# ON THE UPCROSSING AND DOWNCROSSING PROBABILITIES OF A DUAL RISK MODEL WITH PHASE-TYPE GAINS

BY

ANDREW C.Y. NG

ABSTRACT

In this paper, we consider the dual of the classical Cramér-Lundberg model when gains follow a phase-type distribution. By using the property of phase-type distribution, two pairs of upcrossing and downcrossing barrier probabilities are derived. Explicit formulas for the expected total discounted dividends until ruin and the Laplace transform of the time of ruin under a variety of dividend strategies can then be obtained without the use of Laplace transforms.

KEYWORDS

Ruin theory; dual risk model; phase-type distribution; dividend strategy; upcrossing and downcrossing probabilities.

## 1. INTRODUCTION

In insurance mathematics, the surplus in the classical Cramér-Lundberg model at time $t$ can be expressed as

$$U(t) = u + ct - S(t),$$

where $u$ is the initial surplus, $c$ is the premium rate, and $S(t)$ is the aggregate claims by time $t$, usually modeled by a compound Poisson process. Avanzi et al. (2007), Avanzi and Gerber (2008), Gerber and Smith (2008) and Ng (2009) considered the dual to the classical model

$$U(t) = u - ct + S(t).$$

In this model, the premium rate is negative, causing the surplus to decrease. Claims, on the other hand, cause the surplus to increase. Thus the premium rate should be viewed as an expense rate and claims should be viewed as profits or gains. While not very popular in insurance mathematics, this model has appeared in various literature (see Cramér, 1955, Section 5.13; Seal, 1969,

*Astin Bulletin* 40(1), 281-306. doi: 10.2143/AST.40.1.2049230 

pp. 116-119; and Takács, 1967, pp. 152-154). There are many possible interpretations for this model. For example, we can treat the surplus as the amount of capital of a business engaged in research and development. The company pays expenses for research, and occasional profit of random amounts (such as the award of a patent) arises according to a Poisson process. A similar model is used by Bayraktar and Egami (2008) to model the capital of a venture capital investment. Another model is an annuity business. The company issues payments continuously to annuitants, while the gross reserve of an annuitant is released as emerging profit when he dies. Yang and Zhu (2008) generalized the dual model in a regime-switching setting.

One of the current topics of interest in insurance mathematics is the calculation of expected total discounted dividends until ruin, which goes back to Bruno de Finetti (1957), and has also been studied by Bühlmann (1970, Section 6.4) and Gerber (1969, 1972, Sections 7 and 8, 1979, Section 10.1). In Avanzi et al. (2007), the authors considered the dual model under a barrier strategy. They derived the Laplace transform of the total discounted dividends until ruin, and gave explicit formulas for the expected total discounted dividends until ruin when gains follow an exponential or a mixture or exponential distributions by solving integro-differential equations. Ng (2009) studied the dual model under a threshold strategy and derived the corresponding quantities. For more information on dividend problems in ruin models, see, for example, Avanzi (2009) and Albrecher and Thonhauser (2009).

We consider the dual model under a barrier strategy or a threshold strategy, and a combination of both, when gains follow a phase-type distribution. We shall derive explicit formulas for the expected total discounted dividends and the Laplace transform of the time of ruin based on two pairs of discounted upcrossing and downcrossing probabilities. This alternative method, while only works for the phase-type case, has the advantage of avoiding numerical inversion of Laplace transforms as in the case of barrier strategy in Avanzi et al. (2007) and threshold strategy in Ng (2009). Optimal barrier or threshold can also be obtained easily in the presence of explicit formulas. Finally, since the class of phase-type distribution is dense in the set of continuous distributions with positive support, the phase-type distribution can be used to approximate an arbitrary distribution. For details of how this can be implemented, see, for example, Asmussen et al. (1996).

We first begin with the dynamics of the surplus process $\{U(t)\}$. Before a dividend strategy is imposed, the surplus process is

$$U(t) = u - ct + \sum_{i=1}^{N(t)} X_i, \tag{1}$$

where $u$ is the initial surplus, $c$ is the rate of expense, $\{N(t)\}$ is a Poisson process with rate $\lambda$ and $X_i$'s are independently and identically distributed gains. The distribution, density, and moment generating function of the gains are denoted by $F$, $f$, and $M_X$, respectively. The mean of gains is denoted by $\mu$. We do not assume the usual condition of positive loading: the expected drift of the process, $\mathbb{E}U(1)$, can be positive, zero, or negative. That is,

$$c < \lambda\mu \tag{2}$$

may or may not hold. We let $T = \inf\{t : U(t) = 0\}$ be the time of ruin, and when no dividend strategy is imposed, we let

$$\psi(u) = \mathbb{E}[e^{-\delta T} I(T < \infty) \mid U(0) = u]$$

be the Laplace transform of $T$. In actuarial literature, $\psi(u)$ is known as the discounted infinite-horizon ruin probability. When $\delta = 0$, it reduces to the ruin probability. When $\delta > 0$, we can treat $\psi(u)$ as the present value, evaluated under a given valuation force of interest $\delta$, of a contingent claim of 1 payable as soon as ruin occurs. More generally, starting with any initial surplus $x$, $\psi(u)$ is the expected present value of a contingent claim of 1 payable as soon as the surplus reaches $x - u$. By using the relation between life insurance and annuity, the expected present value of a continuous annuity that pays 1 until the surplus process drops from an initial level of $x$ to $x - u$ is independent of $x$ and is given by

$$\frac{1 - \psi(u)}{\delta}. \tag{3}$$

The above can be treated as a special case of the Gerber-Shiu function with penalty function $\bar{s}_{\bar{\delta}\rceil T}$.

Before we discuss the notation for phase-type distribution, we give the convention for vectors and matrices in this paper. All vectors are column vectors. We let $\boldsymbol{e}_i = (0, 0, \ldots, 1, \ldots, 0)'$ be the $i$th unit vector and $\boldsymbol{e} = (1, 1, \ldots, 1)$ the vector of 1. With the exception of $\boldsymbol{e}_i$, $\boldsymbol{k}_i$ is the $i$th element of the vector $\boldsymbol{k}$. Similarly, $[\boldsymbol{A}]_{ij}$ is the $ij$-element of matrix $\boldsymbol{A}$. We let $\boldsymbol{I}$ be the identity matrix. We use $\boldsymbol{A} \otimes \boldsymbol{B}$ and $\boldsymbol{A} \oplus \boldsymbol{B}$ to denote the Kronecker product and sum of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Readers interested in the properties of these two operations may refer to Graham (1981).

For the distribution of the gains, we denote by $\mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{Q})$ the phase-type distribution with initial probability vector $\boldsymbol{\alpha}$ and generator $\boldsymbol{Q}$. Then $\boldsymbol{t} = -\boldsymbol{Qe}$ is the exit rate vector. If $\boldsymbol{\alpha}'\boldsymbol{e} < 1$, then the distribution is defective. A good reference for properties of phase-type distributions is Chapter VIII of Asmussen (2000).

The structure of this paper is as follows. In Section 2, we derive two pairs of upcrossing and downcrossing probabilities and other related auxiliary quantities. In Sections 3, 4 and 5, we apply the two pairs of upcrossing and downcrossing probabilities to obtain the quantities of interest under a barrier and a threshold strategy, and a combination of a barrier and a threshold strategy. Finally, in Section 6, we discuss how our results can be used to calculate the optimal barrier or threshold for the two pure strategies.

## 2. Two Pairs of Upcrossing and Downcrossing Probabilities

In this section we derive two pairs of upcrossing and downcrossing probabilities, which are the key quantities in deriving the expected dividends and many ruin-related quantities when gains follow a phase-type distribuion. Firstly, we need a general result from Ng (2009) which applies to any distribution for the gains. Let

$$\kappa(\theta) = \lambda[M_X(\theta) - 1] - c\theta \tag{4}$$

and $R_\delta$ be the unique non-positive root of $\kappa(\theta) = \delta$. By considering the slope of $\kappa(\theta)$, it can be observed that $R_\delta < 0$ unless both $\delta = 0$ and $c \geq \lambda\mu$. Ng (2009) proved that

$$\psi(u) = e^{R_\delta u}. \tag{5}$$

For $u = 0$, let $T_0 = \inf\{t : U(t) > 0\}$ be the time of recovery. For $y > 0$, the distribution of the ascending ladder height of $\{U(t) : t \geq 0\}$ is

$$\mathbb{P}(T_0 < \infty,\ U(T_0) \leq y) \,|\, U(0) = 0) = \mathbb{E}[I(T_0 < \infty,\ U(T_0) \leq y) \,|\, U(0) = 0].$$

Analogously, we define the distribution of the *discounted* ascending ladder height by

$$L(y) = \mathbb{E}[e^{-\delta T_0} I(T_0 < \infty,\ U(T_0) \leq y) \,|\, U(0) = 0]. \tag{6}$$

**Theorem 1.** *When gains follow* $\mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{Q})$*, the discounted ascending ladder height follows* $\mathrm{PH}(\boldsymbol{\alpha}_+, \boldsymbol{Q})$ *where*

$$\boldsymbol{\alpha}'_+ = -\frac{\lambda}{c}\,\boldsymbol{\alpha}'(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}.$$

*When* $\delta > 0$*, the ascending ladder height is defective.*

**Proof:** Equation (3.3) of Gerber and Shiu (1998) states that in the classical Cramér-Lundberg model, the joint (discounted) distribution

$$F(x, y|0) = \mathbb{E}[e^{-\delta T} I(U(T-) \leq x, |U(T)| \leq y) \,|\, U(0) = 0]$$

has density (noting that $R_\delta$ is the negative of $\rho$ defined in Gerber and Shiu (1998))

$$F(x, y|0) = \frac{\lambda}{c}\, e^{R_\delta x} f(x + y) = \frac{\lambda}{c}\,\boldsymbol{\alpha}' e^{(R_\delta \boldsymbol{I} + \boldsymbol{Q})x} e^{\boldsymbol{Q}y} \boldsymbol{t}.$$

