# OPTIMAL REINSURANCE FOR VARIANCE RELATED PREMIUM CALCULATION PRINCIPLES[1]

BY

MANUEL GUERRA and MARIA DE LOURDES CENTENO


## ABSTRACT

This paper deals with numerical computation of the optimal form of reinsurance from the ceding company point of view, when the cedent seeks to maximize the adjustment coefficient of the retained risk and the reinsurance loading is an increasing function of the variance.

We compare the optimal treaty with the best stop loss policy. The optimal arrangement can provide a significant improvement in the adjustment coefficient when compared to the best stop loss treaty. Further, it is substantially more robust with respect to choice of the retention level than stop-loss treaties.



## KEYWORDS

Adjustment coefficient; optimal reinsurance; stop loss; standard deviation premium principle; variance premium principle.


## 1. INTRODUCTION

The adjustment coefficient plays an important role in risk theory. Its maximization is equivalent to minimization of the upper bound of the probability of ultimate ruin provided by the Lundberg inequality.

In a previous paper (Guerra and Centeno (2008)) we studied the problem of finding the reinsurance policy maximizing the adjustment coefficient of the retained risk. Not surprisingly, the solution depends critically on the premium calculation principle used by the reinsurer. Assuming that the reinsurance premium is convex and satisfies some very general regularity assumptions, it was shown that the optimal reinsurance scheme always exists and it is unique up to "economic equivalence". A necessary optimality condition was found which in principle allows for the computation of the optimal treaty.

[1] This research has been supported by Fundação para a Ciência e a Tecnologia (FCT) – project PTDC/ECO/66693/2006 – through PIDDAC, partially funded by the Portuguese State Budget.

*Astin Bulletin* 40(1), 97-121. doi: 10.2143/AST.40.1.2049220 

In this paper we deal with the case where the reinsurer prices the treaties using a loading which is an increasing function $g$ of the variance of the accepted risk. Important instances of such pricing principles are the variance and the standard deviation principles. In Guerra and Centeno (2008) it was shown that in this case the optimal arrangement is a nonlinear function of a type previously unknown in the reinsurance literature. This function depends on the optimal adjustment coefficient, $R$, and one additional parameter, $\alpha$. The values of $R$ and $\alpha$ must be found simultaneously by solving a pair of nonlinear equations involving the expected value and variance of the ceded risk, which also depend on $(R, \alpha)$. In this paper we show that the numerical computation of $(R, \alpha)$ is feasible. To do this, it is possible to use two-step bisection-type algorithms that are guaranteed to converge, or fast Newton-type algorithms that exhibit local quadratic convergence.

We have three objectives in the present paper: to characterize the functions $g$ that provide convex premium calculation principles, to show that the solution mentioned above can easily be computed by standard numerical methods, and to compare the performance of the optimal treaty with the best stop-loss policy under fairly realistic reinsurance loadings and claim distributions.

Comparison with stop-loss treaties is meaningful because it is by far the most widely used type of aggregate treaty that guarantees existence of the adjustment coefficient for the retained risk in cases where the distribution of the aggregate claims has an heavy tail, as is usually the case in practical applications. Further, there are well known results in the literature showing that stop-loss is the optimal treaty for various types of optimality criteria under different sets of assumptions on the reinsurance premium. Such results go back to Borch (1960) and Arrow (1963) that considered the variance and the expected utility of wealth, respectively, as optimality criteria. Hesselager (1990) proved an equivalent result using the adjustment coefficient as optimality criterion. Some recent results in favor of stop-loss treaties are found in Kaluszka (2004).

The text is organized as follows: Section 2 contains the main assumptions, a rigorous statement of the problem, and a characterization of convex variance-related premium principles. Section 3 contains a short overview of the main results in Guerra and Centeno (2008) concerning specifically the case where the reinsurance loading is an increasing function of the variance. Interested readers are referred to Guerra and Centeno (2008) for a full theoretical analysis of the interrelated problems of maximizing the insurer's expected utility of wealth and maximizing the adjustment coefficient of the retained risk. Some theoretical details which are useful in the computation of optimal treaties are added in the present paper. Section 4 contains an analysis of the main issues arising in the numerical computation of optimal treaties. We show that though the solution given in Section 3 is in an implicit form, it can be numerically computed using classical methods. In Section 5 we compare the optimal policy with the best stop loss policy with respect to a standard deviation principle for two different claim distributions. The distributions are chosen so that the first two moments are identical but the tails are quite different. The results suggest that the optimal policy can offer significant improvement in the value of the adjustment coefficient compared to the best stop-loss treaty and also that its performance is much more robust with respect to the retention level. This is an important feature for practical implementation when the data of the problem cannot be known with full accuracy and hence the chosen treaty is in fact suboptimal.

## 2. Assumptions and preliminaries

Let $Y$ be a non-negative random variable, representing the annual aggregate claims and let us assume that aggregate claims over consecutive periods are i.i.d. random variables. We assume that $Y$ is a continuous random variable, with density function $f$, and that $E[Y^2] < +\infty$. Let $c > E[Y]$ and such that $\Pr\{Y > c\} > 0$, be the corresponding premium income, net of expenses. A map $Z : [0, +\infty) \mapsto [0, +\infty)$ identifies a reinsurance policy. The set of all possible reinsurance programmes is

$$\mathcal{Z} = \{Z : [0, +\infty) \mapsto \mathbb{R} \mid Z \text{ is measurable and } 0 \leq Z(y) \leq y, \forall y \geq 0\}.$$

We do not distinguish between functions which differ only on a set of zero probability. i.e., two measurable functions, $\phi$ and $\phi'$ are considered to be the same whenever $\Pr\{\phi(Y) = \phi'(Y)\} = 1$. Similarly, a measurable function, $Z$, is an element of $\mathcal{Z}$ whenever $\Pr\{0 \leq Z(Y) \leq Y\} = 1$.

For a given reinsurance policy, $Z \in \mathcal{Z}$, the reinsurer charges a premium $P(Z)$ of the type

$$P(Z) = E[Z] + g(Var[Z]), \tag{1}$$

where $g : [0, +\infty) \mapsto [0, +\infty)$ is a continuous function smooth in $(0, +\infty)$ such that $g(0) = 0$ and $g'(x) > 0$, $\forall x \in (0, +\infty)$. Further we assume that $P$ is a convex functional. We call premium calculation principles of this type "variance-related principles". The variance principle and the standard deviation principle are both under these conditions, with $g(x) = \beta x$ and $g(x) = \beta x^{1/2}$, $\beta > 0$, respectively. Convexity of these two principles was proved by Deprez and Gerber (1985), but also follows immediately from Proposition 1, which characterizes convex variance-related premiums.

The net profit, after reinsurance, is

$$L_Z = c - P(Z) - (Y - Z(Y)).$$

We assume that $c$, $P$ and the claim size distribution are such that

$$\Pr\{L_Z < 0\} > 0, \quad \forall Z \in \mathcal{Z}, \tag{2}$$

otherwise there would exist a policy under which the probability of ultimate ruin would be zero. This requires the premium loading to be sufficiently high. Namely,

$$g(Var[Y]) > c - E[Y] \tag{3}$$

must hold. Notice that if (3) fails, then the direct insurer can cede **all the risk** to the reinsurer and still obtain a profit $L_Y = c - P(Y) \geq 0$ with probability equal to one.

For the variance principle, inequality (3) reduces to

$$\beta > (c - E[Y]) / Var[Y], \tag{4}$$

and in the standard deviation principle case the required condition is

$$\beta > (c - E[Y]) / \sqrt{Var[Y]}. \tag{5}$$

In practice, it may be difficult to check analytically whether (2) holds for a particular random variable $Y$ and a particular premium calculation principle $P$. However, a numerical procedure to check (2) is easily imbedded in the algorithm presented in Section 4.1 below.

