

OPTIMAL RISK CONTROL FOR THE EXCESS OF LOSS REINSURANCE POLICIES

BY

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ABSTRACT

The primary objective of the paper is to explore using reinsurance as a risk management tool for an insurance company. We consider an insurance company whose surplus can be modeled by a Brownian motion with drift and that the surplus can be invested in a risky or riskless asset. Under the above Black-Scholes type framework and using the objective of minimizing the ruin probability of the insurer, we formally establish that the excess-of-loss reinsurance treaty is optimal among the class of plausible reinsurance treaties. We also obtain the optimal level of retention as well as provide an explicit expression of the minimal probability of ruin.

KEYWORDS

Excess of loss; Minimal probability of ruin; Stochastic control; Investments; Hamilton-Jacobi-Bellman equation.

1. INTRODUCTION

It is well known that reinsurance can be an effective tool for insurance companies to manage and control their exposure to risk. An appropriate use of reinsurance protects the insurer against any undesirable potential large losses and hence reduces the insurer's earnings' volatilities. In practice, there exists a wide variety of reinsurance strategies. Among them, the proportional reinsurance and the excess-of-loss reinsurance are two of the most widely studied reinsurance strategies. For example, Schmidli [15], [16] considered the proportional reinsurance and determined the optimal proportional reinsurance strategy by minimizing the probability of ruin. Taksar and Markussen [19] extended the analysis by proposing a diffusion model with investment and proportional reinsurance. Schmidt [17] dealt with optimal proportional reinsurance for dependent lines of business. See Promislow and Young [14], Taksar [18], Højgaard and Taksar [8], Asmussen and Taksar [1], and references therein for related studies on proportional reinsurance.

Similar to the proportional reinsurance, the excess-of-loss reinsurance has attracted a significant amount of interests among practitioners and researchers. For example, Asmussen *et al.* [2] explored the excess-of-loss reinsurance and the dividend distribution policy in the context of maximizing the expected present value of the dividends in a diffusion model. Choulli, Taksar and Zhou [5] investigated the case of excess-of-loss reinsurance for an insurance company and solved the problem of risk control and dividend optimization for a financial institution facing a constant liability payment. Centeno [4] dealt with the optimal excess of loss retention limits for two dependent risks. See also Paulsen and Gjessing [13], Irgens and Paulsen [10], Mnif and Sulem [12], Zhang, Zhou and Guo [20], Hürlimann [9] for other related researches on excess-of-loss reinsurance. Motivated by these recent results, the key contribution of this paper is to provide additional analysis on the effective use of excess-of-loss reinsurance strategy as a risk management tool. In particular, we assume that surplus can be invested in a financial market with risky asset or risk-free asset and the excess-of-loss reinsurance is optimally determined by minimizing the ruin probability of the resulting diffusion model.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}$. For the classical Cramér-Lundberg model, the reserve (or surplus) at time t , denoted by P_t , evolves according to

$$P_t = x + pt - \sum_{i=1}^{N(t)} Z_i,$$

where $x \geq 0$ represents the initial level of reserve, $\{N(t)\}$ is a Poisson process with intensity $\beta > 0$, Z_i , $i = 1, 2, \dots$, independent of $\{N(t)\}$, are i.i.d. loss random variables with common continuous distribution F having finite first moment μ_∞ and finite second moment σ_∞^2 , and $p > 0$ is the premium rate. Typically, the premium rate p is determined using the expected value principle; i.e.

$$p = (1 + \eta)\beta\mu_\infty, \quad (1.1)$$

where $\eta > 0$ is the relative safety loading of the insurer.

We now consider a modification of the above classical Cramér-Lundberg model that takes into account the reinsurance. Recall that Z_i is the loss (or claim) random variable insured by a insurer in the absence of reinsurance. When reinsurance exists, we use the notation $\mathcal{H}(Z_i)$ to capture the portion of the claims retained by the insurer for a given Z_i . This implies that $Z_i - \mathcal{H}(Z_i)$ is the residual part of Z_i that is covered by the reinsurer. It is reasonable to assume that $\mathcal{H}(x)$ is an increasing function in x and that $\mathcal{H}(0) = 0$, $0 \leq \mathcal{H}(x) \leq x$. By $\mathcal{D}$ we define as the set of all $\mathcal{H}$ satisfying the above conditions.

Corresponding to a chosen reinsurance policy $\mathcal{H}(Z_i)$, we denote $P_t^{\mathcal{H}}$ as the reserve at time t of a generalized Cramér-Lundberg model in the presence of reinsurance. Then we have

$$P_t^{\mathcal{H}} = x + p^{\mathcal{H}} \cdot t - \sum_{i=1}^{N(t)} \mathcal{H}(Z_i),$$

where $p^{\mathcal{H}}$ is the net premium rate reflecting the reinsurance premium that is payable by the insurer to the reinsurer. Under the assumption that the reinsurer also relies on the expected value principle with a positive constant safety loading θ to determine the reinsurance premium, the net premium rate is given by

$$\begin{aligned} p^{\mathcal{H}} &= (1 + \eta)\beta\mu_{\infty} - (1 + \theta)\beta E[Z_i - \mathcal{H}(Z_i)] \\ &= (1 + \theta)\beta E(\mathcal{H}(Z_i)) - (\theta - \eta)\beta\mu_{\infty}. \end{aligned} \quad (1.2)$$

Note that typically we have $\theta > \eta$.

Without any loss of generality, we assume $\beta = 1$. Then according to Grandell [7], the reserve process can be approximated by a diffusion process $\{R_t\}$ of the following form:

$$dR_t = (\theta E(\mathcal{H}(Z)) - (\theta - \eta)\mu_{\infty})dt + \sqrt{E(\mathcal{H}^2(Z))}dB_t, \quad (1.3)$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion.

