DETERMINING AND ALLOCATING DIVERSIFICATION BENEFITS FOR A PORTFOLIO OF RISKS

BY

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ABSTRACT

A critical problem in financial and insurance risk analysis is the calculation of risk margins. When there are a number of risks, the total risk margin is often reduced to reflect diversification. How large should the "diversification benefit" be? And how should the benefit be allocated to the individual risks? We propose a simple statistical solution. While providing a theoretical analysis, the final expressions are readily implemented in practice.

KEYWORDS

Diversification, Allocated risk margins, Stand-alone risk margins, Capital allocation, Euler allocation, Percentile risk aversion.

1. INTRODUCTION

Risk margins provide a buffer against worse than expected losses. Providing for risk margins costs money – the cost of tying up capital. Three important problems in risk margin calculations are:

- What is an appropriate method to calculate the risk margin for a single risk?
- How does the risk margin for a combined portfolio of risks differ from the sum of risk margins obtained separately for each risk?
- What portion of the overall risk margin in a portfolio of risks should be attributed back to each individual risk?

The first problem relates to risk measurement and is dealt with in Heilman (1989); Wang (1996); McNeil et al. (2005); Furman and Zitikis (2008a) and Choo and De Jong (2009). The second problem relates to diversification and is discussed in this article. The third problem is important in measuring management performance and cost of capital for separate risks, and has received considerable attention in recent literature (Tasche, 2004; Overbeck, 2004; Kalkbrener, 2005; Dhaene et al., 2009) in the context of capital allocation.

Astin Bulletin 40(1), 257-269. doi: 10.2143/AST.40.1.2049228

This article suggests an unified approach dealing with all three problems. For a portfolio of risks, simple formulas are provided for risk margins, the aggregate diversification benefit and the allocation of the same to individual risks. The approach has been mentioned in the literature in various guises (Ruhm et al., 2003; Kreps, 2005; Furman and Zitikis, 2008a,b; Furman and Landsman, 2008). This article provides a single explicit framework, drawing out all relevant implications, generalizations and shortcomings.

Section 2 discusses single risk margins. This is generalized to the multiple risk setting in §3. The relationship to CAPM or standard deviation principle pricing is discussed in §4 whereas connections to other approaches are discussed in §5. Section 6 discusses how the proposed formulas for risk margins and diversification benefits can be applied in practice while §7 gives an empirical illustration. Section 8 provides conclusions.

2. Risk margins and percentile risk aversion

Given a risk or loss $y_i \geq 0$, many practical risk measures can be written as (Heilmann, 1989; Furman and Zitikis, 2008a; Choo and De Jong, 2009).

$$\mathrm{R}(y_i) \equiv \mathrm{E}\{y_i\,\phi(u_i)\}, \quad u_i \equiv F_i(y_i),$$

with corresponding risk margin

$$\dot{m}_i \equiv \mathrm{R}(y_i) - \mu_i = \mathrm{cov}\{y_i, \phi(u_i)\}, \quad \mu_i \equiv \mathrm{E}(y_i). \tag{1}$$

Here F_i is the distribution of y_i, $\phi \geq 0$ is a "percentile aversion function" with $\mathrm{E}\{\phi(u_i)\} = 1$, and cov denotes covariance. Each choice of ϕ implies a "risk measure" which, under specified conditions on ϕ, are coherent – for a detailed treatment see Choo and De Jong (2009).

The percentile aversion function ϕ is subjectively specified to indicate the relative aversion to various percentile outcomes of the risk or loss. The outcomes are adjusted by relative aversion, and the risk measure is the expectation of the risk adjusted outcomes. The single dot on the margin $\dot{m}_i$ indicates the margin is derived on a "stand-alone" basis without reference to other risks in the portfolio and hence possible diversification benefits.