Consider the map $G : \mathbb{R} \times \mathcal{Z} \mapsto [0, +\infty]$ defined by

$$G(R, Z) = \int_0^{+\infty} e^{-RL_Z(y)} f(y)\, dy, \quad R \in \mathbb{R},\ Z \in \mathcal{Z}. \tag{6}$$

Let $R_Z$ denote the adjustment coefficient of the retained risk for a particular reinsurance policy, $Z \in \mathcal{Z}$. $R_Z$ is defined as the strictly positive value of $R$ which solves the equation

$$G(R, Z) = 1, \tag{7}$$

for that particular $Z$, when such a root exists. Equation (7) can not have more than one positive solution. This means the map $Z \mapsto R_Z$ is a well defined functional in the set

$$\mathcal{Z}^+ = \{Z \in \mathcal{Z} : (7) \text{ admits a positive solution}\}.$$

From a mathematical point of view, the problem of finding the reinsurance policy that maximizes the adjustment coefficient of the retained risk can be stated as follows:

**Problem 1.** *Find* $(\hat{R}, \hat{Z}) \in (0, +\infty) \times \mathcal{Z}^+$ *such that*

$$\hat{R} = R_{\hat{Z}} = \max\{R_Z : Z \in \mathcal{Z}^+\}. \ \square$$

We conclude this section with the following characterization of convex variance-related premiums:

**Proposition 1.** *Let* $B = \sup\{Var[Z] : Z \in \mathcal{Z}\}$ *and assume that g is twice differentiable in the interval* $(0, B)$. $P(Z) = E[Z] + g(Var[Z])$ *is a convex functional if and only if*

$$\frac{g''(x)}{g'(x)} \geq -\frac{1}{2x}, \quad \forall x \in (0, B). \ \square \tag{8}$$

**Proof.** The proof below is an adaptation of the proof by Deprez and Gerber (1985) for a related result.

First assume that the map $P : \mathcal{Z} \mapsto \mathbb{R}$ is convex. Fix $Z \in \mathcal{Z} \setminus \{0\}$ and consider the map $t \mapsto P(tZ)$, $t \in [0,1]$. Then

$$\begin{aligned} \frac{d^2}{dt^2} P(tZ) &= \frac{d^2}{dt^2}(tE[Z] + g(t^2 Var[Z])) = \\ &= g''(t^2 Var[Z]) 4t^2 Var[Z]^2 + g'(t^2 Var[Z]) 2 Var[Z]. \end{aligned}$$

Convexity of $P$ implies convexity of the map $t \mapsto P(tZ)$, $t \in [0,1]$. It follows that $\frac{d^2}{dt^2} P(tZ) \geq 0$, i.e.,

$$\frac{g''(t^2 Var[Z])}{g'(t^2 Var[Z])} \geq \frac{-1}{2t^2 Var[Z]}$$

must hold for all $t \in (0,1)$. Since $Z \in \mathcal{Z}$ is arbitrary, inequality (8) follows immediately.

Now, assume that inequality (8) holds and for each $Z$, $W \in \mathcal{Z}$ consider the map

$$t \mapsto \mathcal{P}_{Z,W}(t) = P(Z + t(W - Z)), \quad t \in [0,1].$$

From the definition of convex map, it follows that $Z \mapsto P(Z)$ is convex if and only if for every $Z$, $W \in \mathcal{Z}$ the map $t \mapsto \mathcal{P}_{Z,W}(t)$ is convex. The maps $t \mapsto \mathcal{P}_{Z,W}(t)$ are continuous in $[0,1]$, twice differentiable in $(0,1)$, and

$$\begin{aligned} \mathcal{P}''_{Z,W}(t) = \ & 4g''(Var[Z + t(W - Z)])(Cov[Z, W - Z] + tVar[W - Z])^2 + \\ & + 2g'(Var[Z + t(W - Z)]) Var[W - Z]. \end{aligned}$$

In particular,

$$\begin{aligned} \mathcal{P}''_{Z,W}(0) &= 4g''(Var[Z]) Cov[Z, W - Z]^2 + 2g'(Var[Z]) Var[W - Z] = \\ &= 4g''(Var[Z])\,(Cov[Z, W] - Var[Z])^2 + \end{aligned}$$

$$+ 2g'(Var[Z])(Var[W] - 2Cov[Z,W] + Var[Z]) =$$

$$= 2g'(Var[Z])\left(\frac{2g''(Var[Z])}{g'(Var[Z])}(Cov[Z,W] - Var[Z])^2\right.$$

$$\left.+ Var[W] - 2Cov[Z,W] + Var[Z]\right)$$

By inequality (8), this implies

$$\mathcal{P}''_{Z,W}(0) \geq$$

$$\geq 2g'(Var[Z])\left(\frac{-1}{Var[Z]}(Cov[Z,W] - Var[Z])^2 + Var[W] -\right.$$

$$\left.- 2Cov[Z,W] + Var[Z]\right) =$$

$$= \frac{2g'(Var[Z])}{Var[Z]}(Var[W]Var[Z] - Cov[Z,W]^2).$$

Then, the Cauchy-Schwartz inequality guarantees that

$$\mathcal{P}''_{Z,W}(0) \geq 0, \quad \forall Z, W \in \mathcal{Z}. \tag{9}$$

We conclude the proof by showing that inequality (9) implies the apparently stronger condition

$$\mathcal{P}''_{Z,W}(t) \geq 0, \quad \forall Z, W \in \mathcal{Z}, \ \forall t \in (0,1).$$

In order to do this, note that

$$\begin{aligned} \mathcal{P}_{Z+t(W-Z),W}(s) &= P(Z + t(W-Z) + s(W - (Z + t(W-Z)))) \\ &= \mathcal{P}_{Z,W}(t + s(1-t)) \end{aligned}$$

holds for every $Z, W \in \mathcal{Z}$, $t, s \in (0,1)$ and $t, s \in (0,1)$ implies $t + s(1-t) \in (0,1)$. It follows that

$$\frac{d^2}{ds^2}\mathcal{P}_{Z+t(W-Z),W}(s)\bigg|_{s=0} = \frac{d^2}{ds^2}\mathcal{P}_{Z,W}(t + s(1-t))\bigg|_{s=0} = (1-t)^2\,\mathcal{P}''_{Z,W}(t),$$

which concludes the proof. ■

**Remark 1.** *Condition (8) holds as an equality for the standard deviation principle and the left hand side of (8) is zero for the variance principle. Hence both principles are convex.*

## 3. Optimal reinsurance policies for variance related premiums

In Guerra and Centeno (2008) we proved that, provided the assumptions stated in Section 2 hold, Problem 1 admits a solution. Further we proved that, provided $Z \equiv 0$ is not optimal, there is an optimal treaty $Z \in \mathcal{Z}$ satisfying

$$y = Z(y) + \frac{1}{R}\ln\frac{Z(y)+\alpha}{\alpha}, \quad \forall y \geq 0, \tag{10}$$

where $R$ and $\alpha$ are strictly positive constants.

For each $R > 0$, $\alpha > 0$, the map $\gamma(z) = z + \frac{1}{R}\ln\frac{z+\alpha}{\alpha}$ satisfies

$$\gamma(0) = 0, \quad \gamma'(z) > 1, \quad \forall z \geq 0.$$

Hence (10) defines one unique treaty $Z_{R,\alpha} \in \mathcal{Z}$. Since $\lim_{\alpha\to+\infty} Z_{R,\alpha}(y) = 0$ and $\lim_{\alpha\to+\infty} Z_{R,\alpha}(y) = y$ hold for every $y > 0$, $R > 0$, we set $Z_{R,0} \equiv 0$, $Z_{R,+\infty} = Y$, for every $R > 0$.

In what follows $v \in [0, +\infty)$ denotes the number

$$v = \sup\{y : \Pr\{Y \leq y\} = 0\},$$

and we consider the function

$$h(R,\alpha) = \alpha + E[Z_{R,\alpha}] - \frac{1}{2g'(Var[Z_{R,\alpha}])}. \tag{11}$$

In Guerra and Centeno (2008) we provide a rigorous proof of the following result:

**Theorem 1.** *Suppose all assumptions stated in Section 2 hold. Then, $\mathcal{Z}^{+}$ is non-empty, a solution to Problem 1 always exists, and the following statements hold:*

*a) When $g'$ is a bounded function in a neighborhood of zero, there is an optimal treaty $Z(y)$ satisfying* (10)*, where $R$ and $\alpha$ are positive numbers solving* (7) *and*

$$h(R,\alpha) = 0, \tag{12}$$

*with $h(R,\alpha)$ defined by* (11).

*When $g'$ is unbounded in any neighborhood of zero, then either a contract satisfying* (7), (10)-(12) *is optimal or the optimal treaty is $Z(y) = 0$, $\forall y$ (no reinsurance at all) and no solution to* (7), (10)-(12) *exists in* $(0,+\infty)^2$.

*b) If $v = 0$, the solution is unique. If $v > 0$ then all solutions are of the form $Z(y) + x$, where $Z(y)$ is the treaty described in a) and $x$ is any constant such that $-Z(v) \leq x \leq v - Z(v)$.* □

Theorem 1 evokes some simple remarks:

**Remark 2.** *Equality* (10) *shows that the optimal treaty satisfies* $\lim_{y \to +\infty} Z(y) = +\infty$. *Therefore,* $\lim_{y \to +\infty} (y - Z(y)) = \lim_{y \to +\infty} \frac{1}{R} \ln \frac{Z(y)+\alpha}{\alpha} = +\infty$ *holds, i.e., the retained risk* $Y - Z(Y)$ *is unbounded whenever* $Y$ *is unbounded. This shows that under the optimal treaty the direct insurer always retains some part of the tail of the risk distribution. Of course, the retained tail must always be "light" because the theorem guarantees existence of a positive adjustment coefficient for the retained risk.*

**Remark 3.** *If the optimal treaty is not unique (i.e., if* $v > 0$*) then any two optimal treaties differ only by a constant. Since* $P(Z + x) = x + P(Z)$ *holds for every constant* $x$*, we see that* $L_{Z+x} = L_Z$ *also holds. Therefore all optimal treaties are indifferent from the economic point of view.*

**Remark 4.** *Although* $Z_{R,\alpha}$ *is defined only in an implicit form, the distribution function can be easily calculated. As the right-hand side of* (10) *is strictly increasing with respect to* $Z$*, the distribution function of* $Z_{R,\alpha}$ *is*

$$F_{Z_{R,\alpha}}(\zeta) = \Pr\left\{Y \leq \zeta + \frac{1}{R} \ln \frac{\zeta + \alpha}{\alpha}\right\} = F\left(\zeta + \frac{1}{R} \ln \frac{\zeta + \alpha}{\alpha}\right). \tag{13}$$

*Therefore the density function is*

$$f_{Z_{R,\alpha}}(\zeta) = f\left(\zeta + \frac{1}{R} \ln \frac{\zeta + \alpha}{\alpha}\right) \frac{1 + R(\zeta + \alpha)}{R(\zeta + \alpha)}. \tag{14}$$

Theorem 1 leaves some ambiguity about the number of roots of equation (12), for fixed $R \in (0, \infty)$. We will show that this equation has at most one solution.