Let us now consider two special cases of the above generalized Cramér-Lundberg model by making additional assumption on the structure of the reinsurance strategy $\mathcal{H}(Z)$. In particular we consider the proportional reinsurance and the excess-of-loss reinsurance. For the proportional reinsurance with a proportional constant $0 < a < 1$, we have $\mathcal{H}(Z_i) = aZ_i$ so that $E(\mathcal{H}(Z)) = a\mu_{\infty}$ and $E(\mathcal{H}^2(Z)) = a^2\sigma_{\infty}^2$. Under this specification, the diffusion process (1.3) simplifies to

$$dR_t = (\theta a\mu_{\infty} - (\theta - \eta)\mu_{\infty})dt + a\sigma_{\infty}dB_t. \quad (1.4)$$

For the excess-of-loss reinsurance with retention level a ; i.e. $\mathcal{H}(Z_i) = \min(Z_i, a) = Z_i \wedge a$, we have

$$\mu(a) \triangleq E[Z \wedge a] = \int_0^a \bar{F}(x)dx, \quad (1.5)$$

$$\sigma^2(a) \triangleq E[(Z \wedge a)^2] = \int_0^a 2x\bar{F}(x)dx, \quad (1.6)$$

where $\bar{F}(x) = P(Z > x)$. Furthermore, the diffusion process (1.3) becomes

$$dR_t = (\theta\mu(a) - (\theta - \eta)\mu_{\infty})dt + \sigma(a)dB_t. \quad (1.7)$$

We also assume that the reserve is invested in a financial (risky) asset with the price process $\{S_t\}$ governed by

$$dS_t = rS_tdt + \lambda S_t dW_t, \quad (1.8)$$

where $r \geq 0$ is the constant mean rate of return generated by the underlying financial asset, $\lambda \geq 0$ is its volatility, and $\{W_t, t \geq 0\}$ is a standard Brownian motion which is independent of $\{B_t, t \geq 0\}$. Under the special case $\lambda = 0$, there is no randomness on the return of the asset and hence the underlying financial asset corresponds to the risk-free bond or bank account earning risk-free rate of return r .

An admissible control policy π is described by a $\mathcal{F}_t$ -adapted stochastic process $\mathcal{H}_t^\pi(Z)$, where $\mathcal{H}_t^\pi(x)$ is increasing in x satisfying $\mathcal{H}_t^\pi(0) = 0$, $0 \leq \mathcal{H}_t^\pi(x) \leq x$. Here, we clarify that $E(\mathcal{H}_t^\pi(Z)) = E(\mathcal{H}(Z))|_{\mathcal{H}=\mathcal{H}_t^\pi}$ and $E(\mathcal{H}_t^\pi(Z))^2 = E(\mathcal{H}(Z))^2|_{\mathcal{H}=\mathcal{H}_t^\pi}$. Also, we denote by R_t^π the resulting reserve process given an admissible policy π and by R_0^π the initial reserve which is assumed to be $\mathcal{F}_0$ -measurable. Without any loss of generality, we assume that it is equal to a deterministic value x . Consequently, the dynamics of the surplus process can be written as

$$dR_t^\pi = (\theta E(\mathcal{H}_t^\pi(Z)) - (\theta - \eta)\mu_\infty)dt + \sqrt{E(\mathcal{H}_t^\pi(Z))^2} dB_t + rR_t^\pi dt + \lambda R_t^\pi dW_t \quad (1.9)$$

$$R_0^\pi = x. \quad (1.10)$$

Let us now define $\mathcal{A}$ as the set of all admissible policies. Then for any $\pi \in \mathcal{A}$, the time of ruin and the probability of ruin are defined, respectively, as

$$\begin{aligned} T^\pi &= \inf\{t > 0 : R_t^\pi \leq 0\}, \\ \psi^\pi(x) &= P\{T^\pi < \infty \mid R_t^\pi = x\}. \end{aligned} \quad (1.11)$$

Under the above model formulation, our objective is to find an optimal policy π^* which is the solution to the following value function:

$$\inf_{\pi \in \mathcal{A}} \psi^\pi(x). \quad (1.12)$$

Let $\psi^{\pi^*}(x)$ or simply, $\psi(x)$, be the corresponding optimal value function with optimal policy π^* ; i.e.

$$\psi(x) = \psi^{\pi^*}(x) = \inf_{\pi \in \mathcal{A}} \psi^\pi(x). \quad (1.13)$$

We also denote $\psi(x)$ as the minimal probability of ruin.

It is easy to show that (see P121-123 Klebaner [11]) the solution to the SDE (1.9) is given by

$$R_t^\pi = U(t) \left(x + \int_0^t \frac{\theta E(\mathcal{H}_s^\pi(Z)) - (\theta - \eta)\mu_\infty}{U(s)} ds + \int_0^t \frac{\sqrt{E(\mathcal{H}_s^\pi(Z))^2}}{U(s)} dB_s \right), \quad (1.14)$$

where $U(t) = \exp\{(r - \frac{1}{2}\lambda^2)t + \lambda W_t\}$. In our paper, we are interested in the non-cheap reinsurance, i.e. $\theta > \eta$. Otherwise, we can choose $\mathcal{H}_t^\pi(Z) \equiv 0$ so that (1.14) reduces

$$R_t^\pi = xe^{(r-\frac{\lambda^2}{2})t+\lambda W_t}.$$

This in turn implies that ruin can never occur.

The remaining of the paper is organized as follows. In Section 2, we show that the optimal excess-of-loss reinsurance is always better than any other reinsurance. In Section 3, we state the equation which the optimal probability of ruin as a function of the initial reserve x should satisfy. The verification theorem and some analysis on the equation that satisfied by the optimal probability of ruin are also given in this section. In Sections 4 and 5, we give the optimal excess-of-loss reinsurance strategy and the explicit expression of optimal value function for the riskless and risky investment, respectively. We give the conclusion of this paper in Section 6.

2. THE GAIN OF EXCESS-OF-LOSS REINSURANCE

In this section, we formally establish that the optimal excess-of-loss reinsurance is always better than any other reinsurance, as asserted in Theorem 1¹. We first provide the following lemma that will facilitate us in proving Theorem 1.

