim} \left(\sum_{i=1}^{b} \frac{\theta_i}{\theta_1}\right) \frac{\alpha}{\alpha - 1}\, \theta_1 \left(\frac{\lambda t}{1 - \kappa}\right)^{\frac{1}{\alpha}} \\ &= \left(\sum_{i=1}^{b} \theta_i\right) \frac{\alpha}{\alpha - 1} \left(\frac{\lambda t}{1 - \kappa}\right)^{\frac{1}{\alpha}}, \quad \kappa \to 1. \end{aligned}$$

If the aggregated losses $S_1, \ldots, S_d$ are independent, we know from Theorem 3.9 with $C_\lambda = \sum_{i=1}^{b} c_i \lambda_i$ that

$$\bar{F}^+(x) \sim \frac{C_\lambda}{\lambda^+} \bar{F}_1(x) = \frac{1}{\lambda^+} \sum_{i=1}^{b} \left(\frac{\theta_i}{\theta_1}\right)^{\alpha} \lambda_i \left(1 + \frac{x}{\theta_1}\right)^{-\alpha} \sim \frac{1}{\lambda^+} \sum_{i=1}^{b} \theta_i^{\alpha} \lambda_i x^{-\alpha},$$

if $x \to \infty$. In this case the total OpES can be approximated:

$$\begin{aligned} ES_t^+(\kappa) &\overset{(16)}{\sim} \frac{\alpha}{\alpha - 1} F^{+\leftarrow}\left(1 - \frac{1 - \kappa}{\lambda^+ t}\right) \\ &\sim \frac{\alpha}{\alpha - 1} \left(\frac{t \sum_{i=1}^{b} \theta_i^{\alpha} \lambda_i}{1 - \kappa}\right)^{\frac{1}{\alpha}} \\ &\overset{(10)}{\sim} \left(\sum_{i=1}^{b} \left(ES_t^i(\kappa)\right)^{\alpha}\right)^{\frac{1}{\alpha}}, \quad \kappa \to 1. \end{aligned}$$

For identical frequency parameters $\lambda := \lambda_1 = \cdots = \lambda_b$ *we obtain*

$$ES_t^+(\kappa) \sim \frac{\alpha}{\alpha - 1}\left(\frac{\lambda t}{1-\kappa}\right)^{\frac{1}{\alpha}}\left(\sum_{i=1}^{b}\theta_i^{\alpha}\right)^{\frac{1}{\alpha}}, \quad \kappa \to 1.$$

Our results hold for $\alpha > 1$. At first sight this requirement may appear more restrictive with respect to the case of OpVaR, since for the OpVaR the parameter α can be chosen from the interval $(0, \infty)$. The restriction to $\alpha > 1$ in Theorem 2.5 was a result of the Expected Shortfall being an integral of the Value at Risk. However, also the OpVaR cannot provide a "good" risk measure for the case $0 < \alpha < 1$, as shown in the following Example.

Example 3.11. *Consider identical frequency parameters* λ *also in the independent case and suppose that* $0 < \alpha_1 = \ldots = \alpha_b =: \alpha < \alpha_j$, $j = b+1, \ldots, d$ *for* $b \in \{1, \ldots, d\}$ *like in Example 3.10. Denote by* $VaR_{\|}^{+}$ *the total OpVaR of the completely dependent Pareto model and by* $VaR_{\perp}^{+}$ *the OpVaR of the independent Pareto model. Then in Section 3.1.2 of Böcker and Klüppelberg (2006) it is shown that*

$$\frac{VaR_{\perp}^{+}(\kappa)}{VaR_{\|}^{+}(\kappa)} \sim \frac{\left(\sum_{i=1}^{b}\theta_i^{\alpha}\right)^{1/\alpha}}{\sum_{i=1}^{b}\theta_i}\begin{cases} < 1, & \alpha > 1 \\ = 1, & \alpha = 1 \\ > 1, & \alpha < 1. \end{cases}$$