First, let us introduce the functions

$$\Phi_k(R, \alpha) = \int_0^{+\infty} (1 + R(Z_{R,\alpha}(y) + \alpha))^k f(y)\, dy, \quad k \in \mathbb{Z}. \tag{15}$$

These functions are useful to prove the properties below. They are also convenient to deal with issues related to numerical computation of optimal treaties.

**Remark 5.** *Since we assume that* $E[Y^2] < +\infty$, $\Phi_k(R,\alpha)$ *is finite for all* $k \leq 2$, $\alpha > 0$, $R > 0$.

**Remark 6.** *For* $k \geq 0$*, it is clear that* $\Phi_k(R,\alpha)$ *is a linear combination of the moments of* $Z_{R,\alpha}$ *of order* $\leq k$*. A simple computation shows that for the first two moments we have:*

$$E[Z_{R,\alpha}] = \frac{1}{R}\left(\Phi_1(R,\alpha) - (1 + R\alpha)\right), \tag{16}$$

$$Var[Z_{R,\alpha}] = \frac{1}{R^2}\left(\Phi_2(R,\alpha) - \Phi_1(R,\alpha)^2\right). \tag{17}$$

*In many arguments and expressions throughout this paper we use the functions $\Phi_k$ instead of the moments of $Z_{R,\alpha}$ because, due to Proposition 2 below, functions $\Phi_k$ with $k < 0$ turn out naturally. This makes several expressions far simpler when expressed using the functions $\Phi_k$.*

Derivatives of $\Phi_k$ with respect to the parameter $\alpha$ can be easily computed:

**Proposition 2.** *For $k \leq 2$, $R > 0$ the map $\alpha \mapsto \Phi_k(R,\alpha)$ is smooth and*

$$\frac{\partial \Phi_k}{\partial \alpha} = k\left(\frac{1}{\alpha} + R\right)(\Phi_{k-1} - \Phi_{k-2}). \ \square \tag{18}$$

**Proof.** From (10) it follows that

$$\frac{\partial Z_{R,\alpha}(y)}{\partial \alpha} = \frac{Z_{R,\alpha}(y)}{\alpha(1 + R(Z_{R,\alpha}(y) + \alpha)} = \frac{1}{\alpha R} - \frac{1 + \alpha R}{\alpha R} \frac{1}{(1 + R(Z_{R,\alpha}(y) + \alpha)}.$$

Then,

$$\frac{\partial \Phi_k}{\partial \alpha} = \int_0^{+\infty} k(1 + R(Z_{R,\alpha}(y) + \alpha)^{k-1} R\left(\frac{\partial Z_{R,\alpha}}{\partial \alpha} + 1\right) f(y)dy =$$
$$= \frac{k(1 + \alpha R)}{\alpha} \int_0^{+\infty} ((1 + R(Z_{R,\alpha}(y) + \alpha))^{k-1} - (1 + R(Z_{R,\alpha}(y) + \alpha))^{k-2}) f(y)dy,$$

from where (18) follows. ■

Proposition 2 allows us to state the following:

**Proposition 3**

$$\frac{\partial E[Z_{R,\alpha}]}{\partial \alpha} = \frac{1}{R\alpha} - \frac{1 + R\alpha}{R\alpha} \Phi_{-1}(R,\alpha), \tag{19}$$

$$\frac{\partial Var[Z_{R,\alpha}]}{\partial \alpha} = \frac{2(1 + R\alpha)}{R^2\alpha} (\Phi_1(R,\alpha)\Phi_{-1}(R,\alpha) - 1). \ \square \tag{20}$$

Using the previous propositions we are able to prove the following uniqueness result:

**Proposition 4.** *Suppose that $g$ is twice differentiable in the interval $(0,+\infty)$. For each $R \in (0,+\infty)$ (fixed) equation (12) has at most one solution, $\alpha_R > 0$.*

*If such a solution exists, then $\frac{\partial h}{\partial \alpha}(R,\alpha_R) > 0$ holds. Therefore $h(R,\alpha)$ is strictly negative for $\alpha \in (0,\alpha_R)$, and it is strictly positive for $\alpha \in (\alpha_R,+\infty)$.* □

**Proof.** Differentiating $h(R,\alpha)$ with respect to $\alpha$ we get

$$\frac{\partial h}{\partial \alpha}(R,\alpha) = 1 + \frac{\partial E[Z_{R,\alpha}]}{\partial \alpha} + \frac{1}{2}\,\frac{g''(Var[Z_{R,\alpha}])}{(g'(Var[Z_{R,\alpha}]))^2}\,\frac{\partial Var[Z_{R,\alpha}]}{\partial \alpha}. \tag{21}$$

At the points where $h(R,\alpha) = 0$,

$$\frac{1}{2} = (E[Z_{R,\alpha}] + \alpha)\,g'(Var[Z_{R,\alpha}]) \tag{22}$$

and hence

$$\left.\frac{\partial h}{\partial \alpha}(R,\alpha)\right|_{h(R,\alpha)=0} = 1 + \frac{\partial E[Z_{R,\alpha}]}{\partial \alpha} + (E[Z_{R,\alpha}] + \alpha)\,\frac{g''(Var[Z_{R,\alpha}]}{g'(Var[Z_{R,\alpha}])}\,\frac{\partial Var[Z_{R,\alpha}]}{\partial \alpha}. \tag{23}$$

Noticing that $E[Z_{R,\alpha}] + \alpha$ and $\partial Var[Z_{R,\alpha}] / \partial \alpha$ (given by (20)) are positive and using Proposition 1 we have

$$\left.\frac{\partial h}{\partial \alpha}(R,\alpha)\right|_{h(R,\alpha)=0} \geq 1 + \frac{\partial E[Z_{R,\alpha}]}{\partial \alpha} - \frac{(E[Z_{R,\alpha}] + \alpha)}{2Var[Z_{R,\alpha}]}\,\frac{\partial Var[Z_{R,\alpha}]}{\partial \alpha}. \tag{24}$$

Using (16), (17), (19) and (20), we get

$$\begin{aligned}
\left.\frac{\partial h}{\partial \alpha}(R,\alpha)\right|_{h(R,\alpha)=0} &\geq \left(1 + \frac{1}{R\alpha}\right)(1 - \Phi_{-1}) - \frac{E[Z_{R,\alpha}] + \alpha}{\Phi_2 - \Phi_1^2}\left(\frac{1}{\alpha} + R\right)(\Phi_1\Phi_{-1} - 1) = \\
&= \left(1 + \frac{1}{R\alpha}\right)(1 - \Phi_{-1}) - \frac{\frac{1}{R}(\Phi_1 - 1)}{\Phi_2 - \Phi_1^2}\left(\frac{1}{\alpha} + R\right)(\Phi_1\Phi_{-1} - 1) = \\
&= \left(1 + \frac{1}{R\alpha}\right)\left(1 - \Phi_{-1} - \frac{(\Phi_1 - 1)(\Phi_1\Phi_{-1} - 1)}{\Phi_2 - \Phi_1^2}\right) = \\
&= \frac{1 + R\alpha}{R\alpha(\Phi_2 - \Phi_1^2)}\left((1 - \Phi_{-1})(\Phi_2 - \Phi_1^2) - (\Phi_1 - 1)(\Phi_1\Phi_{-1} - 1)\right) = \\
&= \frac{1 + R\alpha}{R\alpha(\Phi_2 - \Phi_1^2)}\left((\Phi_2 - \Phi_1)(1 - \Phi_{-1}) - (\Phi_1 - 1)^2\right).
\end{aligned}$$

Let

$$X_1 = \sqrt{R(Z_{R,\alpha}(Y) + \alpha)(1 + R(Z_{R,\alpha}(Y) + \alpha))},$$

$$X_2 = \sqrt{\frac{R(Z_{R,\alpha}(Y) + \alpha)}{1 + R(Z_{R,\alpha}(Y) + \alpha)}}.$$

For $\alpha > 0$, the unique constants satisfying

$$\Pr\{c_1 X_1 + c_2 X_2 = 0\} = 1$$

are $c_1 = c_2 = 0$. This implies the that Cauchy-Schwartz inequality is strict, i.e.