To obtain $L'(y)$, which is the same as the discounted descending ladder height for the Cramér-Lundberg model, we need to integrate $x$ out. Since $R_\delta \leq 0$, $R_\delta \boldsymbol{I} + \boldsymbol{Q}$ is a subintensity matrix and is invertible, and the rule

$$\int_0^z e^{\boldsymbol{A}x}\mathrm{d}x = \boldsymbol{A}^{-1}(e^{\boldsymbol{A}z} - \boldsymbol{I})$$

applies for $\boldsymbol{A} = R_\delta \boldsymbol{I} + \boldsymbol{Q}$. Moreover, since $e^{(R_\delta \boldsymbol{I}+\boldsymbol{Q})x} \to \boldsymbol{0}$ as $x \to \infty$,

$$L'(y) = -\frac{\lambda}{c}\,\boldsymbol{\alpha}'(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}e^{\boldsymbol{Q}y}\boldsymbol{t} = \boldsymbol{\alpha}_+'\, e^{\boldsymbol{Q}y}\boldsymbol{t}. \tag{7}$$

Again, since $R_\delta \le 0$, every element in $\boldsymbol{\alpha}_+$ is non-negative. By integrating (7) from 0 to $\infty$, we get $L(\infty) = \boldsymbol{\alpha}_+'\boldsymbol{e}$. But since

$$L(\infty) = \mathbb{E}[e^{-\delta T_0} I(T_0 < \infty) \,|\, U(0) = 0] \le 1,$$

$\boldsymbol{\alpha}_+'\boldsymbol{e} \le 1$. In particular, if $\delta > 0$, then $\boldsymbol{\alpha}_+'\boldsymbol{e} < 1$. □

We assume that $\delta > 0$. Since $L$ follows a defective phase-type distribution, the convolution of $L$ can be thought of as connecting the phases of all ladder heights together to form a terminating "continuous-time" Markov chain along the vertical direction of the graph of $\{U(t)\}$. To find the generator of this process, we consider the transition from phase $i$ to phase $j$. There are two possibilities: if transition occurs within a ladder height, then the rate is the $ij$-th element of $\boldsymbol{Q}$, i.e., $[\boldsymbol{Q}]_{ij}$; if the transition occurs by terminating the current ladder height (with exit rate $\boldsymbol{t}_i$) and then entering another one, the new ladder height has to start at $j$ (which occurs with probability $[\boldsymbol{\alpha}_+]_j$) and the rate is $\boldsymbol{t}_i[\boldsymbol{\alpha}_+]_j$. As a result, the generator is

$$\boldsymbol{Q}_+ = \boldsymbol{Q} + \boldsymbol{t}\boldsymbol{\alpha}_+'. \tag{8}$$

The maximum of $\{U(t) : t \ge 0\}$ given that $U(0) = 0$ then follows PH($\boldsymbol{\alpha}_+$, $\boldsymbol{Q}_+$).

Since $\{U(t)\}$ is skip-free downwards, there are two ways for the process to pass through a particular level $x$: the surplus may be decreasing, in which case the surplus would hit $x$ exactly, or it may be jumping, in which case the surplus would shoot through $x$ and end up at a level that may be above $x$ immediately after the jump. We use "downcrossing" and "upcrossing" to distinguish the two ways for the surplus to pass through $x$.

Let $m_x$ be the phase of the gain that causes the surplus process to *first* upcross level $x$, given that the initial surplus is 0. Futher denote by $T_{-b}$ the first time when $\{U(t)\}$ downcrosses level $-b$, and define

$$p_-(b) = \mathbb{E}[e^{-\delta T_{-b}} I(T_{-b} < T_0,\, T_{-b} < \infty) \,|\, U(0) = 0],$$
$$p_i^+(b) = \mathbb{E}[e^{-\delta T_0} I(T_{-b} > T_0,\, m_0 = i,\, T_0 < \infty) \,|\, U(0) = 0].$$

We denote the discounted upcrossing probability vector $(p_1^+(b), \ldots, p_d^+(b))'$ by $\boldsymbol{p}_+(b)$.

**Theorem 2.** *When gains follow* $\mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{Q})$,

$$\boldsymbol{p}'_+(b) = \boldsymbol{\alpha}'_+[\boldsymbol{I} - e^{(R_\delta \boldsymbol{I} + \boldsymbol{Q}_+)b}]\,[\boldsymbol{I} + (R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}\boldsymbol{\alpha}'_+ e^{(R_\delta \boldsymbol{I}+\boldsymbol{Q}_+)b}]^{-1},$$

$$p_-(b) = e^{R_\delta b}[1 + \boldsymbol{\alpha}'_+(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}]\,[1 + \boldsymbol{\alpha}'_+ e^{(R_\delta \boldsymbol{I}+\boldsymbol{Q}_+)b}(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}]^{-1}.$$

**Proof:** The idea is essentially the same as that in Asmussen and Perry (1992). See also Ng and Yang (2006). Firstly, consider

$$[\boldsymbol{\alpha}_+]_i = \mathbb{E}[e^{-\delta T_0} I(T_0 < \infty, m_0 = i) \,|\, U(0) = 0].$$

Partition the event $\{T_0 < \infty, m_0 = i\}$ into

$$\{T_0 < T_{-b}, T_0 < \infty, m_0 = i\} \bigcup \{T_{-b} < T_0 < \infty, m_0 = i\}. \tag{9}$$

Given that $\{U(t)\}$ downcrosses level $-b$, the discounted probability that it *first* upcrosses level $-b$ again with phase $j$ is $[\boldsymbol{\alpha}_+]_j$. Given that the phase at level $-b$ is $j$, the discounted ladder height of the phase process follows $\mathrm{PH}(\boldsymbol{e}_j, \boldsymbol{Q}_+)$. Thus the probability that it hits 0 again with phase $i$ is $\boldsymbol{e}'_j e^{\boldsymbol{Q}_+ b}\boldsymbol{e}_i$, and

$$[\boldsymbol{\alpha}_+]_i = p_i^+(b) + p_-(b)\, \boldsymbol{\alpha}'_+ e^{\boldsymbol{Q}_+ b}\boldsymbol{e}_i. \tag{10}$$

Secondly, consider $\psi(b) = \mathbb{E}[e^{-\delta T_{-b}} I(T_{-b} < \infty) \,|\, U(0) = 0]$. Partition the event $\{T_{-b} < \infty\}$ into

$$\{T_{-b} < T_0 < \infty\} \bigcup \Big[\bigcup_i \{T_0 < T_{-b} < \infty, m_0 = i\}\Big]. \tag{11}$$

The probability of the first event is $p_-(b)$. For the event $\{T_0 < T_{-b} < \infty, m_0 = i\}$, note that $\{T_0 < T_{-b}, m(0) = i\}$ happens with probability $p_i^+(b)$ and we still need to evaluate $\mathbb{P}(T_{-b} < \infty \,|\, T_0 < T_{-b} < \infty, m_0 = i)$. To this end, given that $\{U(t)\}$ first upcrosses 0 with phase $i$, the first discounted ascending ladder height follows $\mathrm{PH}(\boldsymbol{e}_i, \boldsymbol{Q})$. If it terminates at level $x$, then the discounted probability that it will ever downcross $-b$ is $\psi(b + x)$. Thus,

$$\begin{aligned} \mathbb{P}(T_{-b} < \infty \,|\, T_0 < T_{-b} < \infty, m_0 = i) &= \int_0^\infty e'_i\, e^{\boldsymbol{Q}x}\boldsymbol{t}\psi(b+x)\mathrm{d}x \\ = e'_i \int_0^\infty e^{R_\delta(b+x)} e^{\boldsymbol{Q}x}\boldsymbol{t}\mathrm{d}x &= -\,e^{R_\delta b}\sum_i \boldsymbol{e}'_i(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}, \end{aligned} \tag{12}$$

where the second equality follows from (5). Gathering all the results above, we get

$$\psi(u) = p_-(b) - e^{R_\delta b}\sum_i p_i^+(b)\boldsymbol{e}'_i(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}. \tag{13}$$

Rewrite (10) and (13) in matrix form as follows:

$$\boldsymbol{\alpha}_+' = \boldsymbol{p}_+'(b) + p_-(b)\,\boldsymbol{\alpha}_+'\,e^{\boldsymbol{Q}_+ b}; \tag{14}$$

$$e^{R_\delta b} = p_-(b) - e^{R_\delta b}\boldsymbol{p}_+'(b)\,(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}. \tag{15}$$

Substituting (15) into (14), the first asserted result follows. The second asserted result follows by substituting (14) into (15) and solving for $p_-(b)$. □

**Remarks:**

(1) After calculating $p_-(b)$, we can also use (14) to obtain $\boldsymbol{p}_+(b)$ from

$$\boldsymbol{p}_+'(b) = \boldsymbol{\alpha}_+'\,[\boldsymbol{I} - p_-(b)e^{\boldsymbol{Q}_+ b}]. \tag{16}$$

(2) Owing to its frequent appearance, we shall define

$$\eta(b) = [1 + \boldsymbol{\alpha}_+'\,e^{(R_\delta \boldsymbol{I} + \boldsymbol{Q}_+)b}(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}]^{-1}.$$

We can now rewrite the expressions for $p_-$ and $p_+$ as

$$p_-(b) = \frac{\eta(b)}{\eta(0)}\,e^{R_\delta b} \quad \text{and} \quad \boldsymbol{p}_+'(b) = \boldsymbol{\alpha}_+'\left[\boldsymbol{I} - \frac{\eta(b)}{\eta(0)}\,e^{(R_\delta \boldsymbol{I} + \boldsymbol{Q}_+)b}\right].$$

□

Finally, for $x, y \geq 0$, let $T_x$ be the first time that $\{U(t)\}$ upcrosses $x$. Analogous to the definition of $A(a,b|u) = \mathbb{E}[e^{-\delta T_a} I(T_a < T_b) \,|\, U(0) = u]$ and $B(a,b|u) = \mathbb{E}[e^{-\delta T_b} I(T_b < T_a) \,|\, U(0) = u]$ in Gerber and Shiu (1998) in the Cramér-Lundberg model (where $T_x$ is the first time that $\{U(t)\}$ reaches level $x$ because the process is skip-free upward in this case), we define the double-barrier discounted downcrossing and upcrossing probabilities

$$q_-(x,y) = \mathbb{E}[e^{-\delta T_{-x}} I(T_{-x} < T_y,\ T_{-x} < \infty) \,|\, U(0) = 0]$$

and

$$q_i^+(x,y) = \mathbb{E}[e^{-\delta T_x} I(T_x < T_{-y},\ m_x = i,\ T_x < \infty) \,|\, U(0) = 0].$$

Here, $q_-(x,y)$ is the expected present value of a contingent claim of 1 that is made when the surplus downcrosses $-x$ for the first time, provided that it has not upcrossed $y$ in the meantime. Similarly, $q_i^+(x,y)$ is the expected present value of a contingent claim of 1 that is made when the suplus upcrosses $x$ for the first time, provided that that it has not downcrossed $-y$ in the meantime, and when the surplus upcrosses $x$, the phase of the gain is $i$.

