SOME REMARKS ON DELAYED RENEWAL RISK MODELS

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ABSTRACT

Some extensions to the delayed renewal risk models are considered. In particular, the independence assumption between the interclaim time and the subsequent claim size is relaxed, and the classical Gerber-Shiu penalty function is generalized by incorporating more variables. As a result, general structures regarding various joint densities of ruin related quantities as well as their probabilistic interpretations are provided. The numerical example in case of time-dependent claim sizes is provided, and also the usual delayed model with time-independent claim sizes is discussed including a special case with exponential claim sizes. Furthermore, asymptotic formulas for the associated compound geometric tail for the present model are derived using two alternative methods.

KEYWORDS

Delayed renewal risk model, Gerber-Shiu function, Last ladder height, Last interclaim time, Cramer's asymptotic ruin formula.

1. INTRODUCTION

In risk theory, the ordinary (*Sparre Andersen*) renewal risk model is often used for modelling the insurer's surplus process (see e.g. Gerber and Shiu (2005), Grandell (1991), Li and Garrido (2005), Willmot (2007), and references therein). This traditional ordinary renewal risk model has usually assumed that claim sizes are independent and identically distributed (iid) random variables independent of the sequence of iid interclaim times and a claim occurs at time 0. However, major difficulties with these assumptions are raised in some cases. First, if the time elapsed since the last event has an impact on the claim size of the subsequent event (e.g. catastrophe insurance), then this assumption may be a problem for describing the situation precisely. In recent years, several authors such as Albrecher and Boxma (2004), Badescu et al. (2009), Boudreault et al. (2006), and Cossette et al. (2008) have considered various dependency structures between the claim sizes and the interclaim times. Second, in some cases an event occurred some time in the past rather than at time 0 as implicitly

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assumed in the ordinary renewal risk model. In other words, a business (or a system) might have been operating for some time before we start observing the process at time 0, and an event does not necessarily occur at time 0. Therefore, the delayed (*Sparre Andersen*) renewal risk model including the stationary renewal risk model may be alternative to enhance and improve the model to reflect these circumstances by assuming different distribution of the time until the first claim. Certainly, these modified processes revert to the traditional ordinary model upon the occurrence of the first claim. For further details of the traditional delayed and stationary renewal processes, see Cox (1968, Section 2.2), Grandell (1991), Rolski et al. (1999), Ross (1996, Section 3.5), and Willmot and Lin (2001, Section 11.4). Therefore, the model to be considered here is the delayed renewal risk model with time-dependent claim sizes which retains a form of general dependency structure. An application of this model, for instance, is to earthquake insurance. Since larger earthquakes occur less frequently, and also the last observed earthquake may be occurred in the past rather than in the present, specific time-dependent structure for the claim sizes as well as the occurrence of the last main shock before time 0 are necessarily considered for modelling. We illustrate an example of this in Section 4.1.

As in the ordinary renewal risk model, in the traditional delayed (stationary) renewal risk models, the classical Gerber-Shiu expected discounted penalty function introduced by Gerber ad Shiu (1998) has used for a unified study of the ruin related quantities involving the time of ruin, the surplus prior to ruin and the deficit at ruin (e.g. Willmot (2004), Willmot and Dickson (2003), Kim (2007), Kim and Willmot (2010)). However all these quantities are defined at the time when ruin occurs, and there is not enough information monitoring the process before ruin occurs. Therefore, in the present model (the dependent delayed renewal risk model), we study a generalized Gerber-Shiu penalty function considered by Cheung et al. (2010b) in the dependent ordinary renewal risk model. From this generalization which involves introducing new variables defined before ruin, namely the minimum surplus level before ruin and the surplus level near ruin (precisely after the second last claim), we study further ruin related quantities. The motivation for the analysis of quantities defined in relation to the time of ruin (such as the new variables introduced above) is essentially the same as that for the special case considered by Dufresne and Gerber (1988), namely the claim causing ruin. In particular, the event of ruin necessarily involves adverse financial consequences, and any and all information associated with this event is useful for prediction and thus helping to contribute to sound risk management.

Here we briefly review the traditional delayed renewal risk model first. Suppose that an insurer's surplus at time t is defined as $\{U_t;\, t \geq 0\}$ with $U_t = u + ct - \sum_{i=1}^{N_t} Y_i$, and $u \geq 0$ is the initial surplus. The number of claims process $\{N_t;\, t \geq 0\}$ is assumed to be a renewal process, with V_1 the time of the first claim and V_i the time between the $(i-1)$-th and the i-th claim for $i = 2, 3, 4, \ldots$ having probability density function (pdf) $k(t)$, distribution function (df) $K(t) = 1 - \bar{K}(t)$ and Laplace transform $\tilde{k}(s) = \int_0^\infty e^{-st} k(t)dt$. It is assumed that the distribution of the time (from 0) to the first event V_1 is different from that of the others (V_i) having pdf $k_1(t)$, df $K_1(t) = 1 - \bar{K}_1(t)$ and Laplace transform $\tilde{k}_1(s) = \int_0^\infty e^{-st} k_1(t)\,dt$. In the traditional delayed renewal risk model, it is assumed that $\{V_i\}_{i=1}^\infty$ is an iid sequence of positive random variables, and independent of the sequence of iid positive claim sizes $\{Y_i\}_{i=1}^\infty$ with Y_i the size of the ith claim. However, such independence assumption between $\{V_i\}_{i=1}^\infty$ and $\{Y_i\}_{i=1}^\infty$ can be relaxed. More specifically, we suppose that $\{(V_i, Y_i)\}_{i=1}^\infty$ forms a sequence of iid bivariate random vectors distributed as a generic pair (V, Y). In particular, the first pair (V_1, Y_1) has a different joint distribution from the other pairs (V_i, Y_i) for $i = 2, 3, 4, \ldots$. Thus, for $y > 0$, let the conditional distribution of Y given V be $P_t(y) = \Pr(Y \le y \,|\, V = t) = 1 - \bar{P}_t(y)$ with conditional density $p_t(y) = P_t'(y)$. Then the joint density of (V, Y) at (t, y) is given by $p_t(y)k(t)$. And the conditional distribution of $Y_1 | V_1$ is assumed to be $P_{1,t}(y) = 1 - \bar{P}_{1,t}(y)$ and conditional density be $p_{1,t}(y) = P_{1,t}'(y)$. Premiums are paid continuously of rate c which is assumed to satisfy the positive security loading condition $E[cV - Y] > 0$. We shall call such a model a dependent delayed renewal risk model, and for the remainder of the paper, the word 'dependent' is sometimes omitted for brevity. Clearly, when $P_t(y)$ and $P_{1,t}(y)$ does not depend on t, then the dependent renewal risk model reduces to the traditional delayed renewal risk model.

Under the above-mentioned dependent delayed model (or simply delayed model), we are interested in some generalizations of the classical Gerber-Shiu function by incorporating additional variables into the penalty function as in Cheung et al. (2010b) for the dependent ordinary model. To begin, let T_d be the time to ruin defined by $T_d = \inf\{t \ge 0 : U_t < 0\}$ with $T_d = \infty$ if $U_t \ge 0$ for all $t \ge 0$, and $\delta \ge 0$ is a discount factor. The classical Gerber-Shiu discounted penalty function in the delayed model is defined by (e.g. Gerber and Shiu (1998), Willmot (2004))

$$m_{d,\delta,12}(u) = E\left[e^{-\delta T_d} w_{12}\left(U_{T_d^-}, \left|U_{T_d}\right|\right) \mathrm{I}\left(T_d < \infty\right) \middle| U_0 = u\right], \tag{1}$$

where $w_{12}(x, y)$ is a nonnegative function (so-called penalty function) for $x > 0$, $y > 0$, $U_{T_d^-}$ is the surplus prior to ruin, U_{T_d} is the deficit at ruin, and $\mathrm{I}(.)$ is the indicator function. Let us define the minimum surplus before ruin occurs as $X_{T_d} = \inf_{0 \le s < T} U_s$, and the surplus immediately after the second last claim before ruin as $R_{N_{T_d}-1} = u + \sum_{i=1}^{N_{T_d}-1} (cV_i - Y_i)$ to be occurs if $N_{T_d} > 1$ and $R_0 = u$ if $N_{T_d} = 1$. Corresponding to Equations 2 and 3 in Cheung et al. (2010b), including these two variables in (1) respectively results in the generalized penalty functions for the current model given by

$$m_{d,\delta}^*(u) = E\left[e^{-\delta T_d} w^*\left(U_{T_d^-}, \left|U_{T_d}\right|, X_{T_d}, R_{N_{T_d}-1}\right) \mathrm{I}\left(T_d < \infty\right) \middle| U_0 = u\right], \tag{2}$$

and

$$m_{d,\delta}(u) = E\Big[e^{-\delta T_d} w\Big(U_{T_d^-}, \big|U_{T_d}\big|, R_{N_{T_d}-1}\Big) \mathrm{I}\big(T_d < \infty\big) \,\big|\, U_0 = u\Big]. \tag{3}$$