$$E^2[X_1 X_2] < E[X_1^2]\, E[X_2^2].$$

Since

$$\begin{aligned}
\Phi_2 - \Phi_1 &= \int_0^{+\infty} R\,(Z_{R,\alpha}(y) + \alpha)\,(1 + R\,(Z_{R,\alpha}(y) + \alpha))\; f(y)\,dy \;=\; E[X_1^2], \\
1 - \Phi_{-1} &= \int_0^{+\infty} \frac{R\,(Z_{R,\alpha}(y) + \alpha)}{1 + R\,(Z_{R,\alpha}(y) + \alpha)}\; f(y)\,dy \;=\; E[X_2^2], \\
\Phi_1 - 1 &= \int_0^{+\infty} R\,(Z_{R,\alpha}(y) + \alpha)\; f(y)\,dy \;=\; E[X_1 X_2],
\end{aligned}$$

the proof follows from (25). ■

**Remark 7.** *For the variance premium calculation principle we have* $g(x) = \beta x$, $\beta > 0$. *Therefore* $g' \equiv \beta$ *is bounded in a neighborhood of zero. Therefore, Theorem* 1 *guarantees that the optimal reinsurance policy is always a nonzero policy. Since the solution for* (7), (10)-(12) *is unique, it gives indeed the optimal solution (and not any other critical point of the adjustment coefficient).*

*This contrasts with the case of the standard deviation principle where* $g(x) = \beta x^{1/2}$, $\beta > 0$, *and* $g'(x) = \frac{\beta}{2} x^{-1/2}$ *is unbounded in any neighborhood of zero. In this case the optimal policy may be not to reinsure at all, but this can only happen when the tail of the distribution of* $Y$ *is light enough so that the moment generating function of* $Y$ *is finite in some neighborhood of zero. In any case, if the optimal policy is to reinsure, it will be given again by the unique solution of* (7), (10)-(12).

## 4. Numerical calculation of optimal treaties

It can be shown that for any given $y > 0$, $R > 0$ and $\alpha$ (fixed), the Newton algorithm applied to (10) exhibits **global** quadratic convergence. This means that, given $R$ and $\alpha$, the value of $Z_{R,\alpha}(y)$ can be quickly and accurately computed. Hence, from a numerical point of view, the only remaining difficulty is to find the optimal values of $R$ and $\alpha$.

We discuss two algorithms to find numerical solutions of system (7), (10)-(12). The first algorithm (4.1 below) is guaranteed to converge to the solution albeit this convergence is slow. In alternative, we discuss application of Newton algorithm (4.2) which exhibits quadratic convergence but only if the initial guess is sufficiently close to the solution.

Many variants of these algorithms can be devised improving the basic design given below. Most of these variants require the computation of the same basic quantities as the simple designs. Thus, the main point of the following discussion is to show that all the required functions can be computed by standard numerical quadrature algorithms.

In the following, we use $G(R,\alpha)$ to denote $G(R, Z_{R,\alpha})$.

### 4.1. An algorithm with sure convergence

Due to Proposition 4, the solution to Problem 1 can be found in two steps:

**(S1)** For each $R > 0$ find $\alpha_R$, the unique solution of (12) for that particular $R$ ($\alpha_R = 0$ if there is no solution);

**(S2)** Solve the one-variable equation $G(R,\alpha_R) = 1$.

Thus, algorithms that are guaranteed to converge to the solution of Problem 1 can be based on algorithms having the same property for each of the steps **(S1)**, **(S2)** above.

Bisection algorithms are very simple algorithms of this kind.

*Sub-algorithm to compute $\alpha_R$*

The following algorithm to compute $\alpha_R$ takes as inputs an initial guess $\alpha$ for $\alpha_R$, a step size parameter $d \in (0,1)$ and a tolerance parameter $\varepsilon > 0$. The output is an estimate of $\alpha_R$ with truncation error smaller that $\varepsilon$.

**Initialization:** (Find $\alpha_1 < \alpha_2$ such that $h(R,\alpha_1) < 0$, $h(R,\alpha_2) \geq 0$)

  **Case** $h(R,\alpha) \geq 0$: **While** $h(R,\alpha) \geq 0$ **and** $\alpha \geq \varepsilon$: set $\alpha_2 = \alpha$, update $\alpha = d \times \alpha$;

    **When** $h(R,\alpha) < 0$: set $\alpha_1 = \alpha$; **When** $\alpha < \varepsilon$: set $\alpha_1 = 0$, $\alpha_2 = \alpha$.

  **Case** $h(R,\alpha) < 0$: **While** $h(R,\alpha) < 0$: set $\alpha_1 = \alpha$, update $\alpha = \frac{1}{d} \times \alpha$;

    **When** $h(R,\alpha) > 0$: set $\alpha_2 = \alpha$.

**While** $\alpha_2 - \alpha_1 \geq \varepsilon$: Set $\alpha = \frac{\alpha_1 + \alpha_2}{2}$;

  **If** $h(R,\alpha) > 0$: update $\alpha_2 = \alpha$;

  **If** $h(R,\alpha) < 0$: update $\alpha_1 = \alpha$;

**When** $\alpha_2 - \alpha_1 < \varepsilon$: **Output** $\alpha_R = \frac{\alpha_1 + \alpha_2}{2}$.

The scheme above is still consistent in the case where no positive $\alpha$ solves (12) for the particular $R$ being considered. To see this, notice that

$$\lim_{\alpha \to +\infty} h(R,\alpha) = +\infty$$

holds for every $R > 0$. Therefore, Proposition 4 shows that if there is no positive $\alpha$ solving (12), then $h(R,\alpha) > 0$ must hold for every $\alpha > 0$ and that particular $R$. Thus, the scheme will return a value $\alpha_R$ that differs from zero less than the tolerance $\varepsilon$.

*Algorithm to compute R and α*

The algorithm below takes as inputs initial guesses of $R$ and $\alpha$, a step size parameter $d \in (0,1)$, a tolerance parameter $\varepsilon > 0$ and a large parameter $K$ (see below the meaning of this parameter). The output is an estimate of $(R,\alpha)$. The truncation error in the computation of $R$ is smaller that $\varepsilon$, while the estimate of $\alpha$ differs by less than $\varepsilon$ from the true value of $\alpha_R$ (where $R$ is the estimated $R$, not the exact adjustment coefficient).

**Initialization:** (Find $R_1 < R_2$ such that $G(R_1, \alpha_{R_1}) < 1$, $G(R_2, \alpha_{R_2}) \geq 1$)

Use sub-algorithm to find $\alpha_R$;

**Case** $G(R,\alpha_R) \geq 1$: **While** $G(R,\alpha_R) \geq 1$**:** set $R_2 = R$, update $R = d \times R$, use sub-algorithm to update $\alpha_R$;

**When** $G(R,\alpha_R) < 1$: set $R_1 = R$.

**Case** $G(R,\alpha_R) < 1$: **While** $G(R,\alpha_R) < 1$ **and** $R < K$: set $R_1 = R$, update $R = \frac{1}{d} \times R$, use sub-algorithm to update $\alpha_R$;

**When** $G(R,\alpha_R) \geq 1$: set $R_2 = R$; **When** $R \geq K$: set $R_1 = R_2 = K$.

**While** $R_2 - R_1 \geq \varepsilon$: Set $R = \frac{R_1 + R_2}{2}$, use sub-algorithm to find $\alpha_R$;

**If** $G(R,\alpha_R) \geq 1$: update $R_2 = R$;

**If** $G(R,\alpha_R) < 1$: update $R_1 = R$;

**When** $R_2 - R_1 < \varepsilon$: Set $R = \frac{R_1 + R_2}{2}$, use sub-algorithm to find $\alpha_R$,

**Output** $(R,\alpha_R)$.

The parameter $K$ allows for a numerical check of condition (2). Results in Guerra and Centeno (2008) show that if (2) fails, then $G(R,\alpha_R) < 1$ holds for every $R > 0$. Thus, the scheme above will stop in a finite number of steps and yield an estimate $R = K$. By the Lundberg inequality, the corresponding treaty $Z_{K,\alpha_K}$ gives "virtually zero" probability of ultimate ruin.

### 4.2. Newton algorithm

The Newton algorithm applied to system (7), (10)-(12) takes the form outlined below. The required inputs are an initial guess of the solution $(R,\alpha)$ and a tolerance parameter $\varepsilon > 0$.

**(1)** Find $x_1$, $x_2$, solving the system

$$\begin{cases} \dfrac{\partial G}{\partial R}(R,\alpha)x_1 + \dfrac{\partial G}{\partial \alpha}(R,\alpha)x_2 = 1 - G(R,\alpha) \\ \dfrac{\partial h}{\partial R}(R,\alpha)x_1 + \dfrac{\partial h}{\partial \alpha}(R,\alpha)x_2 = -h(R,\alpha) \end{cases}$$

**(2)** Update $R$ and $\alpha$: $R = R + x_1$, $\alpha = \alpha + x_2$;

Repeat steps **(1)-(2)** until $|x_1| < \varepsilon$ and $|x_2| < \varepsilon$.