As in the case of $\boldsymbol{p}_+$, we let $\boldsymbol{q}_+(x,y) = (q_1^+(x,y),\ q_2^+(x,y),\ \ldots,\ q_d^+(x,y))'$.

**Theorem 3.** *The double-barrier discounted probabilities are given by*

$$\boldsymbol{q}_+'(x,y) = \boldsymbol{\alpha}_+' e^{\boldsymbol{Q}_+ x}[\boldsymbol{I} - p_-(x+y)e^{\boldsymbol{Q}_+ y}/p_-(x)] \text{ and } q_-(x,y) = p_-(x+y)/p_-(y).$$

*In particular, for* $0 < u < b$,

$$\boldsymbol{q}_+'(b-u,u) = \boldsymbol{\alpha}_+' e^{\boldsymbol{Q}_+(b-u)} - \frac{\eta(b)e^{R_\delta u}}{\eta(b-u)} \boldsymbol{\alpha}_+' e^{\boldsymbol{Q}_+ b} \text{ and } q_-(u,b-u) = \frac{\eta(b)e^{R_\delta u}}{\eta(b-u)}.$$

**Proof:** Let $x, y \geq 0$. It is obvious that $p_-(x+y) = p_-(y)\, q_-(x,y)$ because $\{U(t)\}$ is skip-free downwards. By considering whether $\{U(t)\}$ will upcross 0 before downcrossing $-x$, we obtain

$$p_i^+(x+y) = p_i^+(x) + p_-(x)\, q_i^+(x,y).$$

Writing in matrix form and using (16), we get the expression for $\boldsymbol{q}_+(x,y)$. The remaining assertions follow from direct substitution. □

## 3. Barrier Strategy

We begin with the dynamics of the surplus process (modified due to dividend payments) $\{W(t)\}$ under a barrier strategy with barrier level $b$. Whenever the surplus upcrosses $b$ due to the arrival of a gain, the excess is paid out immediately as dividends. Let $D(t;b)$ denote the aggregate (undiscounted) dividends by time $t$. Mathematically,

$$W(t) = u - ct + \sum_{i=1}^{N(t)} X_i - D(t;b),$$

where

$$D(t;b) = \max_{0 \leq \tau \leq t} \left( u - c\tau + \sum_{i=1}^{N(\tau)} X_i - b \right)_+ .$$

Let $T = \inf\{t : W(t) = 0\}$ be the time of ruin. The Laplace transfom of $T$ is $\psi(u;b) = \mathbb{E}[e^{-\delta T} \,|\, W(0) = u]$, while the expected total discounted dividends until ruin is

$$V(u;b) = \mathbb{E}\left[ \int_0^T e^{-\delta t} \,\mathrm{d}D(t;b) \,\middle|\, W(0) = u \right].$$

### 3.1. Explicit Formulas for $V(u;b)$ and $\psi(u;b)$

**Theorem 4.** *The expected total discounted dividends until ruin under a barrier strategy is given by*

$$V(u;b) = \begin{cases} \boldsymbol{q}'_+(b-u,u)[V(b;b)\boldsymbol{I}-\boldsymbol{Q}^{-1}]\boldsymbol{e} & \text{for } u<b \\ -\dfrac{\boldsymbol{p}'_+(b)\boldsymbol{Q}^{-1}\boldsymbol{e}}{1-\boldsymbol{p}'_+(b)\boldsymbol{e}} & \text{for } u=b. \\ u-b+V(b;b) & \text{for } u>b \end{cases}$$

**Proof:** The case for $u > b$ is immediate. For $u \le b$, $D(T;\, b) > 0$ only if the surplus upcrosses $b$, and if this ever happens, the excess of the surplus is paid out and the surplus is brought back to level $b$. By conditioning on the phase of the gain when the surplus process upcrosses $b$ and noting that PH$(\boldsymbol{e}_i, \boldsymbol{Q})$ has mean $-\boldsymbol{e}_i'\boldsymbol{Q}^{-1}\boldsymbol{e}$, we have

$$V(u;b) = \sum_i q_i^+(b-u,u)\left[-\boldsymbol{e}_i'\boldsymbol{Q}^{-1}\boldsymbol{e} + V(b;b)\right], \tag{17}$$

which gives the result for $u < b$. By putting $u = b$ and noting that $\boldsymbol{q}_+(0,b) = \boldsymbol{p}_+(b)$, we can solve for $V(b;b)$. □

Similarly, we have an explicit expression for $\psi(u;b)$ under the barrier strategy:

**Theorem 5.** *The Laplace transform of the time of ruin under a barrier strategy is given by*

$$\psi(u;b) = \begin{cases} q_-(u,b-u) + \psi(b;b)\,\boldsymbol{q}'_+(b-u,u)\boldsymbol{e} & \text{for } u<b \\ \dfrac{p_-(b)}{1-\boldsymbol{p}'_+(b)\boldsymbol{e}} & \text{for } u\ge b \end{cases}.$$

**Proof:** First we consider the case $u = b$. The (discounted) probability that ruin occurs before upcrossing $b$ is $p_-(b)$. The Laplace transform of the first time for the surplus process to reach $b$ before ruin is $\boldsymbol{p}'_+(b)\boldsymbol{e}$. Thus, we obtain

$$\psi(b;b) = p_-(b) + \psi(b;b)\,\boldsymbol{p}'_+(b)\boldsymbol{e},$$

which gives the expression for $\psi(b;b)$. For $u < b$, we replace $p_-(b)$ by $q_-(u, b-u)$ and $\boldsymbol{p}_+(b)$ by $\boldsymbol{q}_+(b-u, u)$ and repeat the same argument. □

Now we shall consider the particular case when gains are exponentially distributed.

### 3.2. Exponentially Distributed Gains

Suppose that gains are exponential with mean $1/\beta$. We first determine $R_\delta$, which is the unique negative root of

$$c\theta^2 + (\lambda - c\beta + \delta)\theta - \beta\delta = 0. \tag{18}$$

For notational convenience, we denote the negative and positive root of (18) by $r$ and $s$, respectively. Then

$$\alpha_+ = -\frac{\lambda}{c}(r-\beta)^{-1} = \frac{\lambda}{c(\beta - r)} = 1 + \frac{\delta}{cr},$$

which is less than 1, and

$$Q_+ = -\beta + \beta\alpha_+ = \frac{\beta\delta}{cr} = \frac{cr^2 + (\lambda - c\beta + \delta)r}{cr} = r - (r+s) = -s.$$

By using the expressions for $\alpha_+$, $Q_+$ and (18), $\eta(u)$ and the various discounted upcrossing and downcrossing probabilities are found to be

$$\eta(u) = \frac{(\beta - r)e^{-ru}}{(\beta - r)e^{-ru} - (\beta - s)e^{-su}},$$

$$p_-(u) = \frac{s - r}{(\beta - r)e^{-ru} - (\beta - s)e^{-su}},$$

$$p_+(u) = \frac{\lambda}{c}\,\frac{e^{-ru} - e^{-su}}{(\beta - r)e^{-ru} - (\beta - s)e^{-su}},$$

$$q_-(u, b-u) = \frac{(\beta - r)e^{-r(b-u)} - (\beta - s)e^{-s(b-u)}}{(\beta - r)e^{-rb} - (\beta - s)e^{-sb}},$$

$$q_+(b-u, u) = \frac{\lambda}{c}\,\frac{e^{-rb-(b-u)s} - e^{-sb-(b-u)r}}{(\beta - r)e^{-rb} - (\beta - s)e^{-sb}}.$$

Thus,

$$V(b;b) = \frac{p_+(b)}{\beta[1 - p_+(b)]} = \frac{\lambda}{\beta}\,\frac{e^{-rb} - e^{-sb}}{(\delta + cs)e^{-rb} - (\delta + cr)e^{-sb}}$$

and it follows that for $0 < u < b$,

$$V(u;b) = q_+(b-u,u)[V(b;b) + \beta^{-1}] = \frac{\lambda}{\beta}\,\frac{e^{-rb-(b-u)s} - e^{-sb-(b-u)r}}{(\delta + cs)e^{-rb} - (\delta + cr)e^{-sb}},$$

which is the same as (3.5) of Avanzi et al. (2007).

For $\psi(u;b)$, we first evaluate

$$\psi(b;b) = \frac{p_-(b)}{1 - p_+(b)} = \frac{c(s-r)}{(\delta + cs)e^{-rb} - (\delta + cr)e^{-sb}}$$

and then obtain

$$\psi(u;b) = \frac{(\delta+cs)(\beta-r)e^{-2rb+ru}+(\delta+cr)(\beta-s)e^{-2sb+su}-\lambda(se^{ru}+re^{su})e^{-(r+s)b}}{[(\delta+cs)e^{-rb}-(\delta+cr)e^{-sb}][(\beta-r)e^{-rb}-(\beta-s)e^{-sb}]}$$

for $0 < u < b$.

## 4. Threshold Strategy

We begin with the dynamics of the surplus process $\{W(t)\}$ under a threshold strategy with a threshold level $b$. When $W(t) < b$, no dividends are paid and the surplus decreases at rate $c_1$. But when $W(t) \geq b$, the surplus would decrease at a different rate $c_2 > c_1$ and dividends are paid continuously at rate $c_2 - c_1$. Mathematically,

$$\mathrm{d}W(t) = -c(W(t))\,\mathrm{d}t + \mathrm{d}S(t),$$

where

$$c(x) = \begin{cases} c_1 & \text{when } x \leq b \\ c_2 & \text{when } x > b \end{cases}, \quad \text{and} \quad S(t) = \sum_{i=1}^{N(t)} X_i\,.$$

The expected total discounted dividends until ruin is

$$V(u;b) = \mathbb{E}\left[(c_2-c_1)\int_0^T e^{-\delta t} I(U(t) > b)\,\mathrm{d}t \,\middle|\, W(0) = u\right],$$

while the Laplace transfom of $T$ is $\psi(u;b) = \mathbb{E}[e^{-\delta T} \mid W(0) = u]$.