Using (2) and (3) we can analyze the last ladder height before ruin $Y_{N_{T_d}} = X_{T_d} + |U_{T_d}|$ and the last interclaim time $V_{N_{T_d}} = (U_{T_d^-} - R_{N_{T_d}-1})/c$. See Cheung et al. (2010b) for the ordinary model. If $w^* \equiv 1$ or $w \equiv 1$ in (2) or (3) respectively, the Gerber-Shiu funtion is reduced to

$$\bar{G}_{d,\delta}(u) = E\Big[e^{-\delta T_d}\, \mathrm{I}(T_d < \infty) \,|\, U_0 = u\Big], \tag{4}$$

and again (4) with $\delta = 0$ is equivalent to the ruin probability in the delayed model denoted by $\psi^d(u) = \Pr(T_d < \infty \,|\, U_0 = u) = \bar{G}_{d,0}(u)$. In what follows, the notations without superscript or subscript 'd' indicate the same quantities but defined in the ordinary renewal risk model.

This paper is organized as follows. In Section 2, it is demonstrated that the Gerber-Shiu functions in (2), (3) and (4) may be expressed in terms of the same quantities in the ordinary renewal risk model. Given these results, in Section 3, the discounted joint densities of four variables in the penalty function defined in (2) are derived using the results in the ordinary risk model. Interestingly, examination of the discounted joint densities of the three variables except for X_{T_d} from the previous one with $U_0 = 0$ is sufficient to obtain any other quantities of interest involving those four variables. Therefore, the general form of these joint densities are studied subsequently. In Section 4, we consider some examples assuming specific claim sizes. For the case of time-dependent claims we assume earthquake insurance and compare the last ladder height under the present model to the ordinary renewal risk model. In addition, we also consider the usual delayed model with time-independent claim sizes including exponentially distributed claim sizes with arbitrary interclaim times. Finally, some asymptotic results with regard to (4) are the subject matter of Section 5.

2. General structures

To begin the analysis, we first define the joint density of the time of ruin (t), the surplus prior to ruin (x), the deficit at ruin (y), and the surplus immediately after the second last claim before ruin occurs (v) in the delayed model, given $U_0 = u$. If ruin occurs on the first claim, then the surplus (x) and the time (t) are related by $x = u + ct$, or equivalently $t = (x - u)/c$. Therefore, the joint defective pdf of the surplus (x) and the deficit (y) is given by

$$h_1^d(x, y \,|\, u) = \frac{1}{c} k_1\Big(\frac{x-u}{c}\Big)\, p_{1,\frac{x-u}{c}}(x+y), \quad x > u, y > 0, \tag{5}$$

and in this case $R_{N_{T_d}-1}$ equals u. If ruin occurs on the second or subsequent claims, there is no such linear relationship between the time of ruin and the surplus prior to ruin, and we simply let $h_2^d(t,x,y,v\,|\,u)$ be the joint defective pdf of $(T_d, U_{T_d^-}, |U_{T_d}|, R_{N_{T_d}-1})$ for ruin on subsequent claims. From Cheung et al. (2010b), these joint defective densities in the ordinary renewal risk model with dependent structure are respectively defined by $h_1(x,y\,|\,u) = c^{-1}k(\frac{x-u}{c})\, p_{(x-u)/c}(x+y)$ for $x > u, y > 0$, and $h_2(t,x,y,v\,|\,u)$. Also, "discounted" joint densities of h_1^d and h_2^d are respectively defined as

$$h_{1,\delta}^d(x,y\,|\,u) = e^{-\frac{\delta(x-u)}{c}} h_1^d(x,y\,|\,u), \tag{6}$$

and

$$h_{2,\delta}^d(x,y,v\,|\,u) = \int_0^\infty e^{-\delta t} h_2^d(t,x,y,v\,|\,u)\,dt. \tag{7}$$

Then the discounted joint density of the surplus and the deficit is given by

$$h_\delta^d(x,y\,|\,u) = h_{1,\delta}^d(x,y\,|\,u) + \int_0^x h_{2,\delta}^d(x,y,v\,|\,u)\,dv. \tag{8}$$

We now employ the arguments of Gerber and Shiu (1998) to obtain an expression for $m_{d,\delta}^*(u)$ in (2) as below (e.g. Gerber and Shiu (1998, 2005), Li and Garrido (2005), Kim (2007), Kim and Willmot (2010), Willmot (2007)).

Proposition 1. *In the delayed renewal risk model with time-dependent claim sizes, the Gerber-Shiu function $m_{d,\delta}^*(u)$ defined by (2) may be expressed as*

$$m_{d,\delta}^*(u) = \phi_{d,\delta}\int_0^u m_\delta^*(u-y) f_{d,\delta}(y)\,dy + v_{d,\delta}^*(u), \tag{9}$$

where

$$\begin{aligned} v_{d,\delta}^*(u) = \int_u^\infty \int_0^\infty \Big\{ & w^*(x+u, y-u, u, u)\, h_{1,\delta}^d(x,y\,|\,0) \\ & + \int_0^x w^*(x+u, y-u, u, v+u)\, h_{2,\delta}^d(x,y,v\,|\,0)\,dv \Big\} dx\,dy \end{aligned} \tag{10}$$

interpreting as the contribution due to ruin on the first drop.

Proof: By conditioning on the first drop in surplus below u, we get the following equation for $m_{d,\delta}^*(u)$

$$\begin{aligned} m_{d,\delta}^*(u) = \int_0^u m_\delta^*(u-y) \Big\{ & \int_0^\infty h_{1,\delta}^d(x,y\,|\,0)\,dx \\ & + \int_0^\infty \int_0^x h_{2,\delta}^d(x,y,v\,|\,0)\,dv\,dx \Big\} dy + v_{d,\delta}^*(u). \end{aligned} \tag{11}$$

Let us define

$$\phi_{d,\delta} = \int_0^\infty \int_0^\infty h_\delta^d(x,y|0)\,dxdy, \tag{12}$$

and

$$f_{d,\delta}(y) = \frac{1}{\phi_{d,\delta}} \int_0^\infty h_\delta^d(x,y|0)\,dx, \tag{13}$$

which allows $m^*_{d,\delta}(u)$ in (11) to be expressed as (9). □

As is evident in what follows, the expression (9) is simplified in some special cases. See Cheung et al. (2010b) in the ordinary Sparre Andersen models with dependency. If $w^*(x,y,z,v) = w(x,y,v)$, (9) becomes

$$m_{d,\delta}(u) = \phi_{d,\delta} \int_0^u m_\delta(u-y) f_{d,\delta}(y)\,dy + v_{d,\delta}(u), \tag{14}$$

where

$$\begin{aligned} v_{d,\delta}(u) = \int_u^\infty \int_0^\infty \Big\{ & w(x+u, y-u, u) h^d_{1,\delta}(x,y|0) \\ & + \int_0^x w(x+u, y-u, v+u) h^d_{2,\delta}(x,y,v|0)\,dv \Big\} dxdy. \end{aligned} \tag{15}$$

Also, if $w^*(x,y,z,v) = w_{23}(y,z)$, then from (8), (12) and (13), (9) and (10) respectively simplify to the form which is only dependent on the ladder height density, and therefore the distribution of the last ladder height $X_{T_d} + |U_{T_d}|$ is obtainable from the generic ladder height distribution (e.g. Cheung et al. (2010b, Equation 31)). Further, if $w^*(x,y,z,v) = 1$, (4) satisfies

$$\bar{G}_{d,\delta}(u) = \phi_{d,\delta} \int_0^u \bar{G}(u-y) f_{d,\delta}(y)\,dy + \phi_{d,\delta} \bar{F}_{d,\delta}(u), \tag{16}$$

where $\bar{F}_{d,\delta}(u) = \int_u^\infty f_{d,\delta}(y)\,dy$.