It is well known that if the initial guess $(R,\alpha)$ is sufficiently close to the solution and the Jacobian matrix

$$\begin{pmatrix} \frac{\partial G}{\partial R}(R,\alpha) & \frac{\partial G}{\partial \alpha}(R,\alpha) \\ \frac{\partial h}{\partial R}(R,\alpha) & \frac{\partial h}{\partial \alpha}(R,\alpha) \end{pmatrix} \tag{26}$$

is nonsingular when $(R, \alpha)$ solves (7), (10)-(12), then the Newton algorithm converges quadratically fast and $\varepsilon$ is an approximate upper bound for the truncation error (see, e.g., Quarteroni et al. (2000), Ch. 7).

We will show that if $Z \equiv 0$ (no reinsurance at all) is not optimal and (2) holds, then (26) is nonsingular (see Corollary 1). However, the algorithm may fail to converge when $Z \equiv 0$ is optimal and diverges when (2) fails. Further, numerical experiments suggest that in the cases where the algorithm converges, the choice of an initial guess may be tricky.

A suitable compromise between reliability and efficiency is to start by running algorithm 4.1 with a fairly large tolerance parameter to check condition (2), check that the optimal $\alpha$ is strictly positive and find a sufficiently accurate guess from which the Newton algorithm can be started.

### 4.3. Algorithm requirements

The algorithm outlined in Section 4.1 only requires evaluating the functions $h(R,\alpha)$ and $G(R,\alpha)$. The Newton Algorithm requires evaluating $h(R,\alpha)$, $G(R,\alpha)$ and matrix (26).

We will proceed to show that these functions reduce to sums of integrals which can be computed by standard numerical quadrature methods.

First, we show how $G(R,\alpha)$ and $h(R,\alpha)$ can be represented as functions of $\Phi_1(R,\alpha)$ and $\Phi_2(R,\alpha)$:

**Proposition 5**

$$G(R,\alpha) = \frac{\Phi_1(R,\alpha)-1}{R\alpha} e^{R(P(Z_{R,\alpha})-c)};$$

$$h(R,\alpha) = \frac{\Phi_1(R,\alpha)-1}{R\alpha} - \frac{1}{2g'\left(\frac{\Phi_2(R,\alpha)-\Phi_1(R,\alpha)^2}{R^2}\right)}$$

*(with $P(Z_{R,\alpha})$ expressed as a function of* $\Phi_1(R,\alpha)$ and $\Phi_2(R,\alpha)$ using (16)-(17).

□

**Proof.** It follows straightforwardly from (10) that

$$\begin{aligned} G(R,\alpha) &= e^{R(P(Z_{R,\alpha})-c)} \int_0^{+\infty} e^{R(y-Z_{R,\alpha}(y))} f(y)dy = \\ &= e^{R(P(Z_{R,\alpha})-c)} \int_0^{+\infty} \frac{Z_{R,\alpha}(y)+\alpha}{\alpha} f(y)dy = \\ &= \frac{\Phi_1(R,\alpha)-1}{R\alpha} e^{R(P(Z_{R,\alpha})-c)}. \end{aligned}$$

The second equality is trivially obtained by substituting (16)-(17) in (11). ■

In order to deal with partial derivatives of $G$ and $h$, we introduce the auxiliary functions

$$\Psi_k(R,\alpha) = \int_0^{+\infty} (1 + R(Z_{R,\alpha}(y)+\alpha))^k \ln\left(\frac{Z_{R,\alpha}(y)+\alpha}{\alpha}\right) f(y)dy, \quad R > 0,\ \alpha > 0,\ k \in \mathbb{Z}.$$

In the following we will omit the arguments $(R,\alpha)$, i.e., we denote $Z_{R,\alpha}$ by $Z$, $\Phi_k(R,\alpha)$ by $\Phi_k$, etc. To simplify further, we use $g$, $g'$, $g''$ to denote $g(Var[Z_{R,\alpha}])$, $g'(Var[Z_{R,\alpha}])$ and $g''(Var[Z_{R,\alpha}])$, respectively.

**Proposition 6.** *For $k \leq 2$, $\alpha > 0$ the map $R \mapsto \Phi_k(R,\alpha)$ is smooth and*

$$\frac{\partial \Phi_k}{\partial R} = \frac{k}{R}(\Phi_k - \Phi_{k-1} + \Psi_{k-1} - \Psi_{k-2}). \ \square$$

**Proof.** By differentiating (10) with respect to $R$, we obtain

$$0 = \frac{\partial Z}{\partial R} - \frac{1}{R^2} \ln\left(\frac{Z+\alpha}{\alpha}\right) + \frac{1}{R} \frac{1}{Z+\alpha} \frac{\partial Z}{\partial R}.$$

Solving this with respect to $\frac{\partial Z}{\partial R}$ yields

$$\frac{\partial Z}{\partial R} = \frac{1}{R^2}\left(1 - \frac{1}{1+R(Z+\alpha)}\right) \ln\left(\frac{Z+\alpha}{\alpha}\right).$$

The assumption $E[Y^2] < +\infty$ implies that $\Psi_k$ is finite for all $k \leq 1$, $R > 0$, $\alpha > 0$. It follows that

$$\begin{aligned} \frac{\partial \Phi_k}{\partial R} &= \int_0^{+\infty} k(1 + R(Z+\alpha))^{k-1}\left(Z + \alpha + R\frac{\partial Z}{\partial R}\right) f dy = \\ &= \int_0^{+\infty} k(1 + R(Z+\alpha))^{k-1} \times \end{aligned}$$

$$\times \left( Z + \alpha + \frac{1}{R} \left( 1 - \frac{1}{1 + R(Z+\alpha)} \right) \ln \left( \frac{Z+\alpha}{\alpha} \right) \right) f dy =$$

$$= \frac{k}{R} \int_0^{+\infty} (1 + R(Z+\alpha))^{k-1} (1 + R(Z+\alpha) - 1) \, f dy +$$

$$+ \frac{k}{R} \int_0^{+\infty} (1 + R(Z+\alpha))^{k-1} \left( 1 - \frac{1}{1 + R(Z+\alpha)} \right) \ln \left( \frac{Z+\alpha}{\alpha} \right) f dy =$$

$$= \frac{k}{R} (\Phi_k - \Phi_{k-1} + \Psi_{k-1} - \Psi_{k-2}).$$

■

Using Propositions 5, 2 and 6, long but straightforward computations result in the following expressions for the partial derivatives:

**Proposition 7.** *The partial derivatives of G and h can be represented as*

$$\frac{\partial G}{\partial \alpha} = \frac{(1 + R\alpha) e^{R(P-c)}}{R\alpha^2} \left( 1 - 2 \frac{\Phi_1 - 1}{R} g' \right) (1 - \Phi_1 \Phi_{-1});$$

$$\frac{\partial G}{\partial R} = \frac{e^{R(P-c)}}{R\alpha} \left( (\Phi_1 - 1) \left( P - c + 2 \frac{\Psi_1 - \Psi_0}{R^2} g' \right) + \left( 1 - 2 \frac{\Phi_1 - 1}{R} g' \right) \frac{\Psi_0 - \Psi_{-1}}{R} \Phi_1 \right);$$

$$\frac{\partial h}{\partial \alpha} = \frac{1 + R\alpha}{R\alpha} (1 - \Phi_{-1}) + \frac{1 - (1 + R\alpha)\Phi_{-1}}{R\alpha} \frac{g''}{2(g')^2} + \frac{1 + R\alpha}{R^2 \alpha} (\Phi_1 \Phi_{-1} - 1) \frac{g''}{g'};$$

$$\frac{\partial h}{\partial R} = \left( 1 - \frac{\Phi_1}{R} \frac{g''}{(g')^2} \right) \frac{\Psi_0 - \Psi_{-1}}{R^2} + \frac{\Psi_1 - \Psi_0}{R^3} \frac{g''}{(g')^2}. \ \square$$

Proposition 7 has the following corollary, announced in Section 4.2:

**Corollary 1.** *If* $(R, \alpha)$ solves system (7), (10)-(12) *and* $\alpha > 0$*, then matrix* (26) *admits inverse.* □

**Proof.** Throughout this proof $(R, \alpha)$ is the solution of (7), (10)-(12) with $\alpha > 0$.

Proposition 4 shows that $\frac{\partial h}{\partial \alpha} > 0$. By Proposition 5, equation (12) is equivalent to

$$\frac{\Phi_1 - 1}{R} = \frac{1}{2g'}. \tag{27}$$

Hence, we see from Proposition 7 that $\frac{\partial G}{\partial \alpha} = 0$ holds and we only need to prove that $\frac{\partial G}{\partial R} \neq 0$.