### 4.1. Explicit Formulas for $V(u;b)$ and $\psi(u;b)$

First we derive an explicit formula for $V(u;b)$ when gains follow $\mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{Q})$. Since $\boldsymbol{p}_+$, $p_-$, $\boldsymbol{q}_+$ and $q_-$ are all functions of the rate of expense and the unique non-positive root of $\kappa(\theta) = \delta$ (where $\kappa$ depends on the rate of expense), we shall distinguish the two sets of probabilities and other quantities of interest by the superscripts (1) and (2). When ($i$) appears, the quantity of interest is evaluated using rate of expense $c_i$.

**Theorem 6.** *The expected total discounted dividends until ruin under a threshold strategy is given by*

$$V(u;b) = \begin{cases} \boldsymbol{q}_+^{(1)\prime}(b-u,u)\boldsymbol{g}(b) & \textit{for } 0 \leq u \leq b \\ \boldsymbol{p}_+^{(1)\prime}(b)\boldsymbol{g}(b) & \textit{for } u = b \\ \dfrac{c_2-c_1}{\delta}(1-e^{R_\delta^{(2)}(u-b)}) + e^{R_\delta^{(2)}(u-b)}\boldsymbol{p}_+^{(1)\prime}(b)\boldsymbol{g}(b) & \textit{for } u > b \end{cases}$$

*where*

$$\boldsymbol{g}(b) = \frac{c_2 - c_1}{\delta}\left(\boldsymbol{I} - \frac{(R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}\boldsymbol{p}_+^{(1)\prime}(b)}{1 + \boldsymbol{p}_+^{(1)\prime}(b)(R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}}\right)\left[(R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1} - \boldsymbol{Q}^{-1}\right]\boldsymbol{t}.$$

**Proof:** Consider the calculation of $V$ when $u \le b$. Continuous dividends will be paid as soon as the surplus upcrosses $b$, provided that ruin does not occur in the meantime. The discounted probability of such event, provided that the upcrossing occurs at phase $i$, is the $i$-th element of $\boldsymbol{q}_+^{(1)\prime}(b-u,u)$.

If we let $g_i(b)$ be the expected value of total discounted dividends, provided that at time 0, a gain arrives and causes the surplus to upcross $b$ at phase $i$, and denote by $\boldsymbol{g}(b)$ the vector formed by $g_i(b)$'s, then we have $V(u;b) = \boldsymbol{q}_+^{(1)\prime}(b-u,u)\boldsymbol{g}(b)$. Substitution of $u = b$ then gives $V(b;b) = \boldsymbol{p}_+^{(1)\prime}(b)\boldsymbol{g}(b)$. Moreover, since it follows from (3) and (5) that for $x \ge 0$,

$$V(b+x;b) = \frac{c_2 - c_1}{\delta}\left(1 - e^{R_\delta^{(2)}x}\right) + e^{R_\delta^{(2)}x}V(b;b), \tag{19}$$

we only need to solve for $\boldsymbol{g}(b)$ in order to obtain $V(u;b)$.

By conditioning on the amount of overshoot upon the upcrossing of $b$, we have

$$\boldsymbol{g}(b) = \int_0^\infty e^{\boldsymbol{Q}x}V(b+x;b)\,\mathrm{d}x\,\boldsymbol{t}. \tag{20}$$

By substituting (19) into (20), we get

$$\boldsymbol{g}(b) = \frac{c_2 - c_1}{\delta}\left[(R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t} - \boldsymbol{Q}^{-1}\boldsymbol{t}\right] - (R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}\boldsymbol{p}_+^{(1)\prime}(b)\boldsymbol{g}(b).$$

Finally, by noting that

$$\left[\boldsymbol{I} + (R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}\boldsymbol{p}_+^{(1)\prime}(b)\right]^{-1} = \boldsymbol{I} - \frac{(R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}\boldsymbol{p}_+^{(1)\prime}(b)}{1 + \boldsymbol{p}_+^{(1)\prime}(b)(R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}},$$

we get the expression for $\boldsymbol{g}(b)$, and the expression for $V(u;b)$ can then be obtained readily. □

**Theorem 7.** *The Laplace transform of the time of ruin under a threshold strategy is given by*

$$\psi(u;b) = \begin{cases} q_-^{(1)}(u, b-u) \\ -\boldsymbol{q}_+^{(1)\prime}(b-u,u)(R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}\,\psi(b;b) & \text{for } 0 \le u < b \\ \dfrac{p_-^{(1)}(b)}{1 + \boldsymbol{p}_+^{(1)\prime}(b)(R_\delta^{(2)}\boldsymbol{I} + \boldsymbol{Q})^{-1}\boldsymbol{t}} & \text{for } u = b \\ e^{R_\delta^{(2)}(u-b)}\psi(b;b) & \text{for } u > b \end{cases}.$$

**Proof:** We separate the proof into two cases. First, we assume that $u > b$. For ruin to occur, the surplus must first downcross $b$. If this ever happens at time $T^{(2)}_{b-u}$, the (discounted) ruin probability is simply $e^{-\delta T^{(2)}_{b-u}}\psi(b;b)$. Taking expectation,

$$\psi(u;b) = \psi^{(2)}(u-b)\psi(b;b) = e^{R^{(2)}_\delta(u-b)}\psi(b;b).$$

Then, we assume that $u \le b$. In this case, if ruin happens, it may happen before or after $\{W(t)\}$ first upcrosses $b$. For the latter event, we can further condition on the state of the phase of gain that causes $\{W(t)\}$ to upcross $b$. Gathering all these together, we have

$$\begin{aligned}\psi(u;b) &= q^{(1)}_{-}(u,b-u) + \boldsymbol{q}^{(1)\prime}_{+}(b-u,u)\int_0^\infty e^{x\boldsymbol{Q}}\boldsymbol{t}\psi(b+x;b)\,\mathrm{d}x \\ &= q^{(1)}_{-}(u,b-u) - \boldsymbol{q}^{(1)\prime}_{+}(b-u,u)(R^{(2)}_\delta\boldsymbol{I}+\boldsymbol{Q})^{-1}\boldsymbol{t}\psi(b;b).\end{aligned}$$

Substituting $b = u$ into the above, we obtain the expression for $\psi(b;b)$. □

It is interesting to note the similarity between the formulas for $\psi(u;b)$ under barrier strategy and those under threshold strategy. Replacing $\boldsymbol{e}$ by $(R^{(2)}_\delta\boldsymbol{I}+\boldsymbol{Q})^{-1}\boldsymbol{t}$ in Theorem 5 gives the corresponding discounted ruin probability when $0 \le u \le b$.

Now we shall consider the particular case when gains are exponential.

### 4.2. Exponentially Distributed Gains

We assume that gains are exponentially distributed with mean $1/\beta$. We let the positive and negative root of $\kappa(\theta) = \delta$ when the expense rate is $c_i$ be $s_i$ and $r_i$ for $i = 1$ and 2. Since there is only one phase for the gains,

$$\begin{aligned}g(b) &= \frac{c_2-c_1}{\delta}\,\frac{1}{1+p^{(1)}_{+}(b)(r_2-\beta)^{-1}\beta}\left[(r_2-\beta)^{-1}+\beta^{-1}\right]\beta \\ &= \frac{(c_2-c_1)(-r_2)}{\delta}\,\frac{(\beta-r_1)e^{-r_1 b}-(\beta-s_1)e^{-s_1 b}}{(s_1-r_2)(\beta-r_1)e^{-r_1 b}-(r_1-r_2)(\beta-s_1)e^{-s_1 b}}.\end{aligned}$$

Then it follows from some simple algebra that

$$V(u;b) = \frac{\lambda(c_2-c_1)(-r_2)}{c_1\delta}\,\frac{e^{s_1 u}-e^{r_1 u}}{(s_1-r_2)(\beta-r_1)e^{s_1 b}-(r_1-r_2)(\beta-s_1)e^{r_1 b}}$$

for $u \le b$. This result is identical to that obtained from solving integro-differential equations as in Section 2.1 of Ng (2009).

For the calculation of $\psi(u;b)$, we obtain from Theorem 7 that for $u \geq b$,

$$\psi(u;b) = \frac{(\beta - r_2)(s_1 - r_1)}{(s_1 - r_2)(\beta - r_1)e^{-r_1 b} - (r_1 - r_2)(\beta - s_1)e^{-s_1 b}}\, e^{r_2(u-b)},$$

and for $0 \leq u < b$,

$$\psi(u;b) =$$

$$\frac{(\beta - r_1)\,e^{-r_1(b-u)} - (\beta - s_1)\,e^{-s_1(b-u)}}{(\beta - r_1)\,e^{-r_1 b} - (\beta - s_1)\,e^{-s_1 b}} + \frac{\lambda(\beta - r_1)(\beta - s_1)(e^{s_1 u} - e^{r_1 u})}{(\beta - r_1)e^{s_1 b} - (\beta - s_1)e^{r_1 b}} \cdot \frac{\psi(b;b)}{\beta - r_2}.$$

## 5. A Hybrid Strategy with a Dividend Threshold and a Dividend Barrier

As a further application of the usefulness of the two pairs of upcrossing and downcrossing probabilities, we consider a dividend strategy that can be treated as a hybrid of the models discussed in Sections 3 and 4. Let $b_1$ be a threshold and $b_3 = b_1 + b_2$ be a barrier. Here $b_1$ and $b_3$ function as a refracting and reflecting barrier, respectively. When the surplus is less than $b_1$, it decreases at rate $c_1$ and no dividends are paid. When the surplus is in between $b_1$ and $b_3$, dividends are paid at a constant rate $c_2 - c_1 > 0$ so that it decreases at rate $c_2$. When a gain causes the surplus to upcross $b_3$, dividends are paid to bring the surplus down to $b_3$.

The hybrid dividend strategy introduced above is a generalization of a pure barrier strategy and a pure threshold strategy. It is more realistic than a pure barrier strategy because it may not be desirable for companies to use a switching mechanism of either paying nothing or paying all excess surplus as dividends. The hybrid strategy gives a smooth transition between the two states. At the same time, it is more realistic than a pure threshold strategy because it is unnatural for the surplus of a company or a line of business to be allowed to grow infinitely. When the surplus exceeds a certain level, they will be allocated to other lines of business or paid out to shareholders.