A more general version of (1) is $\mathrm{cov}\{y_i, w(x)\}$ where x is a random variable not necessarily equal to y_i and $\mathrm{E}\{w(x)\} = 1$. This form of the risk margin is extensively studied in Furman and Zitikis (2008a,b) who call

$$\mathrm{E}\{y_i w(x)\} = \mu_i + \mathrm{cov}\{y_i, w(x)\},$$

the "weighted allocation" of y_i on x, or the "weighted premium" of y_i if $x = y_i$. Our focus and specialization has four aspects. First, assume $w = \phi \circ F_x$ where F_x is the distribution of x. This form of w encompasses most interesting and practical cases as illustrated below. Second, our focus is on the risk margin – that is the excess over the mean, not the risk measure nor allocation. This trivial change of focus has important benefits in terms of decomposing diversification benefits as seen in §3. Third, in our development, x is either y_i, as in (1), or $x = \sum_i y_i$, the total risk of a portfolio of risks. Finally, all risks y_i are assumed to have densities.

Table 1 displays three choices of ϕ and the associated risk measure R. Also displayed is the standard deviation κ_ϕ of $\phi(u)$ given u is uniform. The first column in the body of Table 1 is the well known Value-at-Risk (VaR) picking out the appropriate percentile of the distribution. The notations $(u > q)$ and $(u = q)$ indicate the step function and the Dirac-delta function (Lighthill, 1958) centered on q with unit mass, respectively. For $(u = q)$, $\mathrm{E}\{y_i \phi(u_i)\} \to F_i^-(q)$, the VaR at q and in turn $\mathrm{cov}\{y_i, \phi(u_i)\} \to F_i^-(q) - \mu_i$. This last result also follows directly from (1):

$$\dot{m}_i = \mathrm{cov}\{F_i^-(u_i), \phi(u_i)\} = \mathrm{E}\{F_i^-(u_i)\phi(u_i)\} - \mu_i. \tag{2}$$

The first expression on the right is called the "spectral measure" of risk (Acerbi, 2002) decomposing a risk measure as a weighted average of VaRs.

TABLE 1

RISK MEASURES AND CORRESPONDING PERCENTILE RISK AVERSION

	Value-at-Risk at $0 \le q \le 1$	E(max) $r \ge 1$ ind. copies	CTE at $0 \le q \le 1$
$\phi(u)$	$(u = q)$	ru^{r-1}	$\frac{(u > q)}{1-q}$
κ_ϕ	–	$\frac{r-1}{\sqrt{2r-1}}$	$\sqrt{\frac{q}{1-q}}$

The second column in Table 1 is the expected maximum of r independent copies of the same risk. The final column is the conditional tail expectation, also known as the Tail Value-at-Risk or expected shortfall. An extensive discussion of different risk measures is in McNeil et al. (2005) while further examples of risk measures in terms of ϕ are given in Choo and De Jong (2009). Generally $\phi(u)$ has a nonnegative derivative indicating aversion is monotonic in u although this is not the case for VaR. The choice of ϕ in (1) is equivalent to choosing a risk measure in the sense of McNeil et al. (2005). Any monotonic choice of ϕ in (1) yields a coherent risk measure (Choo and De Jong, 2009, §6).

The approach where risky outcomes are weighed according to some function ϕ of the tail probability has been proposed in Quiggin (1982) and extensively studied in Yaari (1987). In the insurance literature it is often associated with calculating the expected value after distorting the distribution function (Wang, 1996).

Generally it is assumed in this article that distribution functions are differentiable and hence have densities. Although this may be technically restrictive, it is easy to organize an approximating density in any practical setting. If a density exists then so does the inverse distribution function and throughout this paper it is generally assumed that inverse distribution functions F^{-} are well defined on $(0, 1)$. This will be the case if F is strictly monotonic.

The risk margin $\dot{m}_i$ defined in (1) can be rewritten as

$$\dot{m}_i = \kappa_\phi \sigma_i \rho_i, \quad \rho_i \equiv \operatorname{cor}\{y_i, \phi(u_i)\}, \tag{3}$$

where κ_ϕ and σ_i are the standard deviation of $\phi(u_i)$ and y_i, respectively, and cor denotes correlation. Here $\sigma_i \rho_i$ is called the "corrected" standard deviation of y_i where the correction factor is the correlation ρ_i. Assuming each y_i has continuous distribution, the factor κ_ϕ is independent of the risk i, depending only on the aversion function ϕ. Note κ_ϕ is interpreted as a measure of conservatism (Choo and De Jong, 2009) since ϕ adjusts the severity or likelihood of different losses based on aversion, magnifying large losses while diminishing small losses. A large value of κ_ϕ indicates a high degree of adjustment and hence conservatism.