Due to (27) and Proposition 7, we have

$$\frac{\partial G}{\partial R} = \frac{e^{R(P-c)}}{R\alpha} (\Phi_1 - 1) \left( P - c + \frac{1}{R} \frac{\Psi_1 - \Psi_0}{\Phi_1 - 1} \right).$$

Using Proposition 5, this reduces to

$$\frac{\partial G}{\partial R} = \frac{G}{R}\left(\ln\left(e^{R(P-c)}\right) + \frac{\Psi_1 - \Psi_0}{\Phi_1 - 1}\right) = \frac{1}{R}\left(\frac{\Psi_1 - \Psi_0}{\Phi_1 - 1} - \ln\frac{\Phi_1 - 1}{R\alpha}\right).$$

Due to Jensen's inequality, we have

$$-\ln\frac{\Phi_1 - 1}{R\alpha} \le \int_0^{+\infty} - \ln\frac{Z(y) + \alpha}{\alpha} f(y)\,dy = -\Psi_0.$$

Substituting above, one obtains

$$\frac{\partial G}{\partial R} \le \frac{1}{R}\frac{\Psi_1 - \Phi_1\Psi_0}{\Phi_1 - 1}.$$

Using Schwartz inequality and an argument analogous to the one used to prove Proposition 4, we see that this implies $\frac{\partial G}{\partial R} < 0$. ■

### 4.4. Computation of integrals

Propositions 5 and 7 show that the implementation of the algorithm described in Section 4.1 depends only in our ability to compute the integrals $\Phi_1$ and $\Phi_2$, while implementation of the Newton algorithm requires also computation of $\Phi_{-1}$, $\Psi_1$, $\Psi_0$ and $\Psi_{-1}$.

At first glance, these integrals may seem hard to compute because equation (10) defines $Z_{R,\alpha}$ only in implicit form. The following proposition removes this difficulty by giving an explicit representation of $\Phi_k$ and $\Psi_k$.

**Proposition 8.** *The functions $\Phi_k$, $\Psi_k$ can be represented as the integrals:*

$$\Phi_k(R,\alpha) = \frac{1}{R}\int_0^{+\infty} \frac{(1 + R(\zeta + \alpha))^{k+1}}{\zeta + \alpha} f\left(\zeta + \frac{1}{R}\ln\frac{\zeta + \alpha}{\alpha}\right) d\zeta, \tag{28}$$

$$\Psi_k(R,\alpha) = \frac{1}{R}\int_0^{+\infty} \frac{(1 + R(\zeta + \alpha))^{k+1}}{\zeta + \alpha} \ln\left(\frac{\zeta + \alpha}{\alpha}\right) f\left(\zeta + \frac{1}{R}\ln\frac{\zeta + \alpha}{\alpha}\right) d\zeta. \quad \square \tag{29}$$

**Proof.** Using the change of variable $y = \zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}$, $\zeta \in [0, +\infty[$, we obtain

$$\begin{aligned}
\Phi_k &= \int_0^{+\infty} (1 + R(Z(y) + \alpha))^k f(y)\,dy = \\
&= \int_0^{+\infty} (1 + R(\zeta + \alpha))^k f\left(\zeta + \frac{1}{R}\ln\frac{\zeta + \alpha}{\alpha}\right)\left(1 + \frac{1}{R(\zeta + \alpha)}\right) d\zeta = \\
&= \frac{1}{R}\int_0^{+\infty} \frac{(1 + R(\zeta + \alpha))^{k+1}}{\zeta + \alpha} f\left(\zeta + \frac{1}{R}\ln\frac{\zeta + \alpha}{\alpha}\right) d\zeta.
\end{aligned}$$

The proof of equality (29) is similar. ■

Since the integrand functions in (28)-(29) are explicit, there are several numerical integration schemes that can be applied. The choice of a particular one will depend on the properties of the particular distribution of $Y$ being considered. The interested reader may consult Quarteroni et al. (2000), Ch. 9 and 10, and references therein.

To proceed further, we will consider the case when the density $f$ is a continuous function in $[0,+\infty)$ with the possible exception of a finite set of points, where it has right and left limits (possibly infinite). Thus, the density $f$ can be unbounded but only in the neighborhoods of a finite number of points of discontinuity. Also, we assume that there is some constant $\varepsilon > 0$ such that

$$\lim_{y \to +\infty} y^{3+\varepsilon} f(y) = 0 \tag{30}$$

Notice that our blanket assumption that $E[Y^2] < +\infty$ guarantees that if $\lim_{y \to +\infty} y^3 f(y)$ exists, then it must be zero. The case that we are considering now is fairly generic and presents the advantage that the integrals (28)-(29) can be further reduced to integrals of continuous functions in compact intervals.

To see this, notice that there is a partition $0 = a_0 < a_1 < \ldots < a_m < +\infty$ such that the map $\zeta \mapsto f\left(\zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right)$ is continuous and bounded in $[a_m, +\infty)$ and for each $i \in \{1,2,\ldots,m\}$ it is continuous in one semiclosed interval, $(a_{i-1}, a_i]$ or $[a_{i-1}, a_i)$. Thus, we only need to reduce integrals over the intervals $[a_m, +\infty)$ and $(a_{i-1}, a_i]$ or $[a_{i-1}, a_i)$, $i = 1,2,\ldots,m$.

If condition (30) holds then the condition

$$\lim_{y \to +\infty} y^{2+\varepsilon} \ln(y) f(y) = 0 \tag{31}$$

also holds. Using the change of variable $\zeta = t^{-\frac{1}{\varepsilon}} - 1$, we obtain

$$\begin{aligned}
&\int_{a_m}^{+\infty} \frac{(1 + R(\zeta + \alpha))^{k+1}}{\zeta + \alpha} f\left(\zeta + \frac{1}{R}\ln\left(\frac{\zeta+\alpha}{\alpha}\right)\right) d\zeta = \\
&\qquad = \frac{1}{\varepsilon}\int_0^{(1+a_m)^{-\varepsilon}} \frac{(1 + R(t^{-\frac{1}{\varepsilon}} - 1 + \alpha))^{k+1}}{t^{-\frac{1}{\varepsilon}} - 1 + \alpha} \times \\
&\qquad\qquad \times f\left(t^{-\frac{1}{\varepsilon}} - 1 + \frac{1}{R}\ln\left(\frac{t^{-\frac{1}{\varepsilon}} - 1 + \alpha}{\alpha}\right)\right) t^{-\frac{1}{\varepsilon}(1+\varepsilon)} dt,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\int_{a_m}^{+\infty} \frac{(1 + R(\zeta + \alpha))^{k+1}}{\zeta + \alpha} \ln\left(\frac{\zeta+\alpha}{\alpha}\right) f\left(\zeta + \frac{1}{R}\ln\left(\frac{\zeta+\alpha}{\alpha}\right)\right) d\zeta = \\
&\qquad = \frac{1}{\varepsilon}\int_0^{(1+a_m)^{-\varepsilon}} \frac{(1 + R(t^{-\frac{1}{\varepsilon}} - 1 + \alpha))^{k+1}}{t^{-\frac{1}{\varepsilon}} - 1 + \alpha} \ln\left(\frac{t^{-\frac{1}{\varepsilon}} - 1 + \alpha}{\alpha}\right) \times \\
&\qquad\qquad \times f\left(t^{-\frac{1}{\varepsilon}} - 1 + \frac{1}{R}\ln\left(\frac{t^{-\frac{1}{\varepsilon}} - 1 + \alpha}{\alpha}\right)\right) t^{-\frac{1}{\varepsilon}(1+\varepsilon)} dt.
\end{aligned} \tag{33}$$

We can check that the integrand on the right-hand side of (32) is bounded for $k \leq 2$ and the integrand on the right-hand side of (33) is bounded for $k \leq 1$.

Now, consider the case when $\zeta \mapsto f\left(\zeta + \frac{1}{R}\ln\frac{\zeta+\alpha}{\alpha}\right)$ is continuous in $(a_{i-1}, a_i]$ (resp., $[a_{i-1}, a_i)$) but

$$\lim_{y \to \left(a_{i-1} + \frac{1}{R}\ln\frac{a_{i-1}+\alpha}{\alpha}\right)^+} f(y) = +\infty \qquad \left(\text{resp.,} \lim_{y \to \left(a_i + \frac{1}{R}\ln\frac{a_i+\alpha}{\alpha}\right)^-} f(y) = +\infty\right).$$

Since $f$ is integrable, it follows that

$$\lim_{y \to \left(a_{i-1} + \frac{1}{R}\ln\frac{a_{i-1}+\alpha}{\alpha}\right)^+} f(y)\sqrt{y - \left(\alpha_{i-1} + \frac{1}{R}\ln\frac{\alpha_{i-1}+\alpha}{\alpha}\right)} = 0$$

$$\left(\text{resp.,} \lim_{y \to \left(a_i + \frac{1}{R}\ln\frac{a_i+\alpha}{\alpha}\right)^-} f(y)\sqrt{\alpha_i + \frac{1}{R}\ln\frac{\alpha_i+\alpha}{\alpha} - y} = 0\right)$$

must hold.

Using the change of variable $\zeta = a_{i-1} + (a_i - a_{i-1})t^2$ (resp., $\zeta = a_i - (a_i - a_{i-1})t^2$), we transform the integrals

$$\int_{a_{i-1}}^{a_i} \frac{(1 + R(\zeta+\alpha))^{k+1}}{\zeta+\alpha} f\left(\zeta + \frac{1}{R}\ln\left(\frac{\zeta+\alpha}{\alpha}\right)\right)d\zeta,$$

$$\int_{a_{i-1}}^{a_i} \frac{(1 + R(\zeta+\alpha))^{k+1}}{\zeta+\alpha} \ln\left(\frac{\zeta+\alpha}{\alpha}\right) f\left(\zeta + \frac{1}{R}\ln\left(\frac{\zeta+\alpha}{\alpha}\right)\right)d\zeta$$

into integrals of continuous functions over the interval $[0, 1]$.