Lemma 1. *For any fixed function $\mathcal{H} \in \mathcal{D}$, let a_1 be a constant satisfying*

$$E(Z \wedge a_1)^2 - E[\mathcal{H}^2(Z)] = 0,$$

then

$$E(Z \wedge a_1) - E[\mathcal{H}(Z)] \geq 0.$$

Proof: Let us begin the proof by introducing functions $\hbar(a)$ and $\ell(a)$ as

$$\hbar(a) = E(Z \wedge a)^2 - E[\mathcal{H}^2(Z)],$$

$$\ell(a) = E(Z \wedge a) - E[\mathcal{H}(Z)].$$

Obviously, both $\hbar(a)$ and $\ell(a)$ are increasing functions in a . Let a_2 be the root of $\ell(a) = 0$. We can easily prove that

$$\hbar(a_1) = 0, \ell(a_1) \geq 0 \Leftrightarrow \ell(a_2) = 0, \hbar(a_2) \leq 0.$$

Consequently, to complete the proof it is sufficient to establish $\hbar(a_2) \leq 0$. Noting that

$$\begin{aligned} Z \wedge a_2 - \mathcal{H}(Z) &\geq 0, \quad \text{for } Z \leq \mathcal{H}^{-1}(a_2), \\ Z \wedge a_2 - \mathcal{H}(Z) &\leq 0, \quad \text{for } Z \geq \mathcal{H}^{-1}(a_2), \end{aligned}$$

¹ The authors are grateful to the anonymous referee for encouraging us to address the more general result as stated in Theorem 1. We point out a similar problem is also investigated by Zhou and Cai [21]. The proof given in this paper, however, is different from that in Zhou and Cai [21].

we have

$$\begin{aligned}
 \hat{h}(a_2) &= E(Z \wedge a_2)^2 - E[\mathcal{H}^2(Z)] \\
 &= E[(Z \wedge a_2 - \mathcal{H}(Z))(Z \wedge a_2 + \mathcal{H}(Z))I_{Z \leq \mathcal{H}^{-1}(a_2)}] \\
 &\quad + E[(Z \wedge a_2 - \mathcal{H}(Z))(Z \wedge a_2 + \mathcal{H}(Z))I_{Z \geq \mathcal{H}^{-1}(a_2)}] \\
 &\leq 2a_2 E[(Z \wedge a_2 - \mathcal{H}(Z))I_{Z \leq \mathcal{H}^{-1}(a_2)}] + 2a_2 E[(Z \wedge a_2 - \mathcal{H}(Z))I_{Z \geq \mathcal{H}^{-1}(a_2)}] \\
 &= 2a_2 E[(Z \wedge a_2 - \mathcal{H}(Z))] \\
 &= 0,
 \end{aligned}$$

as required. $\square$

Theorem 1. *For all x*

$$\psi(x) \leq \psi_{\mathcal{H}}(x), \quad (2.1)$$

where $\psi_{\mathcal{H}}(x)$ is the ruin probability in any reinsurance function $\mathcal{H}$.

Proof. Let $\mathcal{H}_t^\pi(Z) \in \mathcal{D}$ be any fixed reinsurance function. We have $0 \leq E[\mathcal{H}_t^\pi(Z)]^2 \leq \sigma_\infty^2$. We can choose a feedback control $a_{\pi_1}(t)$ in the excess-of-loss model in such a way that

$$\sigma^2(a_{\pi_1}(t)) = E[Z \wedge a_{\pi_1}(t)]^2 = E[\mathcal{H}_t^\pi(Z)]^2. \quad (2.2)$$

From above lemma, we have

$$\mu(a_{\pi_1}(t)) = E[Z \wedge a_{\pi_1}(t)] \geq E[\mathcal{H}_t^\pi(Z)],$$

and hence

$$\theta\mu(a_{\pi_1}(t)) - (\theta - \eta)\mu_\infty \geq \theta E[\mathcal{H}_t^\pi(Z)] - (\theta - \eta)\mu_\infty.$$

Finally, by the expression (1.14) of R_t^π , we conclude that $\psi(x) \geq \psi_{\mathcal{H}}(x)$. $\square$

Thus, in the following sections we only focus on the Cramér-Lundberg model with the excess-of-loss reinsurance.

3. THE HJB EQUATION AND THE VERIFICATION THEOREM

In this section, we demonstrate that the dynamic programming approach as described in Fleming and Soner [6] can be used to obtain the solution to minimizing the ruin probability. We begin our analysis by first stating the following lemma and theorem. Lemma 2 formally gives an obvious property

associated with $\psi(x)$. The proof of this lemma is trivial and hence is omitted. Theorem 2 can be found in Fleming and Soner [6].

Lemma 2. *The function $\psi(x)$ defined by (1.13) is nonincreasing.*

Theorem 2. *Assume that $\psi(x)$ defined by (1.13) is twice continuous differentiable on $(0, \infty)$. Then $\psi(x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:*

$$\inf_{a \in [0, N]} \left\{ \frac{1}{2} (\sigma^2(a) + \lambda^2 x^2) \psi''(x) + [\theta \mu(a) - (\theta - \eta) \mu_\infty + rx] \psi'(x) \right\} = 0 \quad (3.1)$$

with boundary conditions

$$\psi(0) = 1, \quad (3.2)$$

$$\psi(\infty) = 0, \quad (3.3)$$

and $N = \sup\{y : F(y) < 1\} \leq \infty$.

We emphasize that in general, the value function may not be smooth enough to satisfy the HJB equation (3.1). It, however, still satisfies (3.1) in a viscosity sense (see Fleming and Soner [6]).

The remaining of this paper is devoted to solving the HJB equation (3.1) subject to the boundary conditions (3.2) and (3.3). To do this, the following lemma is essential to the proof of the verification theorem:

Lemma 3. *For any policy π , $M > 0$, and by defining*

$$T_M^\pi = \inf\{t > 0 : R_t^\pi \notin [0, M]\},$$

then $T_M^\pi < \infty$ a.s.

