ANALYTICAL PRICING OF THE UNIT-LINKED ENDOWMENT WITH GUARANTEES AND PERIODIC PREMIUMS

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ABSTRACT

We consider the unit-linked endowment with guarantee and periodic premiums, where at each premium payment date the insurance company invests a certain fraction of the premium into a risky reference portfolio. In the dual random environment of stochastic interest rates with deterministic volatilities and mortality risk, and for a fixed guarantee, simple analytical lower and upper bounds for the fair periodic premium are explicitly derived. We also consider contracts with guaranteed minimum benefits that vary over time and we obtain tight lower and upper bounds for both fair periodic premiums and guaranteed minimum benefits that increase over time. The numerical illustrations of our results reveal that the analytical bounds are very tight. Moreover, the simple, fast and very reliable analytical numerical calculations with controlled accuracy avoid time consuming Monte Carlo calculations and are almost always preferred by practitioners. Some analytical closed-form solutions for one- and two-year maturity dates are also stated.

KEYWORDS

Unit-linked endowment, guaranteed minimum death and accumulation benefits, fair premium, convex lower and upper bounds

1. INTRODUCTION

The specific feature of a unit-linked life insurance contract is the fact that the benefit payable at expiration depends upon the market value of some reference portfolio. A unit-linked endowment with guarantee is a unit-linked contract, which additionally provides for a guaranteed minimum benefit payable on either death or survival at maturity date. In contrast to traditional insurance the benefit is random but part of the investment risk is covered by the insurer. This product combines mortality risk (uncertain payment date) and investment risk (uncertain investment performance). Since the random benefit is the greater of the value of some reference portfolio and some guaranteed minimum payment,

Astin Bulletin 40(2), 631-653. doi: 10.2143/AST.40.2.2061131

the payoff of the contract is equal to the guaranteed amount plus a non-negative bonus. This bonus corresponds to a call option on the reference portfolio with the guaranteed amount as exercise price, which has similarities with an Asian option. The present contribution focuses on the unit-linked endowment with guarantee and periodic premiums, where at each premium payment date the insurance company invests a certain fraction of the premium into a risky reference portfolio.

In the literature single and periodic unit-linked contracts have been analyzed among others in Brennan and Schwartz (1976/79a/79b), Boyle and Schwartz (1977), Corby (1977), Delbaen (1990), Aase and Persson (1994), Persson (1994), Bacinello and Ortu (1993a/93b/94), Nielsen and Sandmann (1995/96/2002). A basic textbook on the modelling and risk management of investment guarantees for equity-linked life insurance is Hardy(2003) and a recent survey article is Bacinello (2007). In this already long development, one notes that the periodic premium situation with stochastic interest rate dynamics has only been discussed since Bacinello and Ortu (1994) and Nielsen and Sandmann (1995). The latter authors have applied extensive Monte Carlo simulations to the pricing problem. Furthermore, they have shown existence and uniqueness of the periodic premium and other interesting properties in Nielsen and Sandmann (1996). The most recent contributions have treated this problem in a more general context. Bacinello et al. (2009a/b) apply the Least Squares Monte Carlo approach to find the fair periodic premium when an additional surrender option is embedded into the contract. In their analysis the reference fund dynamics includes stochastic volatility and jumps and the interest rates follows the Cox-Ingersoll-Ross square root process. However, the lack of analytical tractability beyond the Monte Carlo method has only been scarcely discussed. Costabile et al. (2009) use a bivariate recombining lattice, which describes the joint evolution of interest rates and equity value to compute the periodic premium in the presence of a surrender option. We focus on the original problem and show that it is possible to find tight lower and upper bounds for the fair periodic premium, which are based on stochastic ordering convex approximations derived from comonotonic random sums extensively discussed since Kaas et al. (2000) and Dhaene et al. (2002) among others. Our needs relies on the developments in Vanduffel et al. (2005a/b/2008a). Since the bonus has similarities with an Asian option, the considered approach has potential to benefit in future from the most recent developments in this area (e.g. Vanduffel et al. (2008b)).

The paper is organized as follows. Section 2 introduces shortly the unit-linked endowment with guarantee and summarizes the required definitions and notations used throughout. The market price of the periodic premium without mortality risk but within a financial market model, which includes stochastic interest rates and a risky reference fund, is determined in Section 3. Due to the assumption that the mortality process is independent of the financial market process, the determination of the periodic premium under mortality risk is obtained in Section 4 as implicit solution of an equation containing the call-option prices determined in Section 3. In Section 5 we determine simple analytical lower and upper bounds for the fair periodic premium for the general case of deterministic bond price volatilities. The numerical illustration of our results reveals rather tight bounds for the fair periodic premium. Moreover, the simple, fast and very reliable analytical numerical calculations with controlled accuracy avoid time consuming Monte Carlo calculations and are almost always preferred by practitioners. Analytical closed-form solutions for one- and two-year maturity dates are also stated. Last but not least, Section 6 extends the results of Section 5 to guaranteed minimum benefits that vary over time. Using explicit closed-form analytical formulas, we obtain tight lower and upper bounds for both fair periodic premiums and guaranteed minimum benefits that increase over time. The latter product type with increasing guaranteed minimum benefits is attractive from the policyholder's point of view and is increasingly demanded on the market. Since there has been a lack of analytical tractability beyond the Monte Carlo method for many years, the proposed method closes this gap, a fact which is of great practical value. Indeed, in recent years the main life insurance players have introduced such products. For example, AXA Winterthur has been the first Swiss life insurer to launch a private pension contract of this type in May 2006.