Hence the margin $\dot{m}_i$ is a standard deviation type risk margin calculated as a multiple κ_ϕ of the corrected standard deviation $\sigma_i \rho_i$. Thus while σ_i captures all aspects of the variability of y_i, ρ_i represents the proportion of relevance to the risk manager and hence the corrected standard deviation $\sigma_i \rho_i$ is the appropriate measure of loss variability.

The correlation ρ_i can be written as

$$\rho_i = \operatorname{cor}\{F_i^{-}(u_i), \phi(u_i)\},$$

where F_i^{-} is the inverse of the distribution function F_i of y_i. Hence ρ_i measures the linearity of $F_i^{-}(u_i)$ when plotted against $\phi(u_i)$, that is, the similarity in shape between F_i^{-} and ϕ after accounting for location and scale.

To explain the role and importance of the correction factor ρ_i consider two losses with the same standard deviation but different distributions. Then the difference in risk margins is due to the correction factor ρ_i. Next, consider two different aversion functions ϕ with the same κ_ϕ applied to a single loss distribution. Again the correction factor accounts for the difference in risk margins. In both of these cases, ρ_i "corrects" for the shape of the risk distribution and aversion. Lastly, the correction factor ρ_i is invariant to location and scale transformations of the loss.

If $\rho_i = 1$ then $\dot{m}_i = \kappa_\phi \sigma_i$ and the risk margin is a multiple of the standard deviation, the standard deviation principle of risk margin setting. However the condition $\rho_i = 1$ holds if and only if $\phi(u_i) = a + bF_i^{-}(u_i)$ implying ϕ and κ_ϕ depend on F_i and hence on the risk i. This dependence can be viewed as a shortcoming of the standard deviation principle. Indeed if $\phi = a + bF_i^{-}$ then the standard deviation of $\phi(u_i)$ is $\kappa_\phi = b\sigma_i$ and the condition $\mathrm{E}\{\phi(u_i)\} = 1$ implies $a + b\mu_i = 1$ and hence

$$\phi(u_i) = 1 + b\{F_i^{-}(u_i) - \mu_i\} = 1 + \kappa_\phi \frac{F_i^{-}(u_i) - \mu_i}{\sigma_i},$$

which can be negative. Further with $\kappa_\phi = b\sigma_i$ the risk margin is proportional to the variance yielding the variance principle of risk margin setting. Hence the assumption that $\rho_i = 1$ leads to inconsistencies implying the standard deviation and variance principle of risk margin setting cannot be reconciled with the percentile aversion framework. The standard deviation principle is further discussed in §4.

3. Allocated risk margins and diversification benefits

Applying the §2 risk margin calculation to the total risk $x \equiv \sum_i y_i$ yields the total risk margin

$$\dot{m}_x \equiv \text{cov}\{x, \phi(u_x)\} = \text{cov}\left\{\sum_i y_i, \phi(u_x)\right\} = \sum_i \ddot{m}_i \equiv \ddot{m}_+, \quad u_x \equiv F_x(x), \quad (4)$$

where

$$\ddot{m}_i \equiv \text{cov}\{y_i, \phi(u_x)\} = \kappa_\phi \sigma_i \rho_{ix} = \dot{m}_i \frac{\rho_{ix}}{\rho_i}, \quad \rho_{ix} \equiv \text{cor}\{y_i, \phi(u_x)\}. \quad (5)$$

The double dot notation indicates the risk margins $\ddot{m}_i$ are based on a "stand-together" or overall basis. The risk margin $\ddot{m}_i$ is called the allocated margin for risk i and is a multiple κ_ϕ of the corrected standard deviation $\sigma_i \rho_{ix}$. Note the measure of conservatism κ_ϕ is the same but the correction to the standard deviation σ_i is the correlation of the risk with the aversion adjusted total $\phi(u_x)$.