Proof: The above lemma can be proved by making some modifications to the proof of Lemma 6.1 of Taksar and Markussen [19]. The necessary steps are as follows: first, replace $\theta(a)$ and $\psi(a, x)$ in the proof of Lemma 6.1 of Taksar and Markussen [19] by $\theta(a) = \theta \mu(a) - (\theta - \eta) \mu_\infty$ and $\psi(a, x) = \sqrt{\sigma^2(a) + \lambda^2 x^2}$, respectively. Here $\mu(a)$ and $\sigma^2(a)$ are defined in (1.5) and (1.6), respectively. Second, set $\sigma^* = \psi(a^*, 0)$ and choose a^* such that $\theta(a^*) = 0$. Third, the above definition of $\theta(a)$ implies $\theta(a) \leq \eta \mu_\infty$ and hence by substitution, equation (6.2) of Taksar and Markussen [19] leads to

$$\Theta(\eta_{k+1}) - \Theta(\eta_k) \leq \eta \mu_\infty / (\sigma^*)^2. \quad (3.4)$$

Finally, the remaining of the proof is identical to the proof of Lemma 6.1 of Taksar and Markussen [19]. $\square$

Remark: The above lemma leads to the following results: for any arbitrary strategy π with probability one either ruin occurs or R_t^π diverges to infinity as $t \rightarrow \infty$.

We now present a verification theorem which establishes that the classic solution to the HJB equation yields the solution to the optimization problem.

Theorem 3. *Assume that $f(x) \in C^2$ satisfies (3.1)-(3.3). Then the value function $\psi(x)$ given by (1.12) coincides with $f(x)$. Furthermore, if $a^*(x)$ is such that*

$$\frac{1}{2} (\sigma^2(a^*(x)) + \lambda^2 x^2) \psi''(x) + [\theta \mu(a^*(x)) - (\theta - \eta) \mu_\infty + rx] \psi'(x) = 0 \quad (3.5)$$

for all x . Then the feedback form policy $a_{\pi^*}(s) = a^*(R_s^{\pi^*})$, where $R_s^{\pi^*}$ is the corresponding solution to (1.9)-(1.10), is optimal, i.e. $f(x) = \psi(x) = \psi^{\pi^*}(x)$.

Proof. Let $a_\pi(t)$ be any admissible strategy. By Itô formula, we have

$$\begin{aligned} E[f(R_{T^\pi \wedge t}^\pi)] &= f(x) + \int_0^{T^\pi \wedge t} [\theta \mu(a_\pi(s)) - (\theta - \eta) \mu_\infty + r R_s^\pi] f'(R_s^\pi) ds \\ &\quad + \int_0^{T^\pi \wedge t} \frac{1}{2} f''(R_s^\pi) [(\lambda R_s^\pi)^2 + \sigma^2(a_\pi(s))] ds, \\ &\geq f(x), \end{aligned} \quad (3.6)$$

since $f(x)$ solves (3.1).

Note that

$$E[f(R_{T^\pi \wedge t}^\pi)] = E[f(R_{T^\pi \wedge t}^\pi) I(T^\pi < t)] + E[f(R_{T^\pi \wedge t}^\pi) I(T^\pi \geq t)] \rightarrow \psi^\pi(x)$$

as $t \rightarrow \infty$. This implies that together with (3.6), we have $\psi^\pi(x) \geq f(x)$. If we were to use the strategy $a_{\pi^*}(x)$, we attain the equality in (3.6) and therefore $\psi(x) = f(x)$. $\square$

In what follows we will analyze the HJB equation (3.1) and give two lemmas that will be used to construct the decreasing convex smooth function $\psi(x)$. By differentiating (3.1) with respect to a , we obtain the following infimum function:

$$a(x) = -\theta \frac{\psi'(x)}{\psi''(x)}. \quad (3.7)$$

Furthermore, substituting the above result into (3.1) yields

$$g(a, x) \psi''(x) = 0,$$

where

$$g(a, x) = a \left(1 - \frac{\eta}{\theta} \right) \mu_{\infty} - a \mu(a) - \frac{r a x}{\theta} + \frac{1}{2} [\sigma^2(a) + \lambda^2 x^2]. \quad (3.8)$$

This implies that solving HJB equation (3.1) boils down to solving $g(a, x) = 0$. The following lemma states some conditions for which $g(a, x) = 0$ yields a unique positive solution $\bar{a}(x)$.

Lemma 4. *Fix $x > 0$. If one of the following conditions is satisfied*

- (i) $\lambda = 0$, $r > 0$, $x < \frac{(\theta - \eta)\mu_{\infty}}{r}$
- (ii) $\lambda \neq 0$

then $g(a, x) = 0$ admits a unique positive solution $\bar{a}(x)$.

Proof. Note that for $x > 0$, we have $g''(a, x) = -\bar{F}(a) \leq 0$ and $\lim_{a \rightarrow \infty} g(a, x) = -\infty$.

(i) Under the prescribed conditions $\lambda = 0$, $r > 0$, and $x < \frac{(\theta - \eta)\mu_{\infty}}{r}$, it is easy to verify that $g(0, x) = 0$, and $g'(0, x) = (1 - \eta/\theta)\mu_{\infty} - rx/\theta > 0$. Hence there exists a unique positive solution $\bar{a}(x)$ such that $g(\bar{a}(x), x) = 0$.

(ii) The condition $\lambda \neq 0$ implies that $g(0, x) = \frac{\lambda^2 x^2}{2} > 0$. Consequently, there exists a unique positive solution $\bar{a}(x)$ such that $g(\bar{a}(x), x) = 0$. $\square$

Recall that N was defined earlier as the maximum claim amount. Then the relationship between $a^*(x)$, $\bar{a}(x)$ and N can be summarized as follows:

$$a^*(x) = \begin{cases} \bar{a}(x) & \text{if } \bar{a}(x) \leq N \\ N & \text{otherwise.} \end{cases}$$

Note that when $\bar{a}(x) > N$, the optimal retention level $a^*(x)$ is set to N . This is to be expected since the insurer cannot reinsure claim that is larger than the actual claim. The following lemma gives an equivalent condition for $\bar{a}(x) > N$.