3. Associated defective densities

In this section, we study, using the integral relationship result of $m^*_{d,\delta}(u)$ given by (9), the discounted joint densities of various variables in the penalty function. We begin with a discussion of the discounted joint density of $(U_{T_d^-}, |U_{T_d}|, X_{T_d}, R_{N_{T_d}-1})$.

Corollary 1. *In the delayed renewal risk model, the discounted joint density of* $(U_{T_d^-}, |U_{T_d}|, X_{T_d}, R_{N_{T_d}-1})$ *at* (x,y,z,v) *is defined as follows:*

(1) If ruin occurs on the first drop caused by

(a) the first claim:
$h^d_{1,\delta}(x-u, y+u|0)$ *for* $x > u,\ y > 0,\ z = u, v = u$, *and*

(b) claims other than the first:
$h^d_{2,\delta}(x-u, y+u, v-u|0)$ *for* $x > u, y > 0, z = u, u < v < x$.

(2) If ruin occurs on the second drop caused by

(a) the next claim after the first drop:
$\phi_{d,\delta} f_{d,\delta}(u-z) h_{1,\delta}(x-z, y+z|0)$ *for* $x > z, y > 0, 0 < z < u, v = z$, *and*

(b) subsequent claims after the first drop:
$\phi_{d,\delta}\ f_{d,\delta}(u-z) h_{2,\delta}(x-z,\ y+z,\ v-z|0)$ *for* $x > z,\ y > 0,\ 0 < z < u,\ z < v < x$.

(3) If ruin occurs on drops (other than the first two drops) caused by

(a) the next claim after the drop:
$\{\int_z^u \phi_{d,\delta}\ f_{d,\delta}(u-l) g_\delta(l-z)/(1-\phi_\delta)\, dl\}\ h_{1,\delta}(x-z,\ y+z|0)$ *for* $x > z,\ y > 0,\ 0 < z < u,\ v = z$, *and*

(b) subsequent claims after the drop:
$\{\int_z^u \phi_{d,\delta} f_{d,\delta}(u-l) g_\delta(l-z)/(1-\phi_\delta)\, dl\} h_{2,\delta}(x-z, y+z, v-z|0)$ *for* $x > z,\ y > 0,\ 0 < z < u,\ z < v < x$.

Proof: First, with a choice of $w^*(x, y, z, v) = e^{-s_1 x - s_2 y - s_3 z - s_4 v}$ as in (2), from (9) and (10) the Gerber-Shiu function satisfies

$$m^*_{d,\delta}(u) = \phi_{d,\delta} \int_0^u m^*_d(u-y) f_{d,\delta}(y)\, dy + e^{-s_3 u}\, v_{d,\delta}(u), \tag{17}$$

where $v_{d,\delta}(u)$ from (15) is given by

$$\begin{aligned} v_{d,\delta}(u) = \int_0^\infty \int_u^\infty e^{-s_1 x - s_2 y} \Big\{ & e^{-s_4 u} h^d_{1,\delta}(x-u, y+u|0) \\ & + \int_u^x e^{-s_4 u} h^d_{2,\delta}(x-u, y+u, v-u|0)\, dv \Big\} dx\, dy. \end{aligned} \tag{18}$$

Using the expression for $m^*_\delta(u)$ given by Cheung et al. (2010b, Section 3) leads the integral on the right-hand side in (17) to

$$\phi_{d,\delta} \int_0^u m^*_\delta(u-y) f_{d,\delta}(y)\, dy = \int_0^u \Big[\int_0^\infty \int_l^\infty e^{-s_1 x - s_2 y - s_3 l - s_4 l} h_{1,\delta}(x-l, y+l|0)\, dx\, dy$$

$$+ \int_0^\infty \int_l^\infty \int_l^x e^{-s_1 x - s_2 y - s_3 l - s_4 v} h_{2,\delta}(x-l, y+l, v-l|0)\, dv\, dx\, dy$$

$$+ \int_0^l \int_0^\infty \int_z^\infty e^{-s_1 x - s_2 y - s_3 z - s_4 z} \Big\{ h_{1,\delta}(x-z, y+z|0) \frac{g_\delta(l-z)}{1-\phi_\delta} \Big\} dx\, dy\, dz$$

$$+\int_0^l\int_0^\infty\int_z^\infty\int_z^x e^{-s_1x-s_2y-s_3z-s_4v}\left\{h_{2,\delta}(x-z,y+z,v-z\,|\,0)\,\frac{g_\delta(l-z)}{1-\phi_\delta}\right\}dvdxdydz\Bigg]$$
$$\phi_{d,\delta}f_{d,\delta}(u-l)\,dl. \tag{19}$$

Combining the above and (18) with a multiplication of e^{-s_3u} yields the Laplace-Stieltjes transform of $(T_d, U_{T_d^-}, |U_{T_d}|, X_{T_d}, R_{N_{T_d}-1})$. With an interchange of the order of integration followed by Laplace Stieltjes transform inversion with respect to (s_1, s_2, s_3, s_4), Corollary 1 is proved. We distinguish between the three cases according to the number of drops causing ruin. If ruin occurs on the first drop in surplus below an initial level u, then there are two possibilities; ruin occurs on the first claim or the subsequent claims. The second term on the right-hand side in (17), namely $e^{-s_3u}v_{d,\delta}(u)$ represents these two cases. Hence from (15) with $w(x,y,v) = e^{-s_1x-s_2y-s_4v}$, it follows that 1(a) and 1(b) are obtained respectively. If ruin occurs not on the first drop, then these cases are explained by the integral terms on the right-hand side in (17). Thus from (19), we can obtain four different situations corresponding to ruin on the drop (second or subsequent to this) caused by the (next or not next) claim after the drop. And the joint densities in these four cases are given by 2(a), 2(b) and 3(a), 3(b) respectively. See Figure 1 for graphs depicting the six different cases contributing to this discounted joint densities. □

Note that probabilistic interpretations for the above cases are also available. For example, in cases 3(a) and 3(b), $\phi_{d,\delta}f_{d,\delta}(u-l)$ appears in common which can be interpreted as the size of the first drop being $(u-l)$ not causing ruin. After this first drop, the surplus process is same as the ordinary process with an initial surplus l. This is followed by an arbitrary number of drops (≥ 1) which brings the surplus process from l to z, as explained by the term $g_\delta(l-z)/(1-\phi_\delta)$. Here, l is arbitrary for $z < l < u$ and with a level of surplus z, ruin immediately occurs on the next claim represented by $h_{1,\delta}$ for 3(a) or on the subsequent claim represented by $h_{2,\delta}$ for 3(b).

Furthermore, we know that $\phi_{d,\delta}$ in (12) and $f_{d,\delta}(y)$ in (13) can be obtained by $h^d_{2,\delta}(x,y,v\,|\,0)$ since $h^d_{1,\delta}(x,y\,|\,u)$ is readily known by using (5) and (6). Therefore, from Corollary 1, note that $h^d_{2,\delta}(x,y,v\,|\,0)$ is sufficient to obtain the joint densities of four variables in the penalty function under the delayed risk model as in the ordinary risk model (see Cheung et al. (2010b)). Thus, we derive this discounted joint density in the following corollary.