Comparing the total of the stand-alone risk margins $\dot{m}_+ \equiv \sum_i \dot{m}_i$ to the total allocated risk margin $\ddot{m}_+ \equiv \sum_i \ddot{m}_i$ shows that the diversification benefit is

$$\dot{m}_+ - \ddot{m}_+ = \sum_i \left(\dot{m}_i - \ddot{m}_i\right) = \kappa_\phi \sum_i \sigma_i \left(\rho_i - \rho_{ix}\right). \quad (6)$$

Expression (6) can also be written as

$$\dot{m}_+ \sum_i \frac{\dot{m}_i}{\dot{m}_+} \left(1 - \frac{\rho_{ix}}{\rho_i}\right). \quad (7)$$

Hence the diversification benefit is the total stand-alone margin $\dot{m}_+$ times a weighted average of the proportionate individual diversification benefits $1 - \rho_{ix}/\rho_i$ where the weights are $\dot{m}_i/\dot{m}_+$, the proportions of the total stand-alone risk margin attributable to risk i. Risks y_i with a relatively large stand-alone risk margin are weighted heavily. Note the above decomposition of the diversification benefit applies to all risk measures induced by ϕ.

The weighted sum of all the weighted individual proportional diversification benefits – that is the sum in (7) – is the proportion of the total stand-alone risk margin $\dot{m}_+$ that has been "diversified away." With comonotonic losses $y_i = g_i(y_1)$ where g_i is increasing. Hence $\phi(u_i) = \phi(u_1) = \phi(u_x)$ implying $\rho_i = \rho_{ix}$ for all i and (7) is zero.

As an example suppose ϕ corresponds to CTE as displayed in Table 1. Then straightforward calculations show

$$\dot{m}_i = \mathrm{E}(y_i \,|\, u_i > q) - \mu_i, \quad \ddot{m}_i = \mathrm{E}(y_i \,|\, u_x > q) - \mu_i,$$

implying

$$\frac{\rho_{ix}}{\rho_i} = \frac{\mathrm{E}(y_i \mid u_x > q) - \mu_i}{\mathrm{E}(y_i \mid u_i > q) - \mu_i},$$

which appears in for example Furman and Zitikis (2008b). Note the simplicity of above derivation compared to say Tasche (2001) as set out in McNeil et al. (2005). Similarly for VaR, the above expressions hold with the conditioning greater than q replaced by equal to q.

4. CAPM and the standard deviation principle

The percentile risk aversion framework discussed above is related to the standard deviation principle used with the CAPM model in finance (Luenberger, 1998). A recent treatment with reference to weighted allocation is contained in Furman and Zitikis (2009). Viewing the CAPM model from the percentile aversion framework leads to interesting insights as discussed below.

With CAPM the standalone risk margin for a risk y_i is a fixed multiple κ of the standard deviation σ_i, ie $\kappa\sigma_i$. Noting the correspondence between κ and κ_ϕ, the CAPM risk margin comes under percentile aversion if $\rho_i = 1$, ie ϕ is linear in the inverse distribution F_i^- of the risk. The overall risk margin for the total risk $x = \sum_i y_i$ is $\kappa\sigma_x$ where σ_x is the standard deviation of x.

Under CAPM the overall risk margin is allocated to individual risks in proportion to their covariances with the total risk x:

$$\frac{\mathrm{cov}(y_i, x)}{\sigma_x^2}\kappa\sigma_x = \kappa\sigma_i \,\mathrm{cor}(y_i, x). \tag{8}$$

These margins add up to the total margin $\kappa\sigma_x$ since $\sigma_x^2 = \sum_i \mathrm{cov}(y_i, x)$. Therefore allocated risk margins obtained using CAPM are similar to those in (5), except that dependence is measured as the correlation between individual risks and the total risk rather than the correlation to the aversion adjusted total. Equality is achieved if $\mathrm{cor}(y_i, x) = \rho_{ix}$, which requires ϕ to be linear in F_x^-, the inverse distribution of x.