Further, if $f$ has continuous derivatives up to order $n$ in the intervals $(0, a_1)$, $(a_1, a_2)$, ..., $(a_{m-1}, a_m)$, $(a_m, +\infty)$, then the same holds for the integrands after the changes of variables introduced above. In that case all the integrals can be computed by Gaussian quadrature or any other standard method based on smooth interpolation. Note that adaptive quadrature based on these methods allows for easy estimates of the truncation error.

## 5. Examples

In this section we give two examples for the standard deviation principle. In the first example we consider that $Y$ follows a Pareto distribution. In the second example we consider a generalized gamma distribution. The parameters of these distributions were chosen such that $E[Y] = 1$ and both distributions have the same variance (which was set to $Var[Y] = \frac{16}{5}$, for convenience of the choice of parameters). Notice that though they have the same mean and variance, the tails of the two distributions are rather different. However, none of them has moment generating function defined in any neighborhood of the origin. Hence the optimal solution must be other than no reinsurance.

In both examples we consider the same premium income $c = 1.2$ and the same loading coefficient $\beta = 0.25$.

**Example 1.** *We consider that $Y$ follows the Pareto distribution*

$$f(y) = \frac{32 \times 21^{32/11}}{(21+11y)^{43/11}}, \quad y > 0.$$

*The first column of Table 1 shows the optimal value of $\alpha$ and the corresponding values of $R$, $E[Z]$, $Var[Z]$, $P(Z)$,* and *$E[L_Z]$, while the second column shows the corresponding values for the best (in terms of the adjustment coefficient) stop loss treaty. The optimal policy improves the adjustment coefficient by* 16.1% *with respect to the best stop loss treaty, at the cost of an increase of* 111% *in the reinsurance premium. However, notice that the relative contribution of the loading to the total reinsurance premium is much smaller in the optimal policy, compared with the best stop loss. Hence, though a larger premium is ceded under the optimal treaty than under the best stop loss, this is made mainly through the pure premium, rather than the premium loading, so the expected profits are not dramatically different.*

*Figure 1 shows the optimal reinsurance arrangement versus the best stop loss treaty $Z_M(y) = \max\{0, y - M\}$. It shows that the improved performance of the optimal policy is achieved partly by compensating a lower level of reinsurance against very high losses (which occur rarely) by reinsuring a substantial part of moderate losses, which occur more frequently but are inadequately covered or not covered at all by the stop-loss treaty.*

*In general, it can be expected that the treaty selected in a practical context is suboptimal. Supposing that the direct insurer is allowed to chose a treaty of the type (10), numerical errors and incomplete knowledge about the distribution of claims ensure that the choice of the value for the parameter $\alpha$ can not be made with complete accuracy. Therefore, it is interesting to see how the adjustment coefficients of treaties of type (10) and stop loss treaties behave as functions of the treaty parameters (resp., $\alpha$ and $M$). For this purpose we present some additional figures.*

TABLE 1

$Y$ – PARETO RANDOM VARIABLE

| | **Optimal Treaty** $\alpha = 1.74411$ | **Best Stop Loss** $M = 67.4436$ |
|---|---|---|
| $R$ | 0.055406 | 0.047703 |
| $E[Z]$ | 0.098018 | 0.001050 |
| $Var[Z]$ | 0.212089 | 0.160269 |
| $P(Z)$ | 0.213151 | 0.101134 |
| $E[L_Z]$ | 0.084867 | 0.099916 |

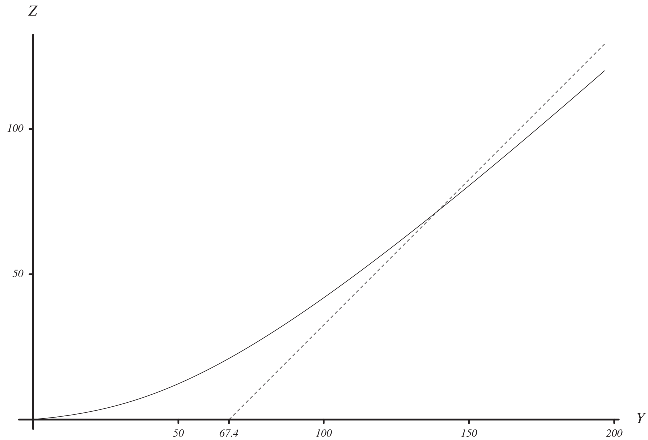


FIGURE 1: Optimal policy (full line) versus best stop loss (dashed line): the Pareto case.

*Figure 2 plots values of the adjustment coefficient against the treaty parameters. In order to make the retained risk to increase in the same direction (from left to right) in both curves, we plot the $\alpha$ parameter of the treaties (10) in inverse scale (i.e., we plot $\frac{1}{\alpha}$). The curves corresponding to both types of treaties have the same overall shape, decreasing smoothly to the right of a well defined maximum. However, notice that the horizontal scales of these curves is not comparable because the parameters M and $\frac{1}{\alpha}$ do not have any common interpretation.*

*In order to make the comparison more meaningful we present two other plots in which the horizontal axis has the same meaning for both treaties. In Figure 3*

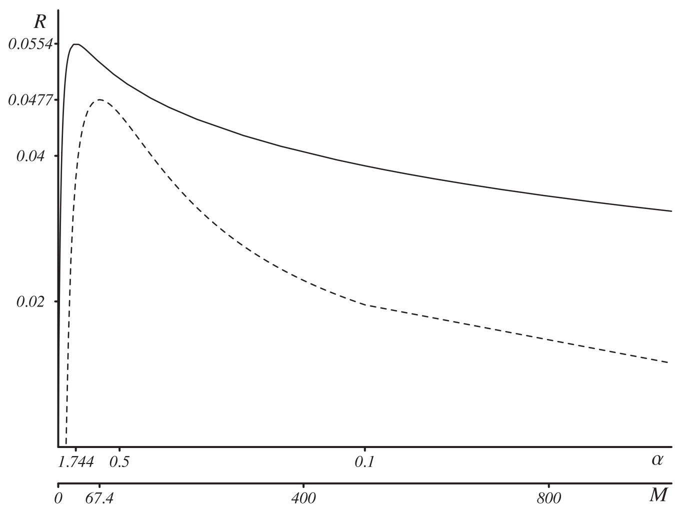


FIGURE 2: Adjustment coefficient as a function of treaty parameter for policies of type (10) (full line) compared with stop loss policies (dashed line) in the Pareto case. In both cases the horizontal axis represents the policy parameter ($\alpha$ and $M$, resp., scales not comparable).

*we show the adjustment coefficient plotted as a function of the ceded risk ($E[Z]$). We see that while stop loss policies exhibit a very sharp maximum corresponding to a small value of $E[Z]$, the policies of type (10) exhibit a broad maximum. The adjustment coefficient of stop loss policies decreases very steeply when $E[Z]$ departs in either way from the optimum (this can be seen in some detail in Figure 4). Such behavior contrasts with policies of type (10) which keep a good performance even when the amount of risk ceded differs substantially from the optimum.*

*The presence of a sharp maximum is due to the fact that when stop loss policies are considered, the expected profit decreases very quickly when the ceded risk increases. By contrast, using policies of type (10) it is possible to cede a larger amount of risk with a moderate decrease in the expected profit.*

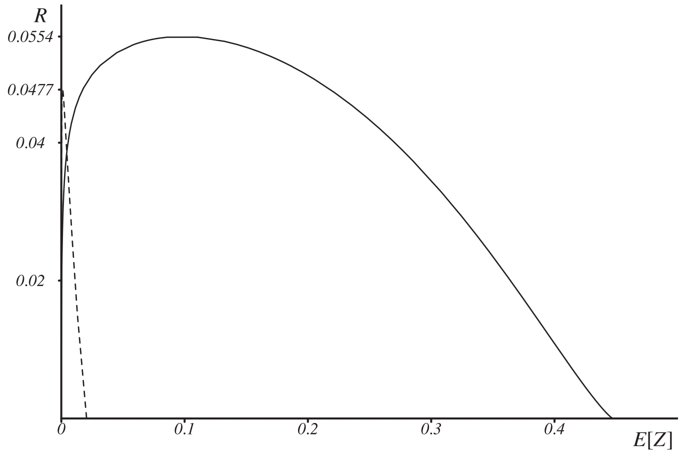


FIGURE 3: Adjustment coefficient as a function of the ceded risk ($E[Z]$) for policies of type (10) (full line) compared with stop loss policies (dashed line) in the Pareto case.