We denote the expected total discounted dividends until ruin by $V(u;b_1,b_2)$ and the Laplace transform of the time of ruin by $\psi(u;b_1,b_2)$. We shall separate $V(u;b_1,b_2)$ into two parts: the discrete part $V_d(u;b_1,b_2)$ denotes the expected total discounted dividends paid out due to upcrossing $b_3$, and the continuous part $V_c(u;b_1,b_2)$ denotes the expected total discounted dividends paid out when the surplus is in between $b_1$ and $b_3$. Obviously, for $u > b_3$,

$$\begin{aligned} V_d(u;b_1,b_2) &= u - b_3 + V_d(b_3;b_1,b_2), \\ V_c(u;b_1,b_2) &= V_c(b_3;b_1,b_2), \\ \psi(u;b_1,b_2) &= \psi(b_3;b_1,b_2). \end{aligned} \tag{21}$$

Thus, we only consider $0 < u \le b_3$ in the following.

Before discussing how $V(u;b_1,b_2)$ and $\psi(u;b_1,b_2)$ can be obtained, we temporarily go back to the case of a dual model without any dividend strategy imposed and establish two auxiliary results.

**Lemma 8.** *For $x, y > 0$, $\mathbb{E}[e^{-\delta(T_{-x} \wedge T_y)} | U(0) = 0] = q_-(x,y) + \boldsymbol{q}_+'(y,x)\boldsymbol{e}$ .*

**Proof:** By noting whether $T_{-x}$ is less than $T_y$ or not, we get

$$\begin{aligned}
&\mathbb{E}[e^{-\delta(T_{-x} \wedge T_y)} | U(0) = 0] \\
&= \mathbb{E}[e^{-\delta T_{-x}} I(T_{-x} < T_y) \,|\, U(0) = 0] + \mathbb{E}[e^{-\delta T_y} I(T_{-x} > T_y) \,|\, U(0) = 0] \\
&= q_-(x,y) + \sum_{i=1}^{d} \mathbb{E}[e^{-\delta T_y} I(T_{-x} > T_y,\, m_y = i) \,|\, U(0) = 0] \\
&= q_-(x,y) + \boldsymbol{q}_+'(y,x)\boldsymbol{e}.
\end{aligned}$$

□

**Lemma 9.** *Let $\boldsymbol{\pi}$ and $\boldsymbol{k}$ be two column vectors. If $\boldsymbol{Q}$ and $\boldsymbol{Q}_+$ have no common eigenvalues, then*

$$\begin{aligned}
&\boldsymbol{\pi}' \int_0^b e^{\boldsymbol{Q}x} \boldsymbol{t} q_-(x, b-x)\,\mathrm{d}x \\
&= \eta(b)\Big[\boldsymbol{\pi}'(R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1} (e^{(R_\delta \boldsymbol{I} + \boldsymbol{Q})b} - \boldsymbol{I})\boldsymbol{t} \\
&\quad + e^{R_\delta b}(\boldsymbol{\pi}' \otimes \boldsymbol{\alpha}_+' e^{\boldsymbol{Q}_+ b}) [\boldsymbol{Q} \oplus (-\boldsymbol{Q}_+)]^{-1} (e^{[\boldsymbol{Q} \oplus (-\boldsymbol{Q}_+)]b} - \boldsymbol{I})(t \otimes (R_\delta \boldsymbol{I} + \boldsymbol{Q})^{-1} \boldsymbol{t})\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{\pi}' \int_0^b e^{\boldsymbol{Q}x} \boldsymbol{t} \boldsymbol{q}_+'(b-x, x)\,\mathrm{d}x\, \boldsymbol{k} \\
&= (\boldsymbol{\pi}' \otimes \boldsymbol{\alpha}_+' e^{\boldsymbol{Q}_+ b})[\boldsymbol{Q} \oplus (-\boldsymbol{Q}_+)]^{-1} (e^{[\boldsymbol{Q} \oplus (-\boldsymbol{Q}_+)]b} - \boldsymbol{I})(\boldsymbol{t} \otimes \boldsymbol{k}) \\
&\quad - (\boldsymbol{\pi}' \int_0^b e^{\boldsymbol{Q}x} \boldsymbol{t} q_-(x, b-x)\,\mathrm{d}x)\, \boldsymbol{\alpha}_+' e^{\boldsymbol{Q}_+ b} \boldsymbol{k}.
\end{aligned}$$

**Proof:** The assumption on $\boldsymbol{Q}$ and $\boldsymbol{Q}_+$ guarantees that $\boldsymbol{Q} \oplus (-\boldsymbol{Q}_+)$ is invertible. The lemma can be proven by using the integration rule

$$\int_0^z \boldsymbol{\alpha}_1' e^{\boldsymbol{A}_1 x} \boldsymbol{\beta}_1 \otimes \boldsymbol{\alpha}_2' e^{\boldsymbol{A}_2 x} \boldsymbol{\beta}_2 \,\mathrm{d}x = (\boldsymbol{\alpha}_1 \otimes \boldsymbol{\alpha}_2)' (\boldsymbol{A}_1 \oplus \boldsymbol{A}_2)^{-1} (e^{(\boldsymbol{A}_1 \oplus \boldsymbol{A}_2)z} - \boldsymbol{I})(\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2)$$

and the relation $\boldsymbol{q}_+'(b-u, u) = \boldsymbol{\alpha}_+' e^{\boldsymbol{Q}_+(b-u)} - q_-(u, b-u)\boldsymbol{\alpha}_+' e^{\boldsymbol{Q}_+ b}$, which is a direct consequence of Theorem 3. □

### 5.1. Explicit formula for $V(u; b_1, b_2)$

In this section we shall derive set of equations to calculate $V(u;b_1,b_2)$. We shall first treat the discrete part $V_d(u;b_1,b_2)$ and derive two equations from which $V_d(b_1; b_1,b_2)$ and $V_d(b_3; b_1,b_2)$ can be evaluated using Lemma 9. With the two pivoting values, $V_d(u;b_1,b_2)$ can be obtained for any $u$. This will be followed by a parallel treatment for the continuous part $V_c(u;b_1,b_2)$.

In the following, we assume the technical condition that $\boldsymbol{Q} \oplus (-\boldsymbol{Q}_+^{(2)})$ is invertible.

**The discrete part $V_d(u; b_1, b_2)$:**

We shall consider two cases, namely, $b_1 < u \le b_3$, and $0 < u \le b_1$.

For $b_1 < u \le b_3$, no discrete dividends are paid until the surplus first upcrosses $b_3$. If the surplus downcrosses $b_1$ before upcrossing $b_3$, then the expected total future discounted discrete dividends at the time of downcrossing is $V_d(b_1; b_1,b_2)$. If the surplus upcrosses $b_3$ before downcrossing $b_1$, a dividend is paid at the time of the upcrossing to bring the surplus back to $b_3$ and the expected total future discounted discrete dividends at that time is $V_d(b_3; b_1,b_2)$. By using a similar argument as in Theorem 4, we obtain

$$\begin{aligned} V_d(b_1+y;b_1,b_2) = {} & q_-^{(2)}(y,b_2-y)\,V_d(b_1;b_1,b_2) \\ & + \boldsymbol{q}_+^{(2)\prime}(b_2-y,y)\big[-\boldsymbol{Q}^{-1}\boldsymbol{e} + \boldsymbol{e}V_d(b_3;b_1,b_2)\big] \end{aligned} \tag{22}$$

for $0 < y \le b_2$. Putting $y = b_2$, we obtain the first relation between $V_d(b_1; b_1,b_2)$ and $V_d(b_3; b_1,b_2)$:

$$\begin{aligned} V_d(b_3;b_1,b_2) = {} & p_-^{(2)}(b_2)\,V_d(b_1;b_1,b_2) \\ & + p_+^{(2)\prime}(b_2)\big[-\boldsymbol{Q}^{-1}\boldsymbol{e} + \boldsymbol{e}V_d(b_3;b_1,b_2)\big]. \end{aligned} \tag{23}$$

For $0 < u \le b_1$, we consider if the surplus downcrosses 0 before upcrossing the threshold $b_1$. In the former case, the total discouted discrete dividends is 0. In the latter case, we further consider if the overshoot above $b_1$ is greater than $b_2$ so that the surplus upcrosses the barrier $b_3$ and a dividend is paid at the time of the upcrossing. Since the amount of the overshoot above $b_1$ follows PH($\boldsymbol{q}_+^{(1)}(b_1 - u, u)$, $\boldsymbol{Q}$), we have

$$\begin{aligned} V_d(u;b_1,b_2) = {} & \int_0^{b_2} V_d(b_1+x;b_1,b_2)\boldsymbol{q}_+^{(1)\prime}(b_1-u,u)e^{\boldsymbol{Q}x}\boldsymbol{t}\,\mathrm{d}x \\ & + \boldsymbol{q}_+^{(1)\prime}(b_1-u,u)e^{\boldsymbol{Q}b_2}\big[-\boldsymbol{Q}^{-1}\boldsymbol{e} + V_d(b_3;b_1,b_2)\boldsymbol{e}\big]. \end{aligned} \tag{24}$$

In particular, if $u = b_1$,

$$V_d(b_1;b_1,b_2) = \int_0^{b_2} V_d(b_1+x;b_1,b_2)\,\boldsymbol{p}_+^{(1)\prime}(b_1)e^{\boldsymbol{Q}x}\boldsymbol{t}\,\mathrm{d}x \\ +\boldsymbol{p}_+^{(1)\prime}(b_1)e^{\boldsymbol{Q}b_2}\left[-\boldsymbol{Q}^{-1}\boldsymbol{e}+V_d(b_3;b_1,b_2)\boldsymbol{e}\right]. \tag{25}$$