Corollary 2. *In the delayed renewal risk model, the discounted joint density of* $(U_{T_d^-}, |U_{T_d}|, R_{N_{T_d}-1})$ *at* (x,y,v) *is defined as:*

$$h^d_{2,\delta}(x,y,v\,|\,u) = h_{1,\delta}(x,y\,|\,v)\,\xi_\delta(u,v), \quad 0 < v < x,\ y > 0, \tag{20}$$

where

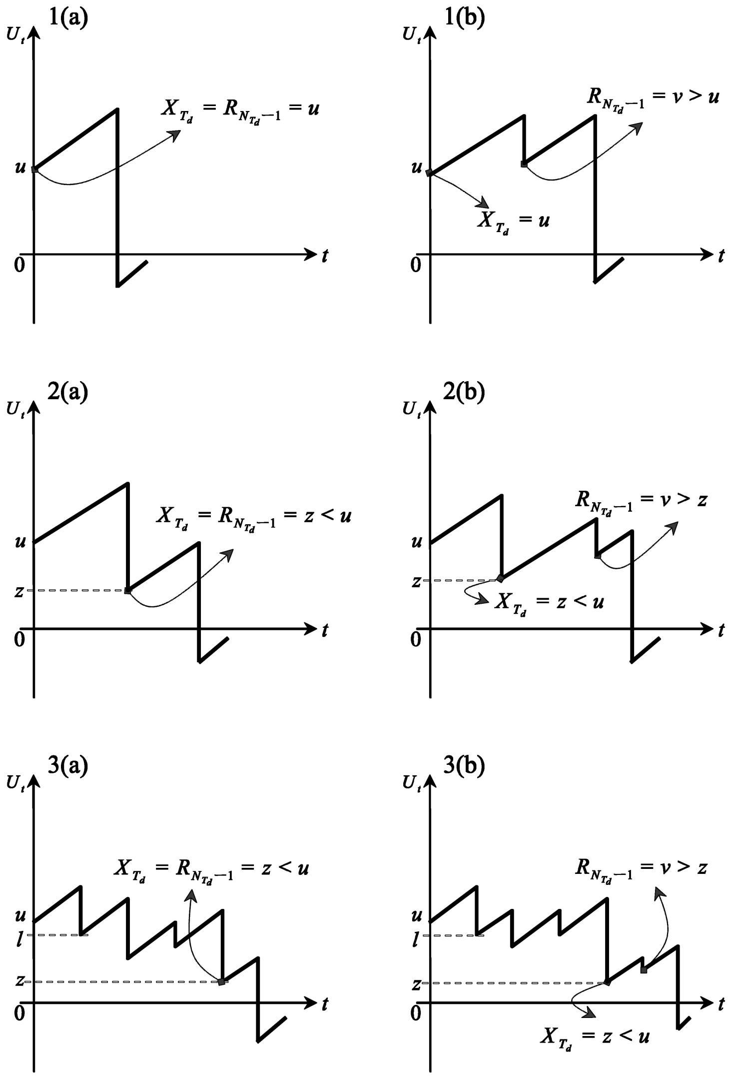

FIGURE 1.

$$\xi_\delta(u,v) = A_\delta(u,v) + \int_0^\infty A_\delta(u,z)\,\tau_\delta(z,v)\,dz, \tag{21}$$

and

$$A_\delta(u,z) = \begin{cases} \int_0^\infty e^{-\delta t} p_{1,t}(u+ct-z)\,dK_1(t), & 0 < z < u \\ \int_{(z-u)/c}^\infty e^{-\delta t} p_{1,t}(u+ct-z)\,dK_1(t), & z > u \end{cases}. \tag{22}$$

Proof: By conditioning on the time and the amount of the first claim in order to identify the components in (14), we have

$$m_{d,\delta}(u) = \beta_{d,\delta}(u) + \int_0^\infty e^{-\delta t} \sigma_{\delta,t}(u+ct)\, dK_1(t), \tag{23}$$

where $\sigma_{\delta,t}(x) = \int_0^x m_\delta(x-y)\, dP_{1,t}(y)$, and

$$\beta_{d,\delta}(u) = \int_0^\infty e^{-\delta t} \int_{u+ct}^\infty w(u+ct, y-u-ct, u)\, dP_{1,t}(y)\, dK_1(t). \tag{24}$$

In other words, by using (6), (24) may be rewritten as

$$\beta_{d,\delta}(u) = \int_u^\infty \int_0^\infty e^{-\delta\left(\frac{x-u}{c}\right)} w(x,y,u)\, h_1^d(x,y \,|\, u)\, dy dx, \tag{25}$$

or equivalently $\beta_{d,\delta}(u) = \int_u^\infty \int_0^\infty w(x,y,u)\, h_{1,\delta}^d(x,y \,|\, u)\, dy dx$. Note that $\beta_{d,\delta}(u)$ may be interpreted as the contribution to the penalty function due to ruin on the first claim. Since $m_{d,\delta}(u)$ in (3) is an expectation, it follows directly that it may be represented as

$$\begin{aligned} m_{d,\delta}(u) &= \int_0^\infty \int_u^\infty w(x,y,u)\, h_{1,\delta}^d(x,y \,|\, u)\, dx dy \\ &\quad + \int_0^\infty \int_0^\infty \int_0^x w(x,y,v)\, h_{2,\delta}^d(x,y,v \,|\, u)\, dv dy dx. \end{aligned} \tag{26}$$

then using (25) it may be reexpressed as

$$m_{d,\delta}(u) = \beta_{d,\delta}(u) + \int_0^\infty \int_0^\infty \int_0^x w(x,y,v)\, h_{2,\delta}^d(x,y,v \,|\, u)\, dv dy dx. \tag{27}$$

Then, comparing (23) and (27) followed by a change of integration leads us to

$$\begin{aligned} &\int_0^\infty \int_0^\infty \int_0^x w(x,y,v)\, h_{2,\delta}^d(x,y,v \,|\, u)\, dv dx dy = \int_0^\infty e^{-\delta t} \sigma_{\delta,t}(u+ct)\, dK_1(t) \\ &= \int_0^\infty e^{-\delta t} \left\{ \int_0^{u+ct} m_\delta(z)\, p_{1,t}(u+ct-z)\, dz \right\} dK_1(t) = \int_0^\infty m_\delta(z)\, A_\delta(u,z)\, dz, \end{aligned} \tag{28}$$

where $A_\delta(u,z)$ given by (22). Similar to (26), $m_\delta(u)$ is also be expressed in terms of the joint defective densities and thus we get

$$\begin{aligned} \int_0^\infty m_\delta(z)\, A_\delta(u,z)\, dz &= \int_0^\infty \Big\{ \int_0^\infty \int_z^\infty w(x,y,z)\, h_{1,\delta}(x,y \,|\, z)\, dx dy \\ &\quad + \int_0^\infty \int_0^\infty \int_0^x w(x,y,v)\, h_{2,\delta}(x,y,v \,|\, z)\, dv dx dy \Big\} A_\delta(u,z)\, dz. \end{aligned}$$

When $w(x,y,v) = e^{-s_1x-s_2y-s_3v}$ on the left-hand side of (28) and on the above equation, equating coefficients of $e^{-s_1x-s_2y-s_3v}$ results in

$$\begin{aligned} h^d_{2,\delta}(x,y,v\,|\,u) = {} & h_{1,\delta}(x,y\,|\,v)A_\delta(u,v) \\ & + \int_0^\infty h_{2,\delta}(x,y,v\,|\,z)A_\delta(u,z)dz, \quad 0 < v < x,\ y > 0. \end{aligned} \tag{29}$$

But $h_{2,\delta}(x,y,v|z) = h_{1,\delta}(x,y\,|\,v)\tau_\delta(z,v)$ for $0 < v < x$ (explicit forms for $\tau_\delta(z,v)$ under certain assumptions on the interclaim time and its probabilistic interpretations are provided by Cheung et al. (2010a), and Willmot and Woo (2010)), we may express (29) as (20). □

As with $\tau_\delta(u,z)$, the function $\xi_\delta(u,v)$ in (21) can also be probabilistically interpreted in the following manner. If $R_{N_{T_d}-1} = v$, the delayed process starting with an initial level u should reach the surplus level v, just after the second last claim before ruin. This transition from u to v in the current process is represented by the function $\xi_\delta(u,v)$ as seen from (20). However, since the first pair (V_1, Y_1) is assumed different from the other pairs, $\xi_\delta(u,v)$ may also be obtained by conditioning on the time and amount of the first claim which is expressible in terms of $A_\delta(u,z)$ in (22). By the definition of $h^d_{2,\delta}$, ruin occurs at $N_{T_d} \geq 2$ and thus if $N_{T_d} = 2$ then the process would be at level v after the first claim explaining the term $A_\delta(u,v)$. Otherwise, for $N_{T_d} > 2$, the process would be at some arbitrary level z after the first claim and then moves from z to v like in the ordinary process with $A_\delta(u,z)\tau_\delta(z,v)$.