2. The unit-linked endowment contract with guarantee

The unit-linked endowment life insurance with asset value guarantee and periodic premium payments is a contract agreement between an insurance company and a policyholder where the buyer pays regularly a predetermined premium to the company. At maturity date or early death of the insured person the contract stipulates as benefit the greater of the value of some reference portfolio and some guaranteed minimum payment. The reference portfolio is usually build up by investing some predetermined percentage of the premium in an investment reference fund subject to financial market volatility.

Throughout the paper the following notations and definitions are used:

n	: number of premium payments
t_i	: premium payment dates, $i = 0, \ldots, n-1$, with $t_0 = 0$
$t_n = T$	: maturity date
$S(t)$	: price at time t of one unit of the reference fund
$X(t)$	: value of the reference portfolio at time t of the endowment contract
$R(t)$	: instantaneous risk-free rate of interest at time t
$P(t,s)$	: price at time t of a zero coupon bond with maturity date $s > t$
G	: guaranteed minimum death benefit (GMDB) respectively guaranteed minimum accumulation benefit (GMAB) at maturity date

P : periodic premium paid at time t_i if the insured is alive, $i = 0, \ldots, n-1$

D : amount invested in the reference portfolio at the premium payment dates, assumed proportional to the premium such that $D = a \cdot P$, $0 \le a \le 1$

$B(t)$: random benefit payable at time t

$V_s(B(t))$: market value at time $s \le t$ of the random benefit $B(t)$

$C_s(X(t),G)$: market value of an European call option at time $s \le t$ to purchase the reference portfolio at time t for the exercise price G

Due to the GMDB and GMAB guarantees, the benefit at time t satisfies the relationship

$$B(t) = \max\{X(t), G\} = G + (X(t) - G)_+. \tag{2.1}$$

The payoff (2.1) is decomposed in a *deterministic payment* G and a *stochastic bonus payment* identical to a *call-option* on the market value of the reference portfolio with exercise price G. This decomposition allows one to view the unit-linked endowment contract with guarantee as a classical life insurance contract with deterministic benefit G subject to a very specific bonus formula. The value $X(t)$ is stochastic and depends upon the price of one unit of the reference fund at time t, the prices of one unit at the past premium payment dates $t_i \le T$ and the amount to be invested in the reference portfolio. It is given by

$$X(t) = D \cdot \sum_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)}, \quad n^*(t) = \min\{i \mid t_i > t\}. \tag{2.2}$$

As usual we will assume that financial and insurance markets are perfectly competitive, frictionless and free of arbitrage opportunities. Furthermore we assume that the mortality process is independent of the financial market process. For a more rigorous mathematical exposé we refer to the original papers by Nielsen and Sandmann (1995/96/2002) as well as to the more recent paper by De Felice and Moriconi (2005). Using (2.1) one observes that the initial market value of the random benefit at time t is given by

$$V_0(B(t)) = P(0,t) \cdot G + C_0(X(t), G). \tag{2.3}$$

In a first step, one is interested in the market price of the call-option in (2.3) in the absence of mortality risk as determined in Section 3. Due to the assumption that the mortality process is independent of the financial market process, the determination of the periodic premium under mortality risk is obtained in Section 4 as implicit solution of an equation containing the call-option prices determined in Section 3.

3. Market price without mortality risk

In a continuous time and complete market framework with filtered probability space $(\Omega, \mathcal{F}, Q)$, where Q is an appropriate arbitrage-free risk neutral measure, the bond price market and the reference fund are assumed to follow a two-factor diffusion model of the type

$$\begin{aligned} \frac{dP(t,s)}{P(t,s)} &= R(t)\,dt + \sigma(t,s)\,dW_1(t), \quad 0 \le t < s, \\ \frac{dS(t)}{S(t)} &= R(t)\,dt + \sigma_S\,\rho \cdot dW_1(t) + \sigma_S\sqrt{1-\rho^2} \cdot dW_2(t), \end{aligned} \tag{3.1}$$

where $W_1(t)$ and $W_2(t)$ are independent Wiener processes under the Q-measure. In Section 5 we assume deterministic bond price volatilities $\sigma(t,s)$. Typically, this situation covers the specifications $\sigma(t,s) = \sigma \cdot (s-t)$ (continuous time limit of the Ho and Lee(1986) model in Nielsen and Sandmann(1995)) and $\sigma(t,s) = \frac{\sigma}{\alpha}(1 - \exp\{-\alpha \cdot (s-t)\}$ (term structure model of Vasicek (1977)). However, these volatility structures have an oversimplified form that is not consistent with the one observed on the market. This inconvenience can be removed. One can consider the extended Vasicek model by Hull and White (1990), which by the way can be fit to the initial term structure of interest rates (e.g. Yolcu (2005), Section 3.2.2). Other interesting models include the humped volatility models in Mercurio and Moraleda (2000/01), Moraleda and Vorst (1997) and Ritchken and Chuang (1999). The volatility of the reference fund is described by the constant σ_S. The correlation coefficient ρ measures the dependence between the bond price and the reference fund price dynamic.

Using the zero coupon bond with maturity T as numeraire for the processes $S(t)$ and $P(t,s)$, $t < s \le T$, one considers the *T-forward risk adjusted measure Q* defined by the Radon-Nikodym derivative

$$\frac{dQ^T}{dQ} = \exp\left\{\int_0^T \sigma(t,T)\,dW_1(t) - \frac{1}{2}\int_0^T \sigma^2(t,T)\,dt\right\}. \tag{3.2}$$

This numeraire change of measure technique, first considered by Jamshidian (1989/91) and Geman et al. (