Substituting (22) into (25), we get

$$\begin{aligned} V_d(b_1;b_1,b_2) = {} & \boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}q_-^{(2)}(x,b_2-x)\,\mathrm{d}x\,V_d(b_1;b_1,b_2) \\ & -\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\,\boldsymbol{Q}^{-1}\boldsymbol{e}\,\mathrm{d}x \\ & +\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\boldsymbol{e}\,\mathrm{d}x\,V_d(b_3;b_1,b_2) \\ & +\boldsymbol{p}_+^{(1)\prime}(b_1)e^{\boldsymbol{Q}b_2}\left[-\boldsymbol{Q}^{-1}\boldsymbol{e}+V_d(b_3;b_1,b_2)\boldsymbol{e}\right]. \end{aligned}$$

Upon rearrangement, we get another relation between $V_d(b_1;b_1,b_2)$ and $V_d(b_3;b_1,b_2)$:

$$\begin{aligned} & \left(\boldsymbol{p}_+^{(1)\prime}(b_1)e^{\boldsymbol{Q}b_2}\boldsymbol{e}+\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\boldsymbol{e}\,\mathrm{d}x\right)V_d(b_3;b_1,b_2) \\ & \quad -\left(1-\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_-^{(2)}(x,b_2-x)\,\mathrm{d}x\right)V_d(b_1;b_1,b_2) \\ & = \boldsymbol{p}_+^{(1)\prime}(b_1)\,\boldsymbol{Q}^{-1}e^{\boldsymbol{Q}b_2}\boldsymbol{e}+\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\,\boldsymbol{Q}^{-1}\boldsymbol{e}\,\mathrm{d}x. \end{aligned} \tag{26}$$

By Lemma 9, all three integrals in (26) can be evaluated. Together with (23), $V_d(b_1;b_1,b_2)$ and $V_d(b_3;b_1,b_2)$ can be solved. An explicit expression for $V_d$ for $b_1 < u < b_3$ is then obtained from (22). For $0 < u < b_1$, we substitute (22) into (24) and get

$$\begin{aligned} V_d(u;b_1,b_2) = {} & \boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}q_-^{(2)}(x,b_2-x)\,\mathrm{d}x\,V_d(b_1;b_1,b_2) \\ & -\boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\,\boldsymbol{Q}^{-1}\boldsymbol{e}\,\mathrm{d}x \\ & +\boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\boldsymbol{e}\,\mathrm{d}x\,V_d(b_3;b_1,b_2) \\ & +\boldsymbol{q}_+^{(1)\prime}(b_1-u,u)e^{\boldsymbol{Q}b_2}\left[-Q^{-1}\boldsymbol{e}+V_d(b_3;b_1,b_2)\boldsymbol{e}\right]. \end{aligned}$$

**The continuous part $V_c(u;b_1,b_2)$:**

We again look at the case of $b_1 < u \le b_3$ first. Continuous dividends are paid before the first time the surplus downcrosses $b_1$ or upcrosses $b_3$. If the surplus upcrosses $b_3$ before downcrossing $b_1$, continuous dividends would not stop and the total expected future discounted dividends at that time is $V_c(b_3; b_1, b_2)$. If the surplus downcrosses $b_1$ before upcrossing $b_3$, the dividend payment stops and the total expected future discounted continuous dividends at that time is $V_c(b_1; b_1, b_2)$. For $0 < y \le b_2$, by combining the arguments in (19) and Lemma 8, we have

$$\begin{aligned} V_c(b_1+y;b_1,b_2) &= \frac{c_2-c_1}{\delta}\left[1-q_-^{(2)}(y,b_2-y)-\boldsymbol{q}_+^{(2)\prime}(b_2-y,y)\boldsymbol{e}\right] \\ &\quad + q_-^{(2)}(y,b_2-y)\,V_c(b_1;b_1,b_2)+\boldsymbol{q}_+^{(2)\prime}(b_2-y,y)\,\boldsymbol{e}V_c(b_3;b_1,b_2) \\ &= \frac{c_2-c_1}{\delta}+\left[V_c(b_1;b_1,b_2)-\frac{c_2-c_1}{\delta}\right]q_-^{(2)}(y,b_2-y) \\ &\quad +\left[V_c(b_3;b_1,b_2)-\frac{c_2-c_1}{\delta}\right]\boldsymbol{q}_+^{(2)\prime}(b_2-y,y)\,\boldsymbol{e}. \end{aligned} \tag{27}$$

Putting $y = b_2$ and rearranging terms, we get the first relation between $V_c(b_1; b_1, b_2)$ and $V_c(b_3; b_1, b_2)$:

$$\begin{aligned} & p_-^{(2)}(b_2)\,V_c(b_1;b_1,b_2)-\left[1-\boldsymbol{p}_+^{(2)\prime}(b_2)\boldsymbol{e}\right]V_c(b_3;b_1,b_2) \\ &= -\frac{c_2-c_1}{\delta}\left[1-p_-^{(2)}(b_2)-\boldsymbol{p}_+^{(2)\prime}(b_2)\boldsymbol{e}\right]. \end{aligned} \tag{28}$$

Then, we consider the case of $0 < u \le b_1$. Similar to the discrete case,

$$\begin{aligned} & V_c(u;b_1,b_2) \\ &= \boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\left(\int_0^{b_2} V_c(b_1+x,\,b_1,b_2)e^{\boldsymbol{Q}x}\boldsymbol{t}\,\mathrm{d}x+\,V_c(b_3;b_1,b_2)e^{\boldsymbol{Q}b_2}\boldsymbol{e}\right). \end{aligned} \tag{29}$$

By using (27),

$$\begin{aligned} & \int_0^{b_2} V_c(b_1+x,b_1,b_2)\,e^{\boldsymbol{Q}x}\boldsymbol{t}\,\mathrm{d}x \\ &= \frac{c_2-c_1}{\delta}(\boldsymbol{e}-e^{\boldsymbol{Q}b_2}\boldsymbol{e})+\left[V_c(b_1;b_1,b_2)-\frac{c_2-c_1}{\delta}\right]\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}q_-^{(2)}(x,b_2-x)\,\mathrm{d}x \\ &\quad +\left[V_c(b_3;b_1,b_2)-\frac{c_2-c_1}{\delta}\right]\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\,\boldsymbol{e}\,\mathrm{d}x. \end{aligned} \tag{30}$$

Substituting (30) into (29) gives

$$
\begin{aligned}
&V_c(u;b_1,b_2)\\
&= \frac{c_2-c_1}{\delta}\,\boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\,[\boldsymbol{e}-e^{\boldsymbol{Q}b_2}\boldsymbol{e}]\\
&\quad+\left[V_c(b_1;b_1,b_2)-\frac{c_2-c_1}{\delta}\right]\boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_0^{b_2}e^{\boldsymbol{Q}x}\boldsymbol{t}q_-^{(2)}(x,b_2-x)\,\mathrm{d}x\\
&\quad+\left[V_c(b_3;b_1,b_2)-\frac{c_2-c_1}{\delta}\right]\boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_0^{b_2}e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\boldsymbol{e}\,\mathrm{d}x\\
&\quad+V_c(b_3;b_1,b_2)\,\boldsymbol{q}_+^{(1)\prime}(b_1-u,u)e^{\boldsymbol{Q}b_2}\boldsymbol{e}.
\end{aligned}
\tag{31}
$$

Putting $u=b_1$ and rearranging terms, we get

$$
\begin{aligned}
&\left(\boldsymbol{p}_+^{(1)\prime}(b_1)e^{\boldsymbol{Q}b_2}\boldsymbol{e}+\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2}e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\,\boldsymbol{e}\mathrm{d}x\right)V_c(b_3;b_1,b_2)\\
&\quad-\left(1-\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2}q_-^{(2)}(x,b_2-x)\,e^{\boldsymbol{Q}x}\boldsymbol{t}\mathrm{d}x\right)V_c(b_1;b_1,b_2)\\
&=\frac{c_2-c_1}{\delta}\Bigg[\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2}e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\,\boldsymbol{e}\mathrm{d}x\\
&\qquad+\boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2}e^{\boldsymbol{Q}x}\boldsymbol{t}q_-^{(2)}(x,b_2-x)\,\mathrm{d}x-\boldsymbol{p}_+^{(1)\prime}(b_1)\big(\boldsymbol{I}-e^{\boldsymbol{Q}b_2}\big)\boldsymbol{e}\Bigg].
\end{aligned}
\tag{32}
$$

Since all integrals in (32) can be evaluated by Lemma 9, we can solve for $V_c(b_1;b_1,b_2)$ and $V_c(b_3;b_1,b_2)$ using (28) and (32). Equations (31) and (27) then give the expressions for $V_c$ for $0<u<b_1$ and $b_1<u<b_3$.

### 5.2. Calculation of discounted ruin probability

Again, we start with the case of $b_1<u\le b_3$. By using the same argument as in Theorem 5, we get

$$
\begin{aligned}
&\psi(b_1+y;b_1,b_2)=\\
&\qquad q_-^{(2)}(y,b_2-y)\,\psi(b_1;b_1,b_2)+\boldsymbol{q}_+^{(2)\prime}(b_2-y,y)\boldsymbol{e}\;\psi(b_3;b_1,b_2)
\end{aligned}
\tag{33}
$$

for $0<y\le b_2$. In particular, by putting $y=b_2$,

$$
\psi(b_3;b_1,b_2)=\;p_-^{(2)}(b_2)\,\psi(b_1;b_1,b_2)+\boldsymbol{p}_+^{(2)\prime}(b_2)\boldsymbol{e}\;\psi(b_3;b_1,b_2). \tag{34}
$$

Then, we consider the case of $0<u\le b_1$. By using the argument as in the calculation of $V_d$ and (33), we get

$$\begin{aligned}
\psi(u;b_1,b_2) &= q_-^{(1)}(u,b_1-u) + \boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\psi(b_1+x;b_1,b_2)\mathrm{d}x \\
&\quad + \boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_\infty^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\mathrm{d}x\,\psi(b_3;b_1,b_2) \\
&= q_-^{(1)}(u,b_1-u) + \boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}q_-^{(2)}(x,b_2-x)\mathrm{d}x\,\psi(b_1;b_1,b_2) \\
&\quad + \boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\boldsymbol{e}\mathrm{d}x\,\psi(b_3;b_1,b_2) \\
&\quad + \boldsymbol{q}_+^{(1)\prime}(b_1-u,u)\,e^{\boldsymbol{Q}b_2}\boldsymbol{e}\,\psi(b_3;b_1,b_2). \qquad (35)
\end{aligned}$$

In particular, when $u = b_1$,

$$\begin{aligned}
\psi(b_1;b_1,b_2) &= p_-^{(1)}(b_1) + \boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}q_-^{(2)}(x,b_2-x)\mathrm{d}x\,\psi(b_1;b_1,b_2) \\
&\quad + \boldsymbol{p}_+^{(1)\prime}(b_1)\int_0^{b_2} e^{\boldsymbol{Q}x}\boldsymbol{t}\boldsymbol{q}_+^{(2)\prime}(b_2-x,x)\boldsymbol{e}\mathrm{d}x\,\psi(b_3;b_1,b_2) \qquad (36) \\
&\quad + \boldsymbol{p}_+^{(1)\prime}(b_1)e^{\boldsymbol{Q}b_2}\boldsymbol{e}\,\psi(b_3;b_1,b_2).
\end{aligned}$$

Since the integrals in (36) can be evaluated using Lemma 9, we can solve for $\psi(b_1;b_1,b_2)$ and $\psi(b_3;b_1,b_2)$ using (34) and (36). Equations (33) and (35) then give the expressions for $\psi$ for $b_1 < u < b_3$ and $0 < u < b_1$.