With the CAPM risk margins (8), the total diversification benefit is

$$\sum_i \kappa\sigma_i - \kappa\sigma_x = \kappa\sum_i \sigma_i\{1-\mathrm{cor}(y_i,x)\} = \kappa\sigma_+ \sum_i \frac{\sigma_i}{\sigma_+}\{1-\mathrm{cor}(y_i,x)\}, \quad (9)$$

where $\sigma_+ \equiv \sum_i \sigma_i$. The final expression is analogous to (7), with $1 - \mathrm{cor}(y_i, x)$ representing the proportional reduction in the risk margin for risk i on account of diversification. Indeed (7) reduces to (9) if again $\rho_i = 1$ and $\rho_{ix} = \mathrm{cor}(y_i, x)$ or equivalently ϕ is linear in both F_i^- and F_x^-. This highlights a shortcoming of CAPM risk margins: in the CAPM framework the aversion function ϕ depends on the risk. This does not seem to be cogent. For example it seems nonintuitive to use say CTE at $q = 0.05$ for one risk and CTE at $q = 0.2$ or E(max) at $r = 10$ for another.

5. Connection to Euler and other allocation rules

It is useful to compare the allocations $\ddot{m}_i$ developed in §3 to those derived from the Euler capital allocation principle (McNeil et al., 2005). Euler allocation can be rationalized in terms of economic justifications (Tasche, 2004) and is often invoked in the context of positive homogeneous risk measures of which the current measures, based on percentile rank aversion, are an example since for scalar $a > 0$,

$$\dot{m}_{ax} \equiv \mathrm{cov}\{ax, (\phi \circ F_{ax})(ax)\} = a\,\mathrm{cov}\{x, \phi(u_x)\} = a\dot{m}_x.$$

Hence risk margins based on percentile rank aversion are homogenous of degree 1. By Euler's theorem, for a scale vector λ and risk vector y, if $\dot{m}_{\lambda' y}$ is the risk margin for $\lambda' y = \sum_i \lambda_i y_i$ then $\dot{m}_{\lambda' y} = \lambda' \ddot{m}_{\lambda' y}$ where $\ddot{m}_{\lambda' y}$ is the vector of derivatives of $\dot{m}_{\lambda' y}$ with respect to each component of λ. The (per unit) Euler allocations associated with $\dot{m}_x$ where $x = 1' y$ are then defined as $\ddot{m}_x$, the vector $\ddot{m}_{\lambda' y}$ evaluated at $\lambda = 1$.

The allocations $\ddot{m}_i$ defined in (5) are Euler allocations[1] – that is $\ddot{m}_x$ has components $\ddot{m}_i$. To show this denote the distribution function of $\lambda' y$ by F_λ and assume F_λ is continuous for all λ. Then from the spectral risk measure representation (2) and Tasche (2001), (see also McNeil et al. (2005, p. 258))

$$\dot{m}_{\lambda' y} = \mathrm{cov}\left\{F_\lambda^-(u), \phi(u)\right\}, \quad \frac{\partial F_\lambda^-(v)}{\partial \lambda} = \mathrm{E}(y \mid u = v),$$

[1] We are indebted to an anonymous referee for pointing out this holds generally.

where $u = F_\lambda(\lambda' y)$. Hence if differentiation can be moved inside the covariance,

$$\ddot{m}_{\lambda' y} = \text{cov}\{\text{E}(y|u), \phi(u)\} = \text{cov}\{y, \phi(u)\}. \tag{10}$$

The final equality follows using iterated covariance using $\text{E}\{\phi(u)|u\} = \phi(u)$ and $\text{cov}\{y, \phi(u)|u\} = 0$. Setting $\lambda = 1$ then $\lambda' y = x$ and $u = u_x$ implying component i of $\ddot{m}_x$ equals $\ddot{m}_i$. Technical conditions permitting the interchange of differentiation and expectation are discussed in Overbeck (2004, p. 310) and Kalkbrener (2005).