In the case $0 < \alpha < 1$, *the total OpVaR allocates more risk to the independent model than to the dependent model,* $VaR_{\perp}^{+}(\kappa) > VaR_{\|}^{+}(\kappa)$ *assuming* κ *close to* 1. *Hence, the Pareto distribution for* $\alpha \in (0,1)$ *is so heavy-tailed that the OpVaR is not subadditive or convex anymore.*

4. Practical relevance

We now discuss the practical relevance of our results. First of all a natural question is whether regularly varying distributions with index $-\alpha$ for $\alpha > 1$ estimate correctly real loss size distributions. Moscadelli examined in Moscadelli (2004) over 45.000 operational losses of 89 banks for the year 2002, categorized according to eight business lines. Due to the scarcity of data, the representation of the few high losses proves to be considerably more complicated. Moscadelli therefore uses extreme value theory, in particular the peaks over threshold method, and assumes that the high loss sizes have a generalized Pareto distribution, where the generalized Pareto distribution ($GPD_{\xi,\beta}$) with form parameter $\xi \in \mathbb{R}$ and scale parameter $\beta > 0$ is defined as

$$GPD_{\xi,\beta}(x) := \begin{cases} 1 - \left(1 + \xi\frac{x}{\beta}\right)^{-\frac{1}{\xi}} & \text{for } \xi \neq 0 \\ 1 - \exp(-x/\beta) & \text{for } \xi = 0, \end{cases}$$

where $x \geq 0$ for $\xi \geq 0$ and $0 \leq x \leq -\beta/\xi$ for $\xi < 0$. The $GPD_{\xi,\beta}$ is regularly varying with parameter $\alpha = 1/\xi$ for $\xi = 0$. In Moscadelli (2004) the parameters (ξ, β) are estimated for every business line by maximum likelihood estimation. The result of this inquiry is that in six out of eight business lines the parameter α is less than 1. If Moscadelli's analysis were an accurate account of the actual operational risk, then the conditions of Theorem 2.5 would be satisfied in 25% of the business lines, since the GPD with parameter $\xi > 0$ has a decreasing Lebesgue density. However, in Nešlehov, Embrechts and Chavez-Demoulin (2006) it is suggested that the aggregation chosen in Moscadelli (2004) is questionable, since the seven loss types are not of the same kind. Therefore the problem of estimating the parameter α is still highly debated and needs further research.

The second problem to be discussed is which kind of measure is the most suitable for the estimation of capital reserves for operational risk.

As a solution Moscadelli suggests in Moscadelli (2004) the risk measure Median Shortfall, which adds the median of the exceedance distribution to the threshold u:

$$MS(u) := u + F_u^{\leftarrow}\left(\frac{1}{2}\right), \quad u > 0,$$

with

$$F_u(x) := \mathbb{P}(X - u \leq x \mid X > u) = \frac{F(x+u) - F(u)}{1 - F(u)}, \quad 0 \leq x < x_F - u, \quad (18)$$

where $x_F \leq \infty$ is the right end point of F. The advantage of the median is that it minimizes the absolute deviation. Reserving equity in the amount of $MS(u)$, a bank presumably can pay half of all losses that exceed u.

In order to include a confidence level κ into the risk measure, we choose VaR as the threshold

$$u = VaR_t(\kappa) = G_t^{\leftarrow}(\kappa),$$

and obtain the following representation of the Median Shortfall in our model:

$$\begin{aligned} MS_t(\kappa) &= VaR_t(\kappa) \\ &\quad + \inf\left\{ y \in \mathbb{R} : \mathbb{P}(S(t) - VaR_t(\kappa) \leq y \mid S(t) > VaR_t(\kappa)) \geq \frac{1}{2} \right\} \\ &\overset{(18)}{=} G_t^{\leftarrow}(\kappa) + \inf\left\{ y \in \mathbb{R} : \frac{G_t\left(y + G_t^{\leftarrow}(\kappa)\right) - G_t\left(G_t^{\leftarrow}(\kappa)\right)}{1 - G_t\left(G_t^{\leftarrow}(\kappa)\right)} \geq \frac{1}{2} \right\} \end{aligned}$$