### 5.3. Numerical illustration

To illustrate the formulas presented in Section 5.1, we consider a dual model with gains following $\mathrm{PH}(\boldsymbol{\alpha},\boldsymbol{Q})$ where

$$\boldsymbol{\alpha} = (0.5,\ \ 0,\ \ 0.25,\ \ 0.25)',$$

and

$$\boldsymbol{Q} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -3/2 & 9/14 \\ 0 & 0 & 7/2 & -11/2 \end{bmatrix}.$$

The mean of the distribtuion is 1.67262. We assume that $\lambda = c_2 = 1$, $c_1 = 0.75$ and $\delta = 6\%$. The drift of the modified surplus process in the lower and upper regime are 0.92262 and 0.67262, respectively. By solving the equation $\kappa(\theta) = \delta$, we get $R_\delta^{(1)} = -0.893124$ and $R_\delta^{(2)} = -0.548103$. By Theorem 1,

$$\boldsymbol{\alpha}_+^{(1)} = (0.352152,\ \ 0.186016,\ \ 0.277652,\ \ 0.094607)'$$

and

$$\boldsymbol{\alpha}_{+}^{(2)} = (0.322976,\ 0.208627,\ 0.271489,\ 0.087439)'.$$

It can be verified that $\boldsymbol{Q}$ and $\boldsymbol{Q}_{+}^{(2)}$ have no common eigenvalues.

Now we compute $V$ for different values of $u$, $b_1$ and $b_2$ and compare that with the barrier strategy. We look at the effect of inserting a refracting layer under the barrier: for a fixed barrier $b_3$, we consider $V(u; (1-\varepsilon)\, b_3, \varepsilon b_3)$, where $\varepsilon = 0$, 1/4, 1/2, 3/4 and 1. The width of the transition region is of length $\varepsilon b_3$. When $\varepsilon = 0$, the hybrid strategy reduces to a barrier strategy. The results for $b_3 = 2$ and $b_3 = 5.57089$ (5.57089 is the optimal barrier when $c = 1$, see Section 6 for the determination of the optimal barrier) for various values of $u$ are shown in Tables 1 and 2. For $u > b_3$, the corresponding $V$ can be obtained from $u - b_3 + V(b_3; (1-\varepsilon)\, b_3, \varepsilon b_3)$.

TABLE 1

VALUES OF $V(u; 2(1-\varepsilon), 2\varepsilon)$ FOR DIFFERENT $u$ AND $\varepsilon$

| $(u, \varepsilon)$ | 0 | 1/4 | 1/2 | 3/4 | 1 |
|---|---|---|---|---|---|
| 0.4 | 2.473 | 2.334 | 2.170 | 1.988 | 1.517 |
| 0.8 | 4.260 | 4.021 | 3.704 | 3.272 | 2.757 |
| 1.2 | 5.569 | 5.258 | 4.817 | 4.264 | 3.775 |
| 1.6 | 6.547 | 6.157 | 5.618 | 5.086 | 4.616 |
| 2.0 | 7.295 | 6.815 | 6.291 | 5.774 | 5.317 |

TABLE 2

VALUES OF $V(u; 5.57089(1-\varepsilon), 5.57089\varepsilon)$ FOR DIFFERENT $u$ AND $\varepsilon$

| $(u, \varepsilon)$ | 0 | 1/4 | 1/2 | 3/4 | 1 |
|---|---|---|---|---|---|
| 1 | 7.604 | 7.613 | 7.466 | 7.035 | 5.420 |
| 2 | 11.151 | 11.164 | 10.951 | 10.138 | 8.815 |
| 3 | 13.063 | 13.079 | 12.806 | 12.020 | 11.058 |
| 4 | 14.332 | 14.349 | 14.048 | 13.421 | 12.655 |
| 5 | 15.364 | 15.380 | 15.114 | 14.568 | 13.899 |

For the cases in Table 1, where $b_3$ is small, the barrier strategy results in the greatest expected total discounted dividends. Expanding the width of the transition region causes $V$ to decrease. While it seems that more dividends would be paid, the effect of ruin causes payments to stop earlier and the overall effect is that $V$ would decrease. It can also be observed that the rate of decrease of $V$ increases with $\varepsilon$. On the contrary, for the cases in Table 2, the barrier strategy does not give the greatest expected total discounted dividends. A hybrid strategy can work better than the two pure strategies.

## 6. The Optimal Dividend Problem

The optimal dividend problem dates back to Bruno de Finetti (1957). For a particular value of initial surplus $u$, we want a dividend strategy that maximizes the expected total discounted dividends until ruin. In the Cramér-Lundberg model, such a strategy is a band strategy. In certain cases (e.g. exponential claims) the optimal strategy reduces to a barrier strategy, and in this case, the problem becomes the determination of an optimal barrier $b^*(u)$ given an initial surplus $u$. For more information on the optimality of the barrier strategy on more general processes, see Loeffen (2007). For the Cramér-Lundberg model, Gerber et al. (2006) proved that $b^*(u)$ does not depend on $u$ using the property that the surplus process is skip-free upwards.

For the dual risk model, Avanzi et al. (2007) pointed out that the optimal dividend strategy is a barrier strategy and that the optimal dividend barrier $b^*$ is independent of $u$ if (2) holds. Moreover, they showed that the left derivative of $V(u;b)$ with respect to $u$, evaluated at $u = b = b^*$, is $V'(b^*; b^*) = 1$ and

$$V(b^*; b^*) = \frac{\lambda\mu - c}{\delta}. \tag{37}$$

If we restrict ourselves to threshold strategies, Ng (2009) showed that under mild conditions, the optimal threshold that maximizes $V(u;b)$ is independent of $u$. Moreover, $V'(b^*; b^*) = 1$ and

$$V(b^*; b^*) = \frac{c_2 - c_1}{\delta} + \frac{1}{R_\delta^{(2)}}. \tag{38}$$

To determine $b^*$ under the barrier strategy, Avanzi et al. (2007) proposed using the inversion of Laplace transform of a function associated with $V(u;b)$. Ng (2009) illustrated how this can be adapted to the threshold strategy.

In view of the results in Section 3 and 4, we have the following variation of the method mentioned in Avanzi et al. (2007) and Ng (2009) to obtain the optimal barrier or threshold for phase-type gains. This variation makes use of the explicit formulas in Theorems 4 and 6 to calculuate $V(b;b)$ directly and does not involve numerical inversion of Laplace transform. Since $V(b;b)$ is increasing, there is a unique $b^*$ such that (37) or (38) holds and this value can be obtained by simple numerical methods. For example, one can plot a graph of $V(b;b)$ and locate the value of $b$ such that (37) or (38) holds. See Figure 1 below for a graphical representation.

In the illustrations below, we use the parameters used in Avanzi et al. (2007) in their Table 4. We assume that gains are Erlang(2) distributed with scale parameter 0.5. That is, $f(x) = 4xe^{-2x}$ for $x > 0$ (which corresponds to $\beta = 2$), $\lambda = 1$ and $\delta = 4\%$.

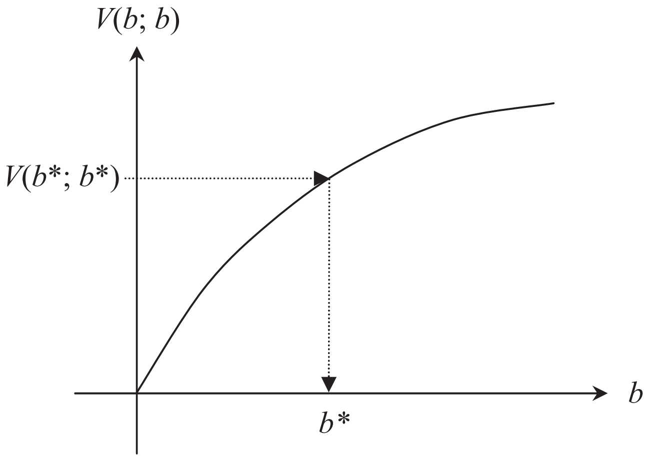


FIGURE 1. The determination of optimal barrier or threshold.