Moreover, using Corollary 2 results in an alternative representation for $m_{d,\delta}(u)$ as follows.

Corollary 3. *In the delayed renewal risk model, the Gerber-Shiu function $m_{d,\delta}(u)$ defined by (3) satisfies*

$$m_{d,\delta}(u) = \beta_{d,\delta}(u) + \int_0^\infty \beta_\delta(v)\xi_\delta(u,v)dv, \tag{30}$$

where $\beta_{d,\delta}(u)$ given by (25), $\beta_\delta(u)$ is defined as $\beta_{d,\delta}(u)$ but with $h^d_1(x,y\,|\,u)$ replaced by $h_1(x,y\,|\,u)$, and $\xi_\delta(u,v)$ given by (21).

Proof: Substitution of (20) into (27) directly yields the above result. □

We point out that Corollary 3 also makes sense intuitively based on the numbers of the claims which causes ruin. If ruin occurs on the first claim with an initial level u, this case may be represented by $\beta_{d,\delta}(u)$. Or if the process first moves from u to v after an arbitrary number of claims (≥ 1) followed by ruin on the subsequent claim from an initial level v, this case may be represented by $\xi_\delta(u,v)\beta_\delta(v)$. In particular, for the ordinary model we know that there is no difference between $\xi_\delta(u,v)$ and $\tau_\delta(u,v)$ while $\beta_{d,\delta}(u)$ is equivalent to $\beta_\delta(u)$, so that (30) is reduced to $m_\delta(u) = \beta_\delta(u) + \int_0^\infty \beta_\delta(v)\, \tau_\delta(u,v)\, dv$ (e.g. Cheung et al. (2010a) for the classical Poisson risk model, Willmot and Woo (2010) for the traditional ordinary renewal risk model with a Coxian class of the interclaim time distributions).

In particular, we may readily find some ruin related quantities with appropriate choices of the penalty functions in (30) since only $\beta_{d,\delta}(u)$ and $\beta_\delta(u)$ contain the penalty function. For example, if $w(x,y,v) = e^{-s(x-v)/c}$, we have the Laplace transform of $V_{N_{T_d}}$ given by $m_{d,0}(u)$ in (3). In this case, with (24) and $\beta_0(u) = \int_0^\infty e^{-st}\, k(t)\bar{P}_t(u+ct)dt$, inverting with respect to s followed by dividing by the ruin probability $\psi^d(u)$ yields the proper density of $V_{N_{T_d}} \,|T_d < \infty$ (denoted by h_V^d) given by

$$h_V^d(t|u) = a_{1,u}(t)k_1(t) + a_{2,u}(t)k(t), \quad t > 0,$$

where $a_{1,u}(t) = \bar{P}_{1,t}(u+ct)/\psi^d(u)$ and $a_{2,u}(t) = \{\int_0^\infty \xi_\delta(u,v)\bar{P}_t(v+ct)dv\}/\psi^d(u)$.

In addition, we may obtain bounds for the last interclaim time when $P_{1,t}(y) = P_t(y) = P(y)$ as follows. First, define $\bar{H}_V^d(t|u) = \int_t^\infty h_V^d(y|u)dy$ and introduce two reliability classes, a new worse (better) than used or NWU (NBU) (i.e. $\bar{K}_1(x+y) \geq (\leq)\, \bar{K}_1(x)\, \bar{K}_1(y)$ for $x, y \geq 0$). See Barlow and Proschan (1981). From Cheung et al. (2010, Theorem 7), if $K_1(t)$ is NWU (NBU), $\bar{K}_1(t) \geq (\leq)\, \bar{K}(t)$ for $t > 0$, and there exists a function $\bar{F}(y)$ on $[0,\infty)$ such that $\bar{P}(x+y) \leq (\geq)\, \bar{P}(x)\, \bar{F}(y)$ for $x, y \geq 0$, then the survival function of $V_{N_{T_d}} \,|T_d < \infty$ satisfies $\bar{H}_V^d(t|u) \leq (\geq)\, \bar{F}(ct)\bar{K}_1(t)$. Depending on the properties of $P(y)$, Cheung et al. (2010) provided three possible choices of $\bar{F}(y)$.

We next turn our attention to the last ladder height $Y_{N_{T_d}} = X_{T_d} + \left|U_{T_d}\right|$. As mentioned previously, if $w^*(x,y,z,v) = w_{23}(y,z) = e^{-s(y+z)}$ in (9) and (10), with the aid of the Laplace transform of the last ladder height in the ordinary model given by Cheung et al. (2010b), inverting with respect to s yields the defective discounted density of $Y_{N_{T_d}}$ (denoted by $f_{d,\delta}(u,y)$) given by

$$f_{d,\delta}(u,y) = \begin{cases} \dfrac{\phi_\delta}{1-\phi_\delta}\left[\bar{G}_{d,\delta}(u-y) - \bar{G}_{d,\delta}(u)\right] f_\delta(y), & y < u \\ \dfrac{\phi_\delta}{1-\phi_\delta}\left[\phi_{d,\delta} - \bar{G}_{d,\delta}(u)\right] f_\delta(y) + \phi_{d,\delta} f_{d,\delta}(y), & y > u \end{cases}. \tag{31}$$

Then the proper survival function $Y_{N_{T_d}}$ given that ruin occurs denoted by $\bar{F}_{d,u}^*(y)$ can be obtained as $\int_y^\infty f_{d,0}(u,x)dx/\psi^d(u)$. Clearly, in the ordinary model (31) reduces to Equation 31 in Cheung et al. (2010b).

In the following section, we illustrate a numerical example in case of the time-dependent claims in the delayed model which contains a comparison of the last ladder height with the ordinary model. And the usual delayed model with the time-independent claims is also presented.

4. Examples

4.1. Time-dependent claims: Earthquake insurance

Let us consider the dependency model in Boudreault et al. (2006), namely the conditional density $p_t(y) = e^{-\beta t} f_1(y) + (1 - e^{-\beta t}) f_2(y)$ for $y > 0$ where f_1 and f_2 are proper densities. Suppose that $f_1(y) = 2.5e^{-2.5y}$, $f_2(y) = 0.5e^{-0.5y}$, $\beta = 1/3$, and $k(t) = te^{-t}$ (i.e. Erlang (2) interclaim times) with $c = 2$ and $\delta = 0$. In this example, if the interclaim time t is large then the time-dependent claim size distribution $p_t(y)$ is more likely to be determined by f_2 than f_1. Here, if the last earthquake before time 0 has occurred 5 years ago, we simply let $k_1(t) = k(t+5)/\bar{K}(5)$ be the residual lifetime distribution corresponding to $k(t)$ and $p_{1,t}(y) = p_{t+5}(y)$. Then from Woo (2010) ϕ_0, $\bar{F}_0(y)$ and $\psi(u)$ can be computed, and in turn an application of Equation 32 and Box I in Cheung et al. (2010b) gives $\bar{F}_u^*(y)$ (the proper survival function of the last ladder height in the ordinary model). For the present model, if $w(x, y, v) = w_2(y)$ and $u = 0$ in (23) and (24), we may obtain the defective density of the deficit as $h_0^d(y|0) = \frac{1}{c}\, k_1 \left(\frac{x-u}{c}\right) \bar{P}_{1,\frac{x-u}{c}}(y) + \int_0^\infty \int_0^{ct} h_0(y|z) p_{1,t}(ct - z) k_1(t)\, dz dt$ where $h_0(y|z)$ is the same as $h_0^d(y|u)$ but defined in the ordinary model. With this $h_0^d(y|0)$, from (12) and (13) we get $\phi_{d,0}$ and $\bar{F}_{d,0}(y)$, and hence $\psi^d(u)$ from (16). Then with the aid of (31) one ultimately finds $\bar{F}_{d,u}^*(y)$. When $u = 0.5$, the comparison of $\bar{F}_{d,u}^*(y)$ with $\bar{F}_u^*(y)$, and also with the generic ladder heights $\bar{F}_{d,0}(y)$ and $\bar{F}_0(y)$ is summarized in Figure 2. In the graph, ‘D’ and ‘O’ indicates the delayed model and the ordinary model respectively.