Lemma 5. *Assume either condition (i) or condition (ii) of Lemma 4 is satisfied, then $N < \bar{a}(x)$ if and only if $N < h(x)$ where $h(x) = \frac{\theta(\sigma_{\infty}^2 + \lambda^2 x^2)}{2(\mu_{\infty}\eta + rx)}$.*

Proof: It follows from the proof of Lemma 4 that $N < \bar{a}(x)$ if and only if $g(N, x) > 0$. The latter inequality becomes

$$g(N, x) = \frac{1}{2} \sigma_{\infty}^2 - N \mu_{\infty} \eta / \theta - r N x / \theta + \frac{1}{2} \lambda^2 x^2 > 0$$

which in turn is equivalent to $N < \frac{\theta(\sigma_{\infty}^2 + \lambda^2 x^2)}{2(\mu_{\infty}\eta + rx)}$, as required. $\square$

4. THE CASE OF $\lambda = 0$

In this section, we continue with our analysis by assuming that the reserve is invested in a risk-free asset such as a bond or a bank account. This corresponds to the case $\lambda = 0$ so that the HJB equation (3.1) simplifies to

$$\inf_{a \in [0, N]} \left\{ \frac{1}{2} \sigma^2(a) \psi''(x) + [\theta \mu(a) - (\theta - \eta) \mu_\infty + rx] \psi'(x) \right\} = 0. \quad (4.1)$$

We are primarily interested in the solution to the above HJB equation for the non-trivial case $r > 0$. The special case with $r = 0$ was studied by Zhang, Zhou and Guo [20].

Suppose the following inequality

$$x \geq \frac{(\theta - \eta) \mu_\infty}{r} \triangleq x_1, \quad (4.2)$$

holds and that we choose $a_\pi(t) \equiv 0$, then by (1.14) we obtain

$$R_t^\pi = e^{rt} \left(x - \frac{(\theta - \eta) \mu_\infty (1 - e^{-rt})}{r} \right).$$

Thus ruin can never occur, i.e. $\psi(x) \equiv 0$ for $x \geq x_1$. The above observation suggests that we only need to construct a decreasing convex function $\psi(x)$ which solves the HJB equation (4.1) for $0 < x \leq x_1$. Furthermore, Lemma 5 asserts that $N < \bar{a}(x)$ if and only if $N < h(x)$ where in this case $h(x) = \frac{\theta \sigma_\infty^2}{2(\mu_\infty \eta + rx)}$ is nonincreasing in x . This implies that it is sufficient to consider the following two subcases: $N < h(0)$ and $N \geq h(0)$. These two subcases are discussed in details in the following two subsections.

4.1. The case of $N < h(0)$

In this special case since $N < h(0) = \frac{\theta \sigma_\infty^2}{2\mu_\infty \eta}$, the equation $h(x) = N$ admits a unique positive root, say x_0 ,

$$x_0 = \frac{\theta \sigma_\infty^2}{2Nr} - \frac{\mu_\infty \eta}{r}. \quad (4.3)$$

Also $\sigma_\infty^2 = \int_0^N 2x \bar{F}(x) dx < 2N \int_0^N \bar{F}(x) dx = 2N\mu_\infty$ so that we easily obtain $x_0 < x_1$.

For $x \leq x_0$, we have $\bar{a}(x) \geq N$ and so we choose N as the optimal retention level. Thus in this case the HJB equation (4.1) becomes

$$\frac{1}{2} \sigma_\infty^2 \psi''(x) + (\eta \mu_\infty + rx) \psi'(x) = 0. \quad (4.4)$$

Solving the above equation for $x \leq x_0$ we obtain

$$\psi(x) = -K_1 \int_0^x \exp \left\{ -\int_0^y \frac{2(\mu_\infty \eta + rs)}{\sigma_\infty^2} ds \right\} dy + K_2.$$

for appropriately chosen constants K_1 and K_2 .

For $x_0 < x < x_1$, $\bar{a}(x) \leq N$ and so we choose $a^*(x) = \bar{a}(x)$ as the optimal retention level. Using the relation $\bar{a}(x) = -\theta \frac{\psi'(x)}{\psi''(x)}$, we get

$$\psi(x) = -K_3 \int_{x_0}^x \exp \left\{ -\int_{x_0}^y \frac{\theta}{\bar{a}(s)} ds \right\} dy + K_4,$$

for constants K_3 and K_4 . Thus the solution to (4.1) can be represented in the following form

$$\psi(x) = \begin{cases} -K_1 \int_0^x \exp \left\{ -\int_0^y \frac{2(\mu_\infty \eta + rs)}{\sigma_\infty^2} ds \right\} dy + K_2, & x \leq x_0 \\ -K_3 \int_{x_0}^x \exp \left\{ -\int_{x_0}^y \frac{\theta}{\bar{a}(s)} ds \right\} dy + K_4 & x_0 < x < x_1, \\ 0, & x \geq x_1. \end{cases} \quad (4.5)$$

From the boundary condition (3.2), it is easy to verify that

$$K_2 = 1. \quad (4.6)$$

Let

$$\begin{aligned} h_1 &= \int_0^{x_0} \exp \left\{ -\int_0^y \frac{2(\mu_\infty \eta + rs)}{\sigma_\infty^2} ds \right\} dy, \\ h_2 &= \exp \left\{ -\int_0^{x_0} \frac{2(\mu_\infty \eta + rs)}{\sigma_\infty^2} ds \right\}, \\ h_3 &= \int_{x_0}^{x_1} \exp \left\{ -\int_{x_0}^y \frac{\theta}{\bar{a}(s)} ds \right\} dy. \end{aligned}$$

Then the continuity of $\psi(x)$ at x_0 , x_1 and the continuity of $\psi'(x)$ at x_0 give the following equations

$$K_4 = 1 - K_1 h_1, \quad K_3 = K_1 h_2, \quad K_4 - h_3 K_3 = 0.$$

Solving for K_1 , K_3 , K_4 , we obtain

$$K_1 = \frac{1}{h_1 + h_2 h_3}, \quad K_3 = \frac{h_2}{h_1 + h_2 h_3}, \quad K_4 = \frac{h_2 h_3}{h_1 + h_2 h_3}. \quad (4.7)$$

We now summarize the key result of this subsection in the following theorem.