Dhaene et al. (2009) considers capital allocation based on the optimization

$$\min_m (c - m)' \Sigma^{-1} (c - m) \quad \text{subject to} \quad 1'm = m_+, \tag{11}$$

where c has components $c_i = \mu_i + \text{cov}(y_i, \psi_i)$ and the ψ_i are positive random variables with $\text{E}(\psi_i) = 1$. Further, Σ is a symmetric positive definite matrix and m is interpreted as a vector of capital allocations. By subtracting μ_i from both c_i and m_i shows (11) can be interpreted as finding risk margins m_i which are closest to $c_i = \text{cov}(y_i, \psi_i)$ and subject to a constraint on the total risk margin. This contrasts with the development in §7 where the total risk margin and the risk margin allocations are jointly determined.

A straightforward calculation shows the solution to (11) is $m = c + k\Sigma 1$ where k is a scalar enforcing the constraint in (11). If in $\psi_i = \phi(u_x)$ for all i then $c_i = \ddot{m}_i$ and hence $m_i = \ddot{m}_i$ if and only if $k = 0$ implying

$$m_+ = \sum_i m_i = \sum_i \ddot{m}_i \equiv \ddot{m}_+ = \text{cov}\{x, \phi(u_x)\}.$$

In other words the risk margins $\ddot{m}_i$ are the solution to (11) if $m_+ = \text{cov}(x, \psi)$ and $\psi_i \equiv \psi \equiv \psi(x) = \phi(u_x)$ with $\text{E}\{\phi(u_x)\} = 1$. This choice for the ψ_i implies the allocation is driven by the total risk x rather than the individual risks y_i.

The special case $\psi_i = 1$ implies $\text{cov}(y_i, \psi_i) = 0$ and hence $m_i = 0$. Together with the assumption Σ diagonal then (11) reduces to a minimization considered by Zaks et al. (2006) with solution $m_i = v_i k$ where v_i is diagonal entry i of Σ.

Tsanakas and Barnett (2003) derive allocations similar and in some cases identical to $\ddot{m}_i$ using game theoretic Aumann-Shapley allocation. Aumann-Shapley values are allocation schemes satisfying economically motivated axioms. (Billera and Heath, 1982; Mirman and Tauman, 1982).

6. Practical risk margin setting

The following restates the standalone and allocated risk margin for an individual risk y_i and the overall diversification benefit:

$$\dot{m}_i = \kappa_\phi \sigma_i \rho_i, \quad \ddot{m}_i = \kappa_\phi \sigma_i \rho_{ix}, \quad \dot{m}_+ - \ddot{m}_+ = \kappa_\phi \sum_i \sigma_i (\rho_i - \rho_{ix}). \tag{12}$$

Obtaining risk margins and the diversification benefit using (12) is straightforward if the joint risk distribution of the risks is known. However difficulties arise with imperfect knowledge of the joint distribution and the dependence structure between individual risks. Any choice of ϕ is also arguable and subjective. The following shows that the formulas in (12) are relatively easily applied in practice by separately obtaining κ_ϕ and σ_i, ρ_i and ρ_{ix} for the risks y_i.

Consider first $\kappa_\phi \geq 0$, a scale factor, controlling the overall level of risk margins for all risks and the diversification benefit. The parameter κ_ϕ reflects the conservatism of the risk manager. To set κ_ϕ, risk managers choose a non-negative value reflecting his (or the regulator's) desired level of conservatism. There is no need to explicitly specify ϕ. Knowledge of ϕ, however, provides guidance on an appropriate value of κ_ϕ. For example, given the results in Table 1, with CTE, q set at 0.8, 0.9 and 0.94 imply κ_ϕ values of 2, 3 and 4, respectively. These same κ_ϕ values are attained with E(max) with r set to 10, 20, and 34, respectively. This suggests κ_ϕ values in the range of 2 to 3, not unlike the range of z-scores often used in statistical studies. Note that these results can be reversed, in that, for example, $\kappa_\phi = 3$ suggest an aversion function $10(u > 0.9)$ or $20u^{19}$. An aversion function, and hence κ_ϕ, can also be elicited using the method outlined in Choo and De Jong (2009).

Second, consider σ_i, the standard deviation of y_i. Information on σ_i is often available from past outcomes, which may be adjusted for views of the future. Standard deviation or volatility is commonly used by risk managers and hence arriving at appropriate values is likely to present relatively little difficulty, compared to say the quantile of a risk. Note the standard deviation of the aggregate risk is not needed.