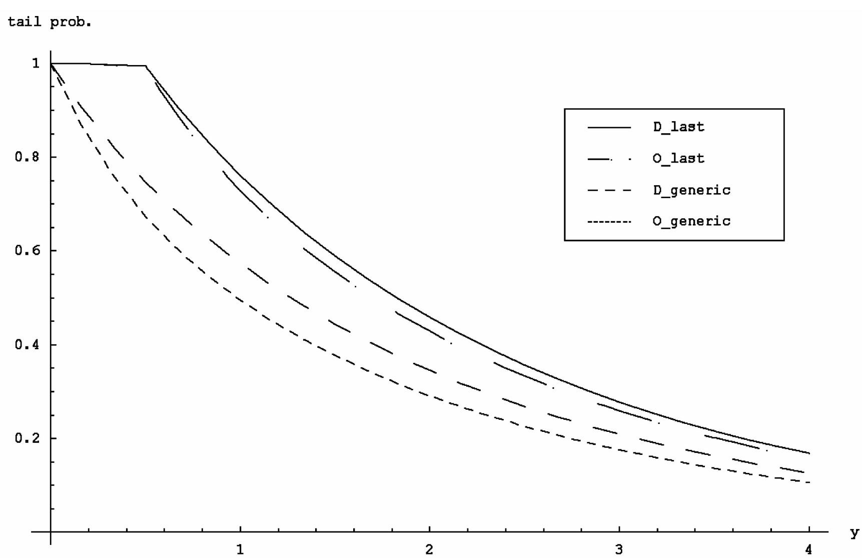

Figure 2.

From Figure 2, there is a distinctive difference among the four ladder heights of our interest. In particular, they can be ordered as $\bar{F}^*_{d,u}(y) \geq \bar{F}^*_u(y) \geq \bar{F}_{d,0}(y) \geq \bar{F}_0(y)$. We remark that the stochastic ordering $\bar{F}^*_u(y) \geq \bar{F}_0(y)$ has been proved by Cheung et al. (2010b) in the ordinary model. More interestingly, under this dependent structure, we may conclude that the insurer is more likely to face the larger severity (drop under the minimum surplus level) in the delayed model compared to the ordinary model. In other words, with the model having no adjustment for the pair of the first event (i.e. ordinary model), the insurer may suffer bigger loss than expected. In addition, we can also check that $\psi^d(u) \geq \psi(u)$ for $u \geq 0$, and the difference between these ruin probabilities may not be significant for a large u. The details are omitted here.

4.2. Time-independent claims

As in Cheung et al. (2010b, Section 5), the results in Section 2 and 3 may be simplified in case of the usual delayed Sparre Andersen model without dependency. The details are given in the Appendix. Here, we may obtain the joint Laplace transform of the five variables $(T_d, U_{T_d^-}, |U_{T_d}|, X_{T_d}, R_{N_{T_d}-1})$, namely $m^*_{d,\delta}(u)$ in (2) with a proper choice of the penalty function, if claim sizes are exponentially distributed. Under the ordinary renewal risk model, this quantity was studied by Cheung et al. (2010b, Section 5). By using the results therein, the joint Laplace transform of those five variables under the delayed renewal risk model is revisited. Suppose $p_1(y) = p(y) = \beta e^{-\beta y}$, with $w^*(x, y, z, v) = e^{-s_1 x - s_2 y - s_3 z - s_4 v}$, we may find

$$
\begin{aligned}
m^*_{d,\delta}(u) = {} & C_{d,\delta}(s_1, s_2, s_3, s_4)(s_1 + s_3 + s_4)\, e^{-(\beta + s_1 + s_3 + s_4)u} \\
& + C_\delta(s_1, s_2, s_3, s_4)\, \phi_{d,\delta}\, \beta e^{-\beta(1-\phi_\delta)u},
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
& C_{d,\delta}(s_1, s_2, s_3, s_4) \\
& = \frac{\beta\left[\phi_{d,\delta}\beta s_3 \tilde{k}(\delta + c\beta + cs_1) + (\phi_\delta \beta + s_1 + s_3 + s_4)\left\{(s_1 + s_4)\tilde{k}_1(\delta + c\beta + cs_1) + \beta k_\delta(s_1, s_4)\right\}\right]}{(\beta + s_2)(s_1 + s_3 + s_4)(\phi_\delta \beta + s_1 + s_3 + s_4)\left\{s_1 + s_4 + \beta \tilde{k}(\delta + c\beta + cs_1 + cs_4)\right\}}.
\end{aligned} \tag{33}
$$

See the Appendix for the details of deriving (32) and (33). For example, we may readily obtain the Laplace transform of $V_{N_{T_d}} = (U_{T_d^-} - R_{N_{T_d}-1})/c$ with the choice of $s_1 = s/c$, $s_4 = -s/c$, and $\delta = 0$ from (51) with (47), (52) and (58) as

$$
\begin{aligned}
& E[e^{-sV_{N_{T_d}}} \mathrm{I}(T_d < \infty) \mid U_0 = u] \\
& = \frac{\tilde{k}(c\beta + s)}{\tilde{k}(c\beta)} \phi_{d,0} e^{-\beta(1-\phi_0)u} + \frac{\tilde{k}_1(c\beta + s)\tilde{k}(c\beta) - \tilde{k}(c\beta + s)\tilde{k}_1(c\beta)}{\tilde{k}(c\beta)} e^{-\beta u},
\end{aligned} \tag{34}
$$

and inversion (34) with respect to s followed by dividing by $\psi^d(u) = \phi_{d,0}\, e^{-\beta(1-\phi_0)u}$ yields the proper density of $V_{N_{T_d}}$ in the delayed renewal risk model as a mixture of Esscher transformed distributions of $K_1(t)$ and $K(t)$, namely $h_V^d(t|u) = (1-a_u)k_e(t) + a_u k_{1,e}(t)$ with $a_u = \tilde{k}_1(c\beta)e^{-\beta u}/\psi^d(u)$, $k_e(t) = e^{-c\beta t}k(t)/\tilde{k}(c\beta)$, and $k_{1,e}(t) = e^{-c\beta t}k_1(t)/\tilde{k}_1(c\beta)$. Since $k_1(t) = k(t)$ and $\phi_{d,0} = \phi_0$ for the ordinary renewal risk model, the second term on the right-hand side of (34) is cancelled out and thus the result agrees with Cheung et al. (2010b).

5. Asymptotic results

In this section, we consider asymptotic results regarding the compound geometric tail in the delayed renewal process, consequently ruin probabilities are also obtained. First, suppose that $\kappa_\delta > 0$ is the adjustment coefficient satisfying $\int_0^\infty e^{\kappa_\delta y} f_\delta(y)\,dy = 1/\phi_\delta$, then we know that the asymptotic result for the compound geometric tail for the ordinary model is given by (e.g. Willmot and Lin (2001, p. 158))

$$\lim_{u\to\infty} e^{\kappa_\delta u}\bar{G}_\delta(u) = C_\delta$$

where $C_\delta = (1-\phi_\delta)\left[\phi_\delta \kappa_\delta \int_0^\infty y e^{\kappa_\delta y} dF_\delta(y)\right]^{-1}$, and that

$$\bar{G}_\delta(u) \le e^{-\kappa_\delta u}, \qquad u \ge 0 \tag{35}$$

by a Lundberg inequality. Here, suppose that $\tilde{p}_{1,t}(-\kappa_\delta) = \int_0^\infty e^{\kappa_\delta y} dP_{1,t}(y) < \infty$, implying that $\lim_{x\to\infty} e^{\kappa_\delta x}\bar{P}_{1,t}(x) = 0$. Also, as (35) holds, by dominated convergence it follows that

$$\begin{aligned}&\lim_{u\to\infty} e^{\kappa_\delta(u+ct)}\left\{\bar{P}_{1,t}(u+ct) + \int_0^{u+ct}\bar{G}_\delta(u+ct-y)\,dP_{1,t}(y)\right\}\\ &= \int_0^\infty \left\{\lim_{u\to\infty} e^{\kappa_\delta(u+ct-y)}\bar{G}_\delta(u+ct-y)\right\} e^{\kappa_\delta y} dP_{1,t}(y) = C_\delta\,\tilde{p}_{1,t}(-\kappa_\delta).\end{aligned}$$

Namely,

$$\lim_{u\to\infty} e^{\kappa_\delta u}\bar{W}_{\delta,t}(u) = C_\delta\,\tilde{p}_{1,t}(-\kappa_\delta), \tag{36}$$

where $\bar{W}_{\delta,t}(u) = \overline{G_\delta * P_{1,t}}(u) = \bar{P}_{1,t}(u) + \int_0^u \bar{G}_\delta(u-y)\,dP_{1,t}(y)$.