Theorem 4. *Suppose that $N < h(0)$. Then $\psi(x)$ defined by (4.5) with K_i , given by (4.6)-(4.7) and x_0, x_1 defined by (4.3), (4.2) respectively is a decreasing twice continuously differentiable convex solution of (3.1)-(3.3). The optimal feedback control function $a^*(x)$ in this case is given by*

$$a^*(x) = \begin{cases} N, & x \leq x_0 \\ \bar{a}(x), & x_0 < x < x_1, \\ 0, & x \geq x_1 \end{cases} \quad (4.8)$$

where $\bar{a}(x)$ is the unique positive solution of $g(a, x)$ defined by (3.8).

4.2. The case of $N \geq h(0)$

In this case $\bar{a}(x) \leq N$ for $x < x_1$ and so we choose $\bar{a}(x)$ as the optimal retention level. Using the same argument as in the preceding subsection yields

$$\psi(x) = -K_5 \int_0^x \exp \left\{ -\int_0^y \frac{\theta}{\bar{a}(s)} ds \right\} dy + K_6,$$

where K_5 and K_6 are appropriately chosen constants that satisfy boundary conditions (3.2) and (3.3). More precisely, we have

$$K_5 = \frac{1}{\int_0^{x_1} \exp \left\{ -\int_0^y \frac{\theta}{\bar{a}(s)} ds \right\} dy} \quad (4.9)$$

and $K_6 = 1$. These calculations yield the following theorem.

Theorem 5. *Supposed that $N \geq h(0) = \frac{\theta \sigma_\infty^2}{2\mu_\infty \eta}$, then*

$$\psi(x) = \begin{cases} 1 - K_5 \int_0^x e^{\int_0^y -\frac{\theta}{\bar{a}(s)} ds} dy, & x < x_1 \\ 0, & x \geq x_1 \end{cases}$$

with K_5 defined by (4.9) is the solution to the HJB equation (4.1) with boundary conditions (3.2) and (3.3). In this case the optimal feedback control function $a^*(x)$ is given by

$$a^*(x) = \begin{cases} \bar{a}(x), & x < x_1 \\ 0, & x \geq x_1 \end{cases}.$$

5. THE CASE OF $\lambda \neq 0$

In the last section, we assume that the reserve is invested in asset that is risk-free. This section relaxes this restriction by assuming that the underlying investment asset in the financial market is risky. This is equivalent to enforcing $\lambda \neq 0$ in (1.8). Under this special case, it is easy to see that the function $h(x)$ is a strictly convex function and attains its minimum at $x_2 = \frac{-\eta\mu_\infty + \sqrt{\eta^2\mu_\infty^2 + r^2\sigma_\infty^2/\lambda^2}}{r}$. To tackle the solution to the HJB equation (3.1), it is useful to divide our analysis into the following three cases, depending on the relative magnitude of N and $h(x)$: (i) $N < h(x_2)$, (ii) $h(x_2) \leq N \leq h(0)$, and (iii) $N > h(0)$. These cases are discussed in the subsequent subsections.

5.1. The case of $N < h(x_2)$

This subsection deals with the special case $N < h(x_2)$. Since $h(x)$ attains its minimum at x_2 for this particular case, Lemma 5 asserts that $\bar{a}(x) > N$ for all $x > 0$ and hence the optimal level of retention is N . Furthermore, it follows from Theorem 2 that $\psi(x)$ is the solution to

$$\frac{1}{2}(\sigma_\infty^2 + \lambda^2 x^2)\psi''(x) + (\eta\mu_\infty + rx)\psi'(x) = 0 \quad (5.1)$$

subject to the boundary conditions (3.2) and (3.3). Solving the above equation gives

$$\psi(x) = -K_1 \int_0^x \exp\left(-\frac{2\eta\mu_\infty}{\lambda\sigma_\infty} \arctan\left(\frac{\lambda y}{\sigma_\infty}\right)\right) \left(\frac{y^2 + \frac{\sigma_\infty^2}{\lambda^2}}{\frac{\sigma_\infty^2}{\lambda^2}}\right)^{-\frac{r}{\lambda^2}} dy + K_2 \quad (5.2)$$

where in this case, $K_2 = 1$ and

$$K_1 = \left[\int_0^\infty \exp\left(-\frac{2\eta\mu_\infty}{\lambda\sigma_\infty} \arctan\left(\frac{\lambda y}{\sigma_\infty}\right)\right) \left(\frac{y^2 + \frac{\sigma_\infty^2}{\lambda^2}}{\frac{\sigma_\infty^2}{\lambda^2}}\right)^{-\frac{r}{\lambda^2}} dy \right]^{-1} \quad (5.3)$$

in order to satisfy the boundary conditions (3.2) and (3.3). It should be emphasized that the boundary conditions (3.2) and (3.3) cannot be satisfied unless integral (5.3) converges. Since $\arctan(x) \rightarrow \pi/2$ as $x \rightarrow \infty$, this implies that the integral in the right hand side of (5.3) converges if and only if $\frac{2r}{\lambda^2} > 1$, or equivalently,

$$r > \frac{\lambda^2}{2}. \quad (5.4)$$

Summarizing these results yields the theorem below.

Theorem 6. *Assume that $N < h(x_2)$ and (5.4) is true. Then the function $\psi(x)$ given by (5.2) with K_1 given by (5.3) and $K_2 = 1$ is a convex decreasing solution to (3.1) to (3.3). The optimal feedback control function $a^*(x)$ in this case is equal to N .*

When (5.4) is not satisfied, then we have the following theorem, which is similar to the result established in Taksar and Markussen [19].