Third, consider

$$\rho_i = \text{cor}\{F_i^-(u_i), \phi(u_i)\} = \text{cor}\{y_i, (\phi \circ F_i)(y_i)\},$$

then ρ_i measures the similarity in shape between F_i^- and ϕ, or equivalently the linearity of $\phi \circ F_i$. Note $\rho_i \geq 0$ since both ϕ and F_i are non-decreasing functions. To obtain ρ_i, the risk manager subjectively chooses a value between zero and one to indicate the extent to which the loss and loss aversion behave similarly as the percentile of the loss varies. A correlation of 1 indicates loss and aversion to the loss move in linear unison and hence $\phi = a + bF_i^-$.

Analogously, the correlation ρ_{ix} measures the similarity between $F_i^-(u_i)$ and $\phi(u_x)$, but is diminished to the extent that the individual risk is not perfectly dependent on the aggregate risk. Negative values are possible if the dependence is negative.

Hence the methods proposed in this paper to obtain risk margins and diversification benefits can be robustly applied in situations where risk distributions are uncertain. The choice of risk measure, ϕ, is a source of subjectivity

(Choo and De Jong, 2009, §11). Hence in practical settings, the efforts expended on applying technically sophisticated methods to say determine the joint distribution may be misplaced. The more subjective approach discussed here appears practically relevant and further research on plausible values for κ_ϕ, ρ_i and ρ_{ix} will be valuable for creating "rules of thumb" in setting risk margins and diversification benefits. Such rules of thumb appear more cogent than say the present rules where correlations are subjectively specified between different risks and from these and standard deviations, the variability of the total risk and risk margins are derived.

7. Correction factors for actual risk distribution

This section displays correction factors ρ_i and ρ_{ix} under a hypothetical joint risk distribution and percentile aversion function. Suppose there are three risks y_1, y_2 and y_3 with the exponential, Pareto, and lognormal distributions, respectively:

$$F_1(y_1) = 1 - e^{-y_1}, \quad F_2(y_2) = 1 - (1 + y_2)^{-10}, \quad F_3(y_3) = \Phi(\log y_3),$$

where Φ is the standard normal distribution. Simplistic distributions are chosen intentionally for illustration purposes. Assume the percentile aversion function is $\phi(u) = 20u^{19}$ implying the standalone risk measure is the expected maximal loss in 20 independent realizations.

Using simulation, the correction factors ρ_i for the three risks, given the specified risk distributions and aversion function are

$$\rho_1 = 0.85, \quad \rho_2 = 0.88, \quad \rho_3 = 0.90.$$

These values suggest the shape of the lognormal is closest to that of the aversion function, followed by the Pareto and exponential. This is confirmed in Figure 1 which displays the inverse distributions and aversion function. The lognormal is less skewed, which is the case for the aversion function, compared to the heavier tailed Pareto and exponential.

In order to compute $\rho_{ix} = \text{cor}\{y_i, \phi(u_x)\}$, the dependence structure of the risks is required. Assume a Clayton copula of the form $(u_1^{-\theta} + u_2^{-\theta} + u_3^{-\theta} - 2)^{-1/\theta}$ where $u_i = F_i(y_i)$ and the θ parameter controls the degree of dependence between the risks. Using simulation, if $\theta = 2$ then

$$\rho_{1x} = 0.48, \quad \rho_{2x} = 0.30, \quad \rho_{3x} = 0.84.$$

Thus for the exponential and Pareto there are substantial diversification benefits with the risk margins reducing by $1 - 0.48/0.85 = 44\%$ and $1 - 0.30/0.88 = 66\%$, respectively. For the lognormal the percentage diversification benefit is 7%.