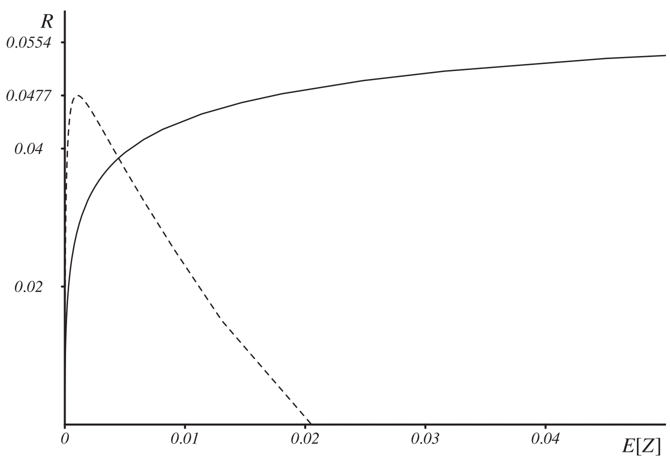


FIGURE 4: Detail of figure 3.

*Figure 5 shows the adjustment coefficient plotted as a function of the expected profit ($E[L_Z]$). Recall that the adjustment coefficient is defined only for policies satisfying $E[L_Z] > 0$ and in our examples $E[L_Z] \leq 0.2$ holds for all $Z \in \mathcal{Z}$. Therefore we see that the policies of type (10) significantly outperform the comparable stop loss policies except for very high or very low values of expected profit (i.e., except in situations of very strong over-reinsurance or sub-reinsurance).*

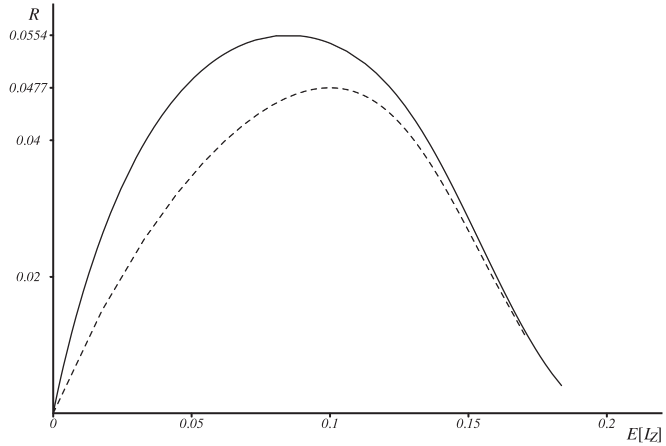


FIGURE 5: Adjustment coefficient as a function of expected profit ($E[L_Z]$) for policies of type (10) (full line) compared with stop loss policies (dashed line) in the Pareto case.

**Example 2.** *In this example, Y follows the generalized gamma distribution with density*

$$f(y) = \frac{b}{\Gamma(k)\theta}\left(\frac{y}{\theta}\right)^{kb-1} e^{-\left(\frac{y}{\theta}\right)^b}, \quad y > 0,$$

*with* $b = 1/3$, $k = 4$ *and* $\theta = 3!/6!$. *Table 2 shows the results for this example. The general features are similar to Example 1 but the improvement with respect to the best stop loss is smaller (the optimal policy increases the adjustment coefficient by about 7.8% with respect to the best stop loss). The optimal policy presents a larger increase in the sharing of risk and profits and a sharp increase in the reinsurance premium (more than seven-fold) with respect to the best stop loss. However, in both cases the amount of the risk and of the profits which are ceded under the reinsurance treaty are substantially smaller than in the Pareto case. Our comments on Example 1 comparing the performance of treaties of type (10) with stop loss treaties remain valid for the present example.*

*Notice that the plots of the adjustment coefficients as functions of the expected profit in the present example (Figure 8) are skewed to the right compared with the corresponding plot in Example 1 (Figure 5). In Figure 8 the stop loss treaty presents a sharper maximum than in Figure 5, while the opposite is true for the treaties of type (10).*

TABLE 2

$Y$ – GENERALIZED GAMMA RANDOM VARIABLE

| | **Optimal Treaty** $\alpha = 0.813383$ | **Best Stop Loss** $M = 47.8468$ |
|---|---|---|
| $R$ | 0.084709 | 0.078571 |
| $E[Z]$ | 0.076969 | 0.000204 |
| $Var[Z]$ | 0.049546 | 0.004951 |
| $P(Z)$ | 0.132616 | 0.017794 |
| $E[L_Z]$ | 0.144353 | 0.182410 |

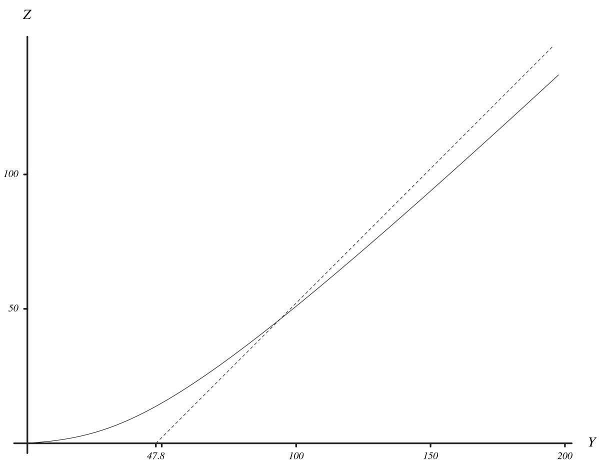


FIGURE 6: Optimal policy (full line) versus best stop loss (dashed line): the generalized gamma case.

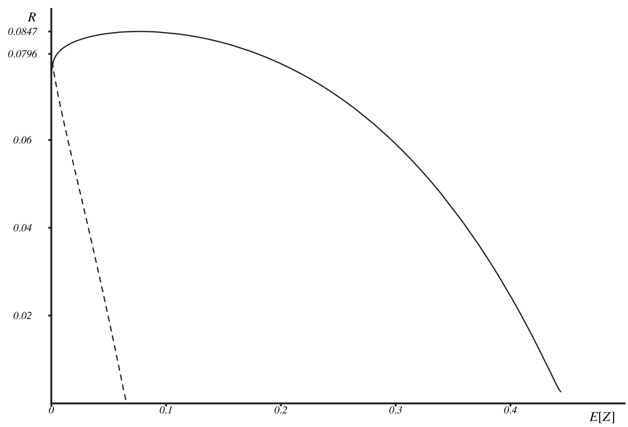


FIGURE 7: Adjustment coefficient as a function of the ceded risk ($E[Z]$) for policies of type (10) (full line) compared with stop loss policies (dashed line) in the generalized gamma case.

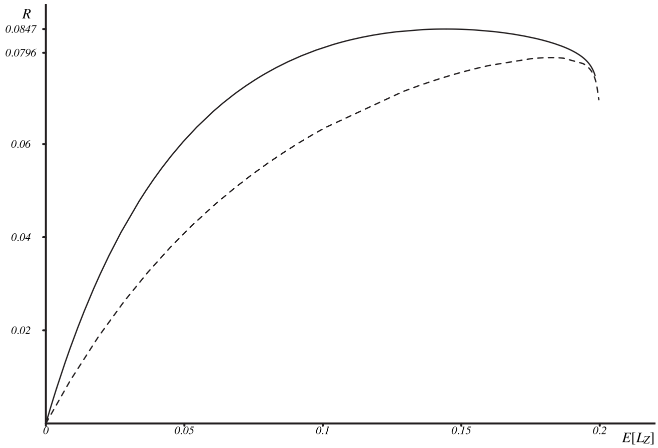


FIGURE 8: Adjustment coefficient as a function of the expected profit ($E[L_Z]$) for policies of type (10) (full line) compared with stop loss policies (dashed line) in the generalized gamma case.

## REFERENCES


ARROW, K.J. (1963) Uncertainty and the Welfare of Medical Care. *The American Economic Review*, **LIII**, 941-973.

BORCH, K. (1960) An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance. *Transactions of the 16th International Congress of Actuaries*, 597-610.

DEPREZ, O. and GERBER, H.U. (1985) On convex principles of premium calculation. *Insurance: Mathematics and Economics*, **4**, 179-189.

GUERRA, M. and CENTENO, M.L. (2008) Optimal Reinsurance policy: the adjustment coefficient and the expected utility criteria. *Insurance Mathematics and Economis*, **42**, 529-539.

HESSELAGER, O. (1990) Some results on optimal reinsurance in terms of the adjustment coefficient. *Scandinavian Actuarial Journal*, **1**, 80-95.

KALUSZKA, M. (2004) An extension of Arrow's result on optimality of a stop loss contract. *Insurance: Mathematics and Economics*, **35**, 524-536.

QUARTERONI, A., SACCO, R. and SALERI, F. (2000) *Numerical Mathematics*. Springer-Verlag, New York.



MANUEL GUERRA
*CEOC and ISEG -T.U.Lisbon*
*R. Quelhas 6, 1200-781 Lisboa, Portugal*
*E-Mail: mguerra@iseg.utl.pt*

MARIA DE LOURDES CENTENO
*CEMAPRE, ISEG -T.U.Lisbon*
*R. Quelhas 6, 1200-781 Lisboa, Portugal*
*E-Mail: lcenteno@iseg.utl.pt*