Now, from (23) with $w(x,y,v) = 1$, (4) has an integral expression as

$$\bar{G}_{d,\delta}(u) = \int_0^\infty e^{-\delta t}\,\bar{W}_{\delta,t}(u+ct)\,dK_1(t). \tag{37}$$

Since (36) holds which implies that $e^{\kappa_\delta u}\bar{W}_{\delta,t}(u)$ is a bounded function of u on $(0,\infty)$. Thus, again by dominated convergence one finds from (37)

$$\lim_{u\to\infty} e^{\kappa_\delta u}\bar{G}_{d,\delta}(u) = \lim_{u\to\infty}\int_0^\infty \left\{e^{\kappa_\delta(u+ct)}\bar{W}_{\delta,t}(u+ct)\right\}e^{-(\delta+c\kappa_\delta)t}dK_1(t)$$
$$= \int_0^\infty C_\delta \tilde{p}_{1,t}(-\kappa_\delta)e^{-(\delta+c\kappa_\delta)t}dK_1(t) = C_\delta E[e^{\kappa_\delta Y_1-(\delta+c\kappa_\delta)V_1}],$$

where $E\left[e^{\kappa_\delta Y_1-(\delta+c\kappa_\delta)V_1}\right] = \int_0^\infty e^{-(\delta+c\kappa_\delta)t}\tilde{p}_{1,t}(-\kappa_\delta)dK_1(t)$.

Hence,

$$\bar{G}_{d,\delta}(u) \sim C_\delta E\left[e^{\kappa_\delta Y_1-(\delta+c\kappa_\delta)V_1}\right]e^{-\kappa_\delta u}, \quad u\to\infty. \tag{38}$$

In particular, for interclaim-independent claim sizes, i.e. $p_{1,t}(y)=p(y)$, we know that

$$E\left[e^{\kappa_\delta Y_1-(\delta+c\kappa_\delta)V_1}\right] = \tilde{p}(-\kappa_\delta)\tilde{k}_1(\delta+c\kappa_\delta) = \frac{\tilde{k}_1(\delta+c\kappa_\delta)}{\tilde{k}(\delta+c\kappa_\delta)}$$

since κ_δ satisfies $\tilde{p}(-\kappa_\delta)\,\tilde{k}(\delta+c\kappa_\delta)=1$ (see Cheung et al. (2010b, Section 4)). Therefore, in this case (38) reduces to

$$\bar{G}_{d,\delta}(u) \sim \frac{C_\delta \tilde{k}_1(\delta+c\kappa_\delta)}{\tilde{k}(\delta+c\kappa_\delta)}e^{-\kappa_\delta u}, \quad u\to\infty.$$

Further for $\delta=0$ we know that $\bar{G}_{d,0}(u)=\psi^d(u)$, and the asymptotic result for $\psi^d(u)$ from the above agrees with Theorem 11.4.3 in Willmot and Lin (2001).

Alternatively, we may directly obtain the asymptotic form in (36) by using the result for the tail of a compound geometric convolution $\bar{W}_{\delta,t}(u)$ which satisfies the defective renewal equation (see Willmot and Cai (2004) and references therein)

$$\bar{W}_{\delta,t}(x) = \phi_\delta\int_0^x \bar{W}_{\delta,t}(x-y)\,dF_\delta(y) + \phi_\delta\bar{F}_\delta(x) + (1-\phi_\delta)\bar{P}_{1,t}(x).$$

It is shown

$$\lim_{u\to\infty} e^{\kappa_\delta u}\bar{W}_{\delta,t}(u) = \frac{(1-\phi_\delta)\int_0^\infty e^{\kappa_\delta y}dP_{1,t}(y)}{\phi_\delta\kappa_\delta\int_0^\infty y e^{\kappa_\delta y}dF_\delta(y)} = C_\delta\tilde{p}_{1,t}(-\kappa_\delta). \tag{39}$$

Furthermore, from (16) it is clear that $\bar{G}_{d,\delta}(u)/\phi_{d,\delta}$ is also the tail of a compound geometric convolution, and thus if $\tilde{f}_{d,\delta}(-\kappa_\delta) = \int_0^\infty e^{\kappa_\delta y}dF_{d,\delta}(y) < \infty$, the same argument used to drive (39) results in

$$\bar{G}_{d,\delta}(u) \sim C_\delta \, \phi_{d,\delta} \, \tilde{f}_{d,\delta}(-\kappa_\delta) e^{-\kappa_\delta u}, \quad u \to \infty. \tag{40}$$

Curiously, comparison of (38) with (40) results in the identity

$$\phi_{d,\delta} \tilde{f}_{d,\delta}(-\kappa_\delta) = E\left[e^{\kappa_\delta Y_1 - (\delta + c\kappa_\delta) V_1}\right], \tag{41}$$

and obviously both sides of (41) equal 1 in the nondelayed case.

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APPENDIX

Here we demonstrate how to obtain $m^*_{d,\delta}(u)$ in (2) in case of interclaim-independent claim sizes. Suppose that $p_{1,t}(y) = p_1(y)$, $\bar{P}_{1,t}(y) = \bar{P}_1(y)$, and $p_t(y) = p(y)$, $\bar{P}_t(y) = \bar{P}(y)$. Then as in Gerber and Shiu (1998), the conditional density of $|U_{T_d}|$ given $T_d = t$, $U_{T_d^-} = x$, $R_{N_{T_d}-1} = v$ for $N_{T_d} \geq 2$ is given by $p(x+y)/\bar{P}(x)$. By using this, one finds that the joint defective density of $(T_d, U_{T_d^-}, |U_{T_d}|, R_{N_{T_d}-1})$ ad its discounted density as in Equation 53 and Equation 54 in Cheung et al. (2010b) (for the ordinary model). Thus in this case (7) may be expressed as $\{p(x+y)/\bar{P}(x)\}\, h^d_{(2),\delta}(x, v\,|\,u)$ where $h^d_{(2),\delta}(x, v\,|\,u)$ is the discounted defective density of $(U_{T_d^-}, R_{N_{T_d}-1})$ for $N_{T_d} \geq 2$. Then, by substituting this expression for $h^d_{2,\delta}(x, y, v\,|\,u)$ into the integral on the right-hand side of (8) one may write

$$h^d_\delta(x,y\,|\,u) = \frac{p_1(x+y)}{\bar{P}_1(x)}\, h^d_{1,\delta}(x\,|\,u) + \frac{p(x+y)}{\bar{P}_1(x)}\, h^d_{2,\delta}(x\,|\,u), \tag{42}$$

where

$$h^d_{1,\delta}(x\,|\,u) = \frac{1}{c}\, e^{-\delta(\frac{x-u}{c})} k_1\left(\frac{x-u}{c}\right) \bar{P}_1(x) \tag{43}$$

and $h^d_{2,\delta}(x\,|\,u) = \int_0^x h^d_{(2),\delta}(x, v\,|\,u)\, dv.$

Therefore, using (42), $\phi_{d,\delta}$ in (12) may be expressed as

$$\phi_{d,\delta} = \int_0^\infty \int_0^\infty h^d_\delta(x,y\,|\,0)\, dy dx = \int_0^\infty h^d_\delta(x\,|\,0)\, dx, \tag{44}$$

where $h^d_\delta(x\,|\,0) = h^d_{1,\delta}(x\,|\,0) + h^d_{2,\delta}(x\,|\,0)$, and also $f_{d,\delta}(y)$ in (13) may be expressed as the mixed density (e.g Willmot (2007), Kim (2007))

$$f_{d,\delta}(y) = \int_0^\infty \left\{\frac{h^d_{1,\delta}(x\,|\,0)}{\phi_{d,\delta}}\right\} \frac{p_1(x+y)}{\bar{P}_1(x)} dx + \int_0^\infty \left\{\frac{h^d_{2,\delta}(x\,|\,0)}{\phi_{d,\delta}}\right\} \frac{p(x+y)}{\bar{P}(x)} dx. \tag{45}$$