If G_t is continuous, we can simplify the second summand

$$\begin{aligned}
&\inf\left\{y \in \mathbb{R} : G_t\left(y + G_t^{\leftarrow}(\kappa)\right) - \kappa > \frac{1-\kappa}{2}\right\} \\
&= \inf\left\{x \in \mathbb{R} : G_t(x) \geq \frac{1+\kappa}{2}\right\} - G_t^{\leftarrow}(\kappa) \\
&= G_t^{\leftarrow}\left(\frac{1+\kappa}{2}\right) - G_t^{\leftarrow}(\kappa)
\end{aligned}$$

and obtain

$$MS_t(\kappa) = G_t^{\leftarrow}\left(\frac{1+\kappa}{2}\right) = VaR_t\left(\frac{1+\kappa}{2}\right).$$

Hence, in the case of a continuous aggregated loss df G_t, the Median Shortfall at confidence level κ equals the Value at Risk at level $\frac{1+\kappa}{2}$, i.e. for $\kappa = 99.9\%$

$$MS_t(0.999) = VaR_t(0.9995).$$

This directly yields that Median Shortfall is not coherent and thus is no ideal candidate for measuring operational risk.

To conclude we remark again that the choice of VaR is not completely satisfactory, since it is too optimistic (see (5)) and not convex. Indicating only the probability of a loss and not the size of it, it may underestimate the "potentially severe tail loss events" (Basel Committee of Banking Supervision (2004), §667). In addition, for $\alpha \in (0,1)$ the mere summation of the OpVaR of the single cells is not an upper bound of the total OpVaR, as the Basel Committee assumes in Basel Committee of Banking Supervision (2004), §669d). This is only accurate if applying a convex risk measure like the ES provided it exists.

A. Regularly varying distribution tails

The class of regularly varying functions has several properties, that we recall here for the reader's convenience. For further details, see Bingham, Goldie and Teugels (1987), Embrechts, Klüppelberg and Mikosch (1997) and Resnick (1987) (especially Theorem 1.7.2 and Proposition 1.5.10 of Bingham, Goldie and Teugels (1987), Lemma 1.3.1 and Appendix A3 of Embrechts, Klüppelberg and Mikosch (1997), Proposition 0.8 of Resnick (1987)).

Proposition A.1.

a) Let $\bar{F} \in \mathcal{R}_{-\alpha}$ be the tail of a df and let X be distributed according to F. Then $\mathbb{E}[X^\beta] < \infty$ if $\beta < \alpha$.

b) Every regularly varying distribution tail is subexponential.

c) Let *$U \in \mathcal{R}_\rho$ with $\rho \in \mathbb{R}$ and f, g positive functions on $(0,\infty)$ with $f(x) \to \infty$, $g(x) \to \infty$, $x \to \infty$, and such that there exists a constant $c \in (0,\infty)$ with*

$$f(x) \sim c \cdot g(x), \ x \to \infty.$$

Then

$$U(f(x)) \sim c^{\rho} U(g(x)), \;\; x \to \infty.$$

d) *Let* $\bar{F} \in \mathcal{R}_{-\alpha}$, $\alpha > 0$, *a distribution tail. Then* $\left(\frac{1}{\bar{F}}\right)^{\leftarrow} \in \mathcal{R}_{1/\alpha}$.

e) *Let* $\bar{F}, \bar{G} \in \mathcal{R}_{-\alpha}$, $\alpha > 0$, *F a df,* $\bar{G}$ *decreasing. If* $\bar{F}(x) \sim c\bar{G}(x)$, $x \to \infty$, *for some* $c = 0$, *then*

$$\left(\frac{1}{\bar{F}}\right)^{\leftarrow}(x) \sim c^{1/\alpha} \left(\frac{1}{\bar{G}}\right)^{\leftarrow}(x), \quad x \to \infty. \tag{19}$$

f) ***(Karamata's Theorem)*** *Let L be slowly varying and* $\rho < -1$. *Then*

$$\int_x^{\infty} t^{\rho} L(t)\, dt \sim \frac{-1}{\rho + 1} x^{\rho + 1} L(x), \;\; x \to \infty. \tag{20}$$

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