Theorem 7. *If $r \leq \lambda^2/2$ then $\psi(x) = 1$.*

Proof: For any strategy π , let us define

$$\psi_M^\pi(x) = P(R_{T_M^\pi}^\pi = 0).$$

where T_M^π was defined in Lemma 3. It is easy to verify that $\psi_M^\pi(x) \rightarrow \psi^\pi(x)$ as $M \rightarrow \infty$ and $\psi_M^\pi(x)$ satisfies (we write below $a(t)$ instead of $a_M^\pi(t)$)

$$\frac{1}{2}(\sigma^2(a(x)) + \lambda^2 x^2) \psi_M^{\pi''}(x) + [\theta \mu(a(x)) - (\theta - \eta) \mu_\infty + rx] \psi_M^{\pi'}(x) = 0 \quad (5.5)$$

with boundary conditions $\psi_M^\pi(0) = 1$ and $\psi_M^\pi(M) = 0$. Solving the above equation yields

$$\psi_M^\pi(x) = 1 - \frac{\int_0^x \exp \left\{ -\int_0^y \frac{2(\theta \mu(a(s)) - (\theta - \eta) \mu_\infty + rs)}{\sigma^2(a(s)) + \lambda^2 s^2} ds \right\} dy}{\int_0^M \exp \left\{ -\int_0^y \frac{2(\theta \mu(a(s)) - (\theta - \eta) \mu_\infty + rs)}{\sigma^2(a(s)) + \lambda^2 s^2} ds \right\} dy}.$$

Note that $a(s) \in [0, N]$. This implies that for any $y > y_1$, we have

$$\begin{aligned} & \int_0^\infty \exp \left\{ -\int_0^y \frac{2(\theta \mu(a(s)) - (\theta - \eta) \mu_\infty + rs)}{\sigma^2(a(s)) + \lambda^2 s^2} ds \right\} dy \\ & \geq \int_0^{y_1} \exp \left\{ -\int_0^y \frac{2(\theta \mu(a(s)) - (\theta - \eta) \mu_\infty + rs)}{\sigma^2(a(s)) + \lambda^2 s^2} ds \right\} dy \\ & + \exp \left\{ -\int_0^{y_1} \frac{2(\theta \mu(a(s)) - (\theta - \eta) \mu_\infty + rs)}{\sigma^2(a(s)) + \lambda^2 s^2} ds \right\} \cdot \int_{y_1}^\infty \exp \left\{ -\int_{y_1}^y \frac{2(\eta \mu_\infty + rs)}{\sigma_\infty^2 + \lambda^2 s^2} ds \right\} dy. \end{aligned}$$

Since the last integral can be expressed as

$$\begin{aligned}
& \int_{y_1}^{\infty} \exp \left\{ - \int_{y_1}^y \frac{2(\eta\mu_{\infty} + rs)}{\sigma_{\infty}^2 + \lambda^2 s^2} ds \right\} dy \\
&= \int_{y_1}^{\infty} \exp \left(\frac{-2\mu_{\infty}\eta}{\lambda\sigma_{\infty}} \left[\arctan \left(\frac{\lambda y}{\sigma_{\infty}} \right) - \arctan \left(\frac{\lambda y_1}{\sigma_{\infty}} \right) \right] \right) \cdot \left(\frac{\lambda^2 y^2 + \sigma_{\infty}^2}{\lambda^2 y_1^2 + \sigma_{\infty}^2} \right)^{-\frac{r}{\lambda^2}} dy,
\end{aligned} \tag{5.6}$$

this suggests that when $r \leq \lambda^2/2$, the integral in (5.6) diverges so that $\psi_M^{\pi}(x) \rightarrow 1$ as $M \rightarrow \infty$. Consequently for any strategy π , $\psi^{\pi}(x) = 1$, and hence $\psi(x) = 1$. $\square$

5.2. The case of $h(x_2) \leq N \leq h(0)$

In this case $h(x) = N$ has two different positive solutions, say x_3 and x_4 , such that

$$x_3 = \frac{Nr - \sqrt{N^2 r^2 - \theta \lambda^2 (\theta \sigma_{\infty}^2 - 2N\mu_{\infty}\eta)}}{\theta \lambda^2}, \tag{5.7}$$

$$x_4 = \frac{Nr + \sqrt{N^2 r^2 - \theta \lambda^2 (\theta \sigma_{\infty}^2 - 2N\mu_{\infty}\eta)}}{\theta \lambda^2}. \tag{5.8}$$

For $x < x_3$ or $x > x_4$, we have $\bar{a}(x) > N$ and so we choose N as the optimal retention level. As in the previous subsection, one can show that the function $\psi(x)$ satisfies equation (5.1). Solving this equation, we obtain for $x < x_3$,

$$\psi(x) = -K_1 \int_0^x g(y) dy + K_2,$$

and for $x > x_4$,

$$\psi(x) = -K_1' \int_{x_4}^x g(y) dy + K_2',$$

where

$$g(y) = \exp \left(- \frac{2\eta\mu_{\infty}}{\lambda\sigma_{\infty}} \arctan \left(\frac{\lambda y}{\sigma_{\infty}} \right) \right) \left(\frac{y^2 + \frac{\sigma_{\infty}^2}{\lambda^2}}{\frac{\sigma_{\infty}^2}{\lambda^2}} \right)^{-\frac{r}{\lambda^2}}. \tag{5.9}$$

For $x_3 \leq x \leq x_4$ we have $\bar{a}(x) \leq N$. This implies that the optimal retention level $a^*(x)$ is equal to $\bar{a}(x)$. In this case the function $\psi(x)$ satisfies the following equation

$$\bar{a}(x) = -\theta \frac{\psi'(x)}{\psi''(x)}. \tag{5.10}$$

Solving above equation we get for $x_3 \leq x \leq x_4$,

$$\psi(x) = -K_3 \int_{x_3}^x \exp\left(-\int_{x_3}^y \frac{\theta}{\bar{a}(s)} ds\right) dy + K_4. \quad (5.11)$$