### 6.1. Barrier Strategy

Let $c = 0.8$. The roots of (4) are $r_0 = -\frac{1}{2}$, $r_1 = \frac{8-3\sqrt{6}}{5}$ and $r_2 = \frac{8+3\sqrt{6}}{5}$. By using the results in Section 2,

$$\boldsymbol{\alpha}_+ = \left(\frac{1}{2}, \frac{2}{5}\right)', \quad \boldsymbol{Q}_+ = \begin{bmatrix} -2 & 2 \\ 1 & -6/5 \end{bmatrix},$$

$$e^{\boldsymbol{Q}_+ u} = \frac{1}{36}\left(e^{r_1 u}\begin{bmatrix} 18-2\sqrt{6} & 10\sqrt{6} \\ 5\sqrt{6} & 18+2\sqrt{6} \end{bmatrix} - e^{r_2 u}\begin{bmatrix} -18-2\sqrt{6} & 10\sqrt{6} \\ 5\sqrt{6} & -18+2\sqrt{6} \end{bmatrix}\right),$$

and

$$p_-(u) = \frac{27}{75e^{-r_0 u} - (24+11\sqrt{6})e^{-r_1 u} - (24-11\sqrt{6})e^{-r_2 u}}.$$

As a result,

$$p_1^+(u) = \frac{150e^{-r_0 u} - (75+25\sqrt{6})e^{-r_1 u} - (75-25\sqrt{6})e^{-r_2 u}}{4\left[75e^{-r_0 u} - (24+11\sqrt{6})e^{-r_1 u} - (24-11\sqrt{6})e^{-r_2 u}\right]}$$

and

$$p_2^+(u) = \frac{120e^{-r_0 u} - (60+35\sqrt{6})e^{-r_1 u} - (60-35\sqrt{6})e^{-r_2 u}}{4\left[75e^{-r_0 u} - (24+11\sqrt{6})e^{-r_1 u} - (24-11\sqrt{6})e^{-r_2 u}\right]}.$$

Then, we arrive at

$$V(b;b) = \frac{420e^{-r_0 b} - (210 + 85\sqrt{6})e^{-r_1 b} - (39 - 16\sqrt{6})e^{-r_2 b}}{2\left[30e^{-r_0 b} + (39 + 16\sqrt{6})e^{-r_1 b} + (39 - 16\sqrt{6})e^{-r_2 b}\right]}.$$

The mean of the Erlang(2) distribution is 1 and $V(b^*; b^*) = 5$ by (37). The value of $b^*$ that satisfies $V(b^*; b^*) = 5$ is found to be 3.65329.

### 6.2. Threshold Strategy

Let $c_1 = 134/225 \approx 0.6$ and $c_2 = 0.8$. The various discounted upcrossing and downcrossing probabilities with subscript (2) are given in the barrier strategy. The roots of (4) are $-1$, $\frac{109 - 5\sqrt{427}}{67}$ and $\frac{109 + 5\sqrt{427}}{67}$. To summarize, we have

$$\boldsymbol{\alpha}_+^{(1)} = \left(\frac{75}{134}, \frac{25}{67}\right)' \quad \text{and} \quad \boldsymbol{Q}_+^{(1)} = \begin{bmatrix} -2 & 2 \\ 75/67 & -84/67 \end{bmatrix}.$$

This gives

$$p_-^{(1)}(u) = \frac{101}{201e^{-r_0 u} - \left(50 + \frac{1255}{\sqrt{427}}\right)e^{-r_1 u} + \left(50 - \frac{1255}{\sqrt{427}}\right)e^{-r_2 u}},$$

from which $p_+^{(1)}$ and $V(b;b)$ can be computed. From (38), $V(b^*; b^*) = 28/9$ and the value of $b^*$ for this to hold is found to be 1.58089.

### 6.3. A Comparsion of Barrier Strategy and Threshold Strategy

In this subsection we compare the two dividend strategies. For $c_1 = 0.2$ and 0.75 (which correspond to $\mu = 0.8$ and 0.25 in Table 4 of Avanzi et al. (2007)), we calculate $b^*$ and $V(b^*; b^*)$ for two values of $c_2$. The results are shown in Tables 3 and 4. In the tables, for each value of $c_2$ and $\delta$, the value of $b^*$ is given first, followed by the value of $V(b^*; b^*)$. The corresponding values under the barrier strategy (which is the limiting case $c_2 \to \infty$) are shown in the last column for comparsion.

Similar to the case of a barrier strategy, $b^*$ is decreasing with $\delta$. Also, the corresponding $b^*$ and $V(b^*; b^*)$ are less than that for the barrier strategy, and they converge to the corresponding values of the barrier strategy. For an analytic proof for this fact in the case of a general distribution for $X_i$'s, readers may refer to Remark 4.1 of Ng (2009).

It is also interesting to compare Tables 2 and 3 in Ng (2009) with Tables 3 and 4. The example in Ng (2009) corresponds to the exponential case with mean 1. While they have the same mean, the variance of the Erlang(2) gains

TABLE 3

VALUES OF $b^*$ AND $V(b^*; b^*)$ FOR $c_1 = 0.2$

| $(\delta, c_2)$ | 0.4 | 4 | 50 | 100 | barrier |
|---|---|---|---|---|---|
| 0.01 | 1.031 | 2.181 | 2.229 | 2.231 | 2.233 |
| | 19.460 | 79.751 | 79.985 | 79.992 | 80 |
| 0.03 | 0.763 | 1.664 | 1.712 | 1.714 | 1.716 |
| | 6.148 | 26.419 | 26.651 | 26.659 | 26.667 |
| 0.06 | 0.584 | 1.330 | 1.378 | 1.379 | 1.381 |
| | 2.842 | 13.088 | 13.318 | 13.326 | 13.333 |
| 0.1 | 0.450 | 1.084 | 1.131 | 1.133 | 1.134 |
| | 1.541 | 7.757 | 7.985 | 7.992 | 8 |

TABLE 4

VALUES OF $b^*$ AND $V(b^*; b^*)$ FOR $c_1 = 0.75$

| $(\delta, c_2)$ | 1.5 | 6 | 25 | 100 | barrier |
|---|---|---|---|---|---|
| 0.01 | 8.534 | 9.341 | 9.430 | 9.448 | 9.454 |
| | 23.561 | 24.850 | 24.969 | 24.992 | 25 |
| 0.03 | 4.037 | 4.806 | 4.895 | 4.913 | 4.919 |
| | 6.995 | 8.184 | 8.302 | 8.326 | 8.333 |
| 0.06 | 2.077 | 2.801 | 2.890 | 2.908 | 2.914 |
| | 2.944 | 4.018 | 4.135 | 4.159 | 4.167 |
| 0.1 | 1.104 | 1.782 | 1.870 | 1.888 | 1.894 |
| | 1.393 | 2.352 | 2.469 | 2.492 | 2.5 |

in the illustration above is 0.5, and the variance of the exponentially distributed gains with mean 1 is 1. It can be seen that the optimal threshold for the Erlang(2) case is less than that of the exponential case, while the optimal total expected discounted dividends for the Erlang(2) case is greater than that of the exponential case. That both the barrier strategy and the threshold strategy exhibit similar behavior is not surprising. Intuitively speaking, the optimal threshold should increase with risk.

## ACKNOWLEDGMENTS

The work described in this paper was partially supported by a grant from the CUHK research committee of the Hong Kong Special Administrative Region, China (Project No. CUHK 2070402). The author thanks the referees for comments and suggestions.

## References

Albrecher, H. and Thonhauser, S. (2009) Optimality Results for Dividend Problems in Insurance. *RACSAM Revista de la Real Academia de Ciencias; Serie A, Matemáticas*, **103(2)**, 295-320.

Asmussen, S. (2000) *Ruin Probabilities*. Singapore: World Scientific.

Asmussen, S., Nerman, O., and Olsson, M. (1996) Fitting Phase-type Distribution via the EM Algorithm. *Scandinavian Journal of Statistics*, **30**, 365-372.

Asmussen, S. and Perry, D. (1992) On Cycle Maxima, First Passage Problems and Extreme Value Theory for Queues, *Stochastic Models*, **8**, 421-458.

Avanzi, B. (2009) Strategies for Dividend Distribution: A Review, *North American Actuarial Journal*, **13(2)**, 217-251.

Avanzi, B. and Gerber, H.U. (2008) Optimal Dividends in the Dual Model with Diffusion, *ASTIN Bulletin* **38(2)**: 653-667.

Avanzi, B., Gerber, H.U. and Shiu, E.S.W. (2007) Optimal Dividends in the Dual Model, *Insurance: Mathematics and Economics*, **41(1)**, 111-123.

Bayraktar, E. and Egami, M. (2008) Optimizing Venture Capital Investment in a Jump Diffusion Model, *Mathematical Methods of Operations Research*, **67(1)**, 21-42.

Bühlmann, H. (1970) *Mathematical Methods in Risk Theory*. New York: Springer-Verlag.

Cramér, H. (1955) *Collective Risk Theory: A Survey of the Theory from the Point of View of the Theory of Stochastic Process*. Stockholm: Ab Nordiska Bokhandeln.

de Finetti, B. (1957) Su Un'impostazione Alternativa della Teoria Collettiva del Rischio, *Transactions of the XVth International Congress of Actuaries 2*, 433-443.

Gerber, H.U. (1969) Entscheidungskriterien für den Zusammengesetzten Poisson-Prozess, *Bulletin de l'Association Suisse des Actuaires*, **1969(2)**, 185-228.

Gerber, H.U. (1972) Games of Economic Survival with Discrete- and Continuous-Income Processes, *Operational Research*, **20(1)**, 37-45.

Gerber, H.U., Lin, X.S. and Yang, H. (2006) A Note on the Dividends-Penalty Identity and the Optimal Dividend Barrier, *ASTIN Bulletin*, **36(2)**, 489-503.

Gerber, H.U. and Shiu, E.S.W. (1998) On the Time Value of Ruin, *North American Actuarial Journal*, **2(1)**, 48-72.

Gerber, H.U. and Smith, N. (2008) Optimal Dividends with Incomplete Information in the Dual Model, *Insurance: Mathematics and Economics*, **43(2)**: 227-233.

Graham, A. (1981) *Kronecker Products and Matrix Calculus with Applications*, Chichester: Ellis Horwood.

Loeffen, R.L. (2008) On Optimality of the Barrier Strategy in de Finetti's Dividend Problem for Spectrally Negative Lévy Processes, **18(5)**, 1669-1680.

Ng, A.C.Y. (2009) On a Dual Model with a Dividend Threshold, *Insurance: Mathematics and Economics*, **44(2)**, 315-324.

Ng, A.C.Y. and Yang, H. (2006) On the Joint Distribution of Surplus Before and After Ruin Under a Markovian Regime Switching Model, *Stochastic Processes and Their Applications*, **116**, 244-266.

Seal, H.L. (1969) *Stochastic Theory of a Risk Business*. New York: Wiley.

Takács, L. (1967) *Combinatorial Methods in the Theory of Stochastic Processes*. New York: Wiley.

Zhu, J. and Yang, H. (2008) Ruin Probabilities of a Dual Markov-Modulated Risk Model, *Communications in Statistics – Theory and Methods*, **37**, 3298-3307.

Andrew C.Y. Ng
*Department of Finance*
*The Chinese University of Hong Kong*
*Shatin, Hong Kong*