Furthermore, we illustrate how to derive (32) and (33). First, from Equation 66 and Equation 67 in Cheung et al. (2010b), we have

$$m^*_\delta(u) = C_\delta(s_1, s_2, s_3, s_4)\left\{(s_1 + s_3 + s_4) e^{-(\beta + s_1 + s_3 + s_4)u} + \phi_\delta \beta e^{-\beta(1-\phi_\delta)u}\right\}, \tag{46}$$

where

$$C_\delta(s_1,s_2,s_3,s_4) = \frac{\beta(\phi_\delta\beta + s_1 + s_4)\tilde{k}(\delta + c\beta + cs_1)}{(\beta + s_2)(\phi_\delta\beta + s_1 + s_3 + s_4)\left\{s_1 + s_4 + \beta\tilde{k}(\delta + c\beta + cs_1 + cs_4)\right\}}. \tag{47}$$

In this case, $v^*_{d,\delta}(u)$ in (10) can be obtained, by some simple algebra, as

$$v^*_{d,\delta}(u) = \frac{\beta\gamma_{d,\delta}(s_1,s_4)}{\beta+s_2}\, e^{-(\beta+s_1+s_3+s_4)u}, \tag{48}$$

where

$$\gamma_{d,\delta}(s_1,s_4) = \tilde{k}_1(\delta+c\beta+cs_1)+\int_0^\infty\int_0^x e^{-s_1x-s_4v}h^d_{(2),\delta}(x,v\,|\,0)\,dvdx. \tag{49}$$

Then, combining (46) and $f_{d,\delta}(y)=p(y)=\beta e^{-\beta y}$ from (45), $m^*_{d,\delta}(u)$ in (9) becomes

$$m^*_{d,\delta}(u) = C_\delta(s_1,s_2,s_3,s_4)\phi_{d,\delta}\,\beta\{e^{-\beta(1-\phi_\delta)u}-e^{-(\beta+s_1+s_3+s_4)u}\}+v^*_{d,\delta}(u), \tag{50}$$

where $v^*_{d,\delta}(u)$ is given by (48).

Next, $\gamma_{d,\delta}(s_1,s_4)$ (or the Laplace transform of $h^d_{(2),\delta}(x,v\,|\,0)$) may be expressed in terms of the Laplace transform of the interclaim times as follows. We simply consider (50) and (48) with $s_2=s_3=0$,

$$m_{d,\delta,14}(u) = C_\delta(s_1,0,0,s_4)\phi_{d,\delta}\,\beta\{e^{-\beta(1-\phi_\delta)u}-e^{-(\beta+s_1+s_4)u}\}+v_{d,\delta,14}(u) \tag{51}$$

where

$$v_{d,\delta,14}(u) = \gamma_{d,\delta}(s_1,s_4)e^{-(\beta+s_1+s_4)u} \tag{52}$$

Then, from (23) and (24) with $w(x,y,v)=e^{-s_1x-s_4v}$ we get $m_{d,\delta,14}(u)$ as

$$m_{d,\delta,14}(u) = \beta_{d,\delta,14}(u)+\int_0^\infty e^{-\delta t}\sigma_{\delta,t}(u+ct)\,dK_1(t), \tag{53}$$

where $\beta_{d,\delta,14}(u)=\tilde{k}_1(\delta+c\beta+cs_1)\,e^{-(\beta+s_1+s_4)u}$ and $\sigma_{\delta,t}(x)=\int_0^x m_{\delta,14}(x-y)\,\beta e^{-\beta y}dy$. With substitution of (46) with $s_2=s_3=0$ into the above equation, the integral on the right-hand side of (53) becomes

$$\int_0^\infty e^{-\delta t}\sigma_{\delta,t}(u+ct)\,dK_1(t) \tag{54}$$
$$= C_\delta(s_1,0,0,s_4)\,\beta\{\tilde{k}_1(\delta+c\beta-\phi_\delta c\beta)e^{-\beta(1-\phi_\delta)u}-\tilde{k}_1(\delta+c\beta+cs_1+cs_4)e^{-(\beta+s_1+s_4)u}\}.$$

Thus, combining the expression for $\beta_{d,\delta,14}(u)$ and (54) leads (53) to

$$m_{d,\delta,14}(u) = \tilde{k}_1(\delta+c\beta+cs_1)\,e^{-(\beta+s_1+s_4)u} \tag{55}$$
$$+\,C_\delta(s_1,0,0,s_4)\,\beta\{\tilde{k}_1(\delta+c\beta-\phi_\delta\beta)e^{-\beta(1-\phi_\delta)u}-\tilde{k}_1(\delta+c\beta+cs_1+cs_4)e^{-(\beta+s_1+s_4)u}\}.$$

But, with $s_1 = s_4 = 0$, one finds $C_\delta(0,0,0,0) = \beta^{-1}$ from (47), and (49) reduces to $\gamma_{d,\delta}(0,0) = \tilde{k}_1(\delta + c\beta) + \int_0^\infty h^d_{2,\delta}(x|0)\,dx = \phi_{d,\delta}$ from (43) and (44). Consequently, from (51) and (52) with $s_1 = s_4 = 0$ we obtain (4) as

$$\bar{G}_{d,\delta}(u) = \phi_{d,\delta}\, e^{-\beta(1-\phi_\delta)u},$$

where

$$\phi_{d,\delta} = \tilde{k}_1(\delta + c\beta - \phi_\delta c\beta), \tag{56}$$

from (55) with $s_1 = s_4 = 0$ and $u = 0$ (e.g. Kim (2007)). Evidently, in the ordinary model, $\bar{G}_\delta(u) = \phi_\delta e^{-\beta(1-\phi_\delta)u}$ with $\phi_\delta = \tilde{k}(\delta + c\beta - \phi_\delta c\beta)$ (e.g. Willmot (2007)).

Now, equating (51) and (55) followed by rearranging yields

$$\gamma_{d,\delta}(s_1,s_4)\, e^{-(\beta+s_1+s_4)u} = \tilde{k}_1(\delta+c\beta+cs_1)\, e^{-(\beta+s_1+s_4)u} + C_\delta(s_1,0,0,s_4)\beta \tag{57}$$
$$\times \left[\left\{\tilde{k}_1(\delta+c\beta-\phi_\delta c\beta)-\phi_{d,\delta}\right\}e^{-\beta(1-\phi_\delta)u} + \left\{\phi_{d,\delta} - \tilde{k}_1(\delta+c\beta+cs_1+cs_4)\right\}e^{-(\beta+s_1+s_4)u}\right].$$

Then, application of (56) to (57) followed by division by $e^{-(\beta+s_1+s_4)u}$ leads to

$$\gamma_{d,\delta}(s_1,s_4) = \tilde{k}_1(\delta+c\beta+cs_1) + C_\delta(s_1,0,0,s_4)\,\beta\left\{\phi_{d,\delta} - \tilde{k}_1(\delta+c\beta+cs_1+cs_4)\right\}.$$

In other words, using (47),

$$\gamma_{d,\delta}(s_1,s_4) = \frac{\phi_{d,\delta}\beta\tilde{k}(\delta+c\beta+cs_1) + (s_1+s_4)\tilde{k}_1(\delta+c\beta+cs_1) + \beta k_\delta(s_1,s_4)}{s_1+s_4+\beta\tilde{k}(\delta+c\beta+cs_1+cs_4)}, \tag{58}$$

where $k_\delta(s_1,s_4) = \tilde{k}_1(\delta+c\beta+cs_1)\,\tilde{k}(\delta+c\beta+cs_1+cs_4) - \tilde{k}(\delta+c\beta+cs_1)\,\tilde{k}_1(\delta+c\beta+cs_1+cs_4)$. Finally, substitution of (48) into (50) together with the use of (58) yields (32). But above $k_\delta(s_1,s_4)$ equals 0 in the ordinary model, (58) and (33) are equivalent respectively to Equation 65 and Equation 67 in Cheung et al. (2010b).