Thus the solution to (3.1) can be represented in the following form

$$\psi(x) = \begin{cases} -K_1 \int_0^x g(y) dy + K_2, & x < x_3 \\ -K_3 \int_{x_3}^x \exp\left(-\int_{x_3}^y \frac{\theta}{\bar{a}(s)} ds\right) dy + K_4, & x_3 \leq x \leq x_4, \\ -K_1' \int_{x_4}^x g(y) dy + K_2', & x > x_4 \end{cases} \quad (5.12)$$

where $g(y)$ is given by (5.9). From the boundary condition (3.2), we have

$$K_2 = 1. \quad (5.13)$$

Now let

$$h_1 = \int_0^{x_3} g(y) dy, \quad h_2 = g(x_3), \quad h_3 = \int_{x_3}^{x_4} \exp\left(-\int_{x_3}^y \frac{\theta}{\bar{a}(s)} ds\right) dy$$

$$h_4 = \frac{\exp\left(-\int_{x_3}^{x_4} \frac{\theta}{\bar{a}(s)} ds\right)}{g(x_4)}, \quad h_5 = \int_{x_4}^{\infty} g(y) dy,$$

then the continuity of $\psi(x)$ and $\psi'(x)$ at x_3, x_4 and the boundary condition (3.3) give the following equations

$$K_4 = 1 - K_1 h_1, \quad K_3 = K_1 h_2, \quad K_2' = K_4 - h_3 K_3, \quad K_1' = h_4 K_3, \quad K_2' = h_5 K_1'.$$

Solving for $K_1, K_3, K_4, K_1', K_2'$, we get

$$K_1 = \frac{1}{h_1 + h_2 h_3 + h_2 h_4 h_5}, \quad K_3 = \frac{h_2}{h_1 + h_2 h_4 + h_2 h_4 h_5},$$

$$K_4 = \frac{h_2 h_3 + h_2 h_4 h_5}{h_1 + h_2 h_3 + h_2 h_4 h_5}, \quad (5.14)$$

$$K_1' = \frac{h_2 h_4}{h_1 + h_2 h_3 + h_2 h_4 h_5}, \quad K_2' = \frac{h_2 h_4 h_5}{h_1 + h_2 h_3 + h_2 h_4 h_5}.$$

Note that h_1, h_2, h_3, h_4 are always finite. The constant h_5 is finite if and only if (5.4) holds.

Theorem 8. Suppose that $h(x_2) \leq N \leq h(0)$ and (5.4) is true. Then $\psi(x)$ defined by (5.12) with K_i, K'_i given by (5.13)-(5.14) and x_3, x_4 defined by (5.7)-(5.8) is a decreasing twice continuously differentiable convex solution of (3.1)-(3.3). The optimal feedback control function $a^*(x)$ in this case is given by

$$a^*(x) = \begin{cases} N, & x < x_3 \\ \bar{a}(x), & x_3 \leq x \leq x_4, \\ N, & x > x_4 \end{cases} \quad (5.15)$$

where $\bar{a}(x)$ is the unique positive root of $g(a, x)$ defined by (3.8).

If (5.4) does not hold, then we can apply the same arguments as in the previous subsection and conclude that Theorem 7 holds.

5.3. The case of $N > h(0)$

In this case the function $h(x) = N$ admits a unique positive root, say x_5 ,

$$x_5 = \frac{Nr + \sqrt{N^2 r^2 - \theta \lambda^2 (\theta \sigma_\infty^2 - 2N\mu_\infty \eta)}}{\theta \lambda^2}.$$

By the property of $h(x)$, we have $\bar{a}(x) \leq N$ for $x \leq x_5$ and $\bar{a}(x) > N$ for $x > x_5$. Therefore we choose the optimal retention level as $a^*(x) = N$ for $x > x_5$ and $a^*(x) = \bar{a}(x)$ for $x \leq x_5$. Using the same arguments as in the previous subsection we can write $\psi(x)$ as

$$\psi(x) = \begin{cases} -K_3 \int_0^x \exp\left(-\int_0^y \frac{\theta}{\bar{a}(s)} ds\right) dy + K_4 & 0 < x \leq x_5, \\ -K_1 \int_{x_5}^x g(y) dy + K_2, & x > x_5 \end{cases}, \quad (5.16)$$

where g is defined by (5.9). The continuity of $\psi(x)$ and $\psi'(x)$ and the boundary conditions (3.2), (3.3) yield

$$K_4 = 1 \quad (5.17)$$

and

$$K_2 = 1 - h_1 K_3, \quad K_1 = K_3 h_2, \quad K_2 = K_1 h_3,$$

where

$$h_1 = \int_0^{x_5} \exp\left(-\int_0^y \frac{\theta}{\bar{a}(s)} ds\right) dy, \quad h_2 = \frac{\exp\left(-\int_0^{x_5} \frac{\theta}{\bar{a}(s)} ds\right)}{g(x_5)}, \\ h_3 = \int_{x_5}^\infty g(y) dy.$$

Solving for K_i , $i = 1, 2, 3$, we have

$$K_1 = \frac{h_2}{h_1 + h_2 h_3}, \quad K_2 = \frac{h_2 h_3}{h_1 + h_2 h_3}, \quad K_3 = \frac{1}{h_1 + h_2 h_3}. \quad (5.18)$$

We remark that h_3 is finite if and only if (5.4) holds.

Theorem 9. *Supposed that $N > h(0)$ and (5.4) holds. The function $\psi(x)$ defined by (5.16) with K_i , $i = 1, \dots, 4$ given by (5.17)-(5.18). Then $\psi(x)$ is a convex decreasing twice differentiable solution to (3.1) to (3.3). The optimal feedback control function $a^*(x)$ in this case is given by*

$$a^*(x) = \begin{cases} \bar{a}(x) & 0 < x \leq x_5 \\ N, & x > x_5 \end{cases}, \quad (5.19)$$

where $\bar{a}(x)$ is the unique positive root of $g(a, x)$ defined by (3.8).

When (5.4) fails, the same arguments as in Section 5.1 can be used to demonstrate that Theorem 7 holds.

6. CONCLUSION

The quest for optimal reinsurance strategies has remained an active area of research among academics and practitioners in the last few decades. The profound interest in the design of reinsurance lies in its potential as a risk management and risk mitigating tool. This paper contributes to the literature by providing additional analysis on the use of reinsurance. In particular, we consider the optimal control problem of the insurance company with investment and reinsurance. By minimizing the probability of ruin of an insurer, we formally show that the optimal excess-of-loss reinsurance is always better than any other reinsurance. More importantly, we also give derive the optimal retention in the excess-of-loss reinsurance strategy and the explicit expression of optimal value function for the riskless and risky investment, respectively.

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