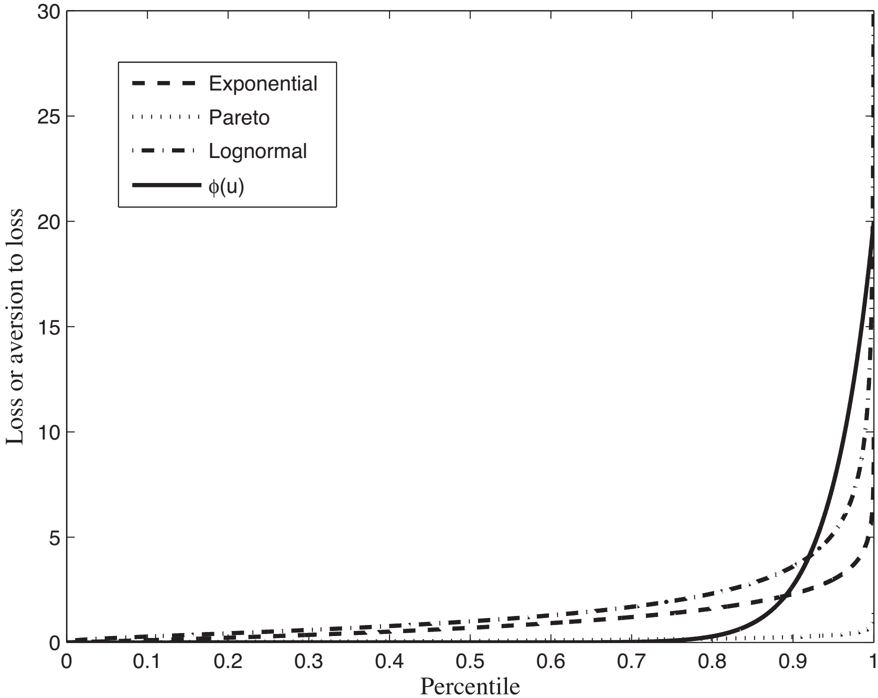

FIGURE 1: Comparison of inverse distributions and aversion function.

If the dependence parameter θ in the copula is increased to 10 then equivalent calculations yield

$$\rho_{1x} = 0.69, \quad \rho_{2x} = 0.60, \quad \rho_{3x} = 0.87.$$

and risk margin reductions for the exponential, Pareto and lognormal of 19%, 32% and 3%, respectively implying higher risk margins than for $\theta = 2$. In this case diversification diminishes as risks become more closely dependent, a phenomenon commonly seen in practice.

Under the CAPM risk margins of §4 there is no aversion adjustment. Corresponding to correction factors arising under percentile risk aversion are correlations as follows. For standalone risk margins, correlations are implicitly one. If $\theta = 2$ then correlations $\mathrm{cor}(y_i, x)$ are 0.66, 0.46 and 0.93, for the exponential, Pareto and lognormal, respectively. Percentage diversification benefits are $1 - 0.66 = 34\%$, $1 - 0.46 = 54\%$ and $1 - 0.93 = 7\%$. For $\theta = 10$ correlations are 0.84, 0.75 and 0.96, and diversification benefits are 16%, 25% and 4%. Note the diversification benefits follow a similar pattern as those obtained under percentile risk aversion, however actual magnitudes differ.

8. Conclusions

This article has discussed the determination and allocation of risk margins for a portfolio of risks. The overall risk margin is set using an aversion function which weighs different outcomes of the total based on their percentile rank. The total risk margin is then decomposed or allocated to the individual risks making up the total. The resulting margins depend on the risk measure, and hence the amount of conservatism, and the correlation of the individual risks with the aversion adjusted percentile rank of the total.

The allocation method of this paper can be viewed as either "top-down" or "bottom-up". Further, the method can be implemented either formally or subjectively. A formal bottom-up approach specifies the marginals and copula of the risks, the consequent derivation of the distribution of the sum, and imposes an analytically specified aversion function from which allocations are derived. A subjective top-down approach is where correlations of the aggregate risk to the percentile aversion function are specified subjectively with similar subjective specifications at the individual risk level. These correlations are then subjectively reconciled and combined with the standard deviations of the risks and a conservatism factor to arrive at the allocation.

It is intriguing to note that in the formulas, both standard deviation and correlation are critical. This occurs despite the fact that there is no assumption related to normality, skewness or linearity. Hence the current formulas temper the critical treatment accorded to both standard deviation and correlation in the recent literature. The development of this paper suggests that both standard deviation and correlation are cogent risk assessment inputs provided input scales are properly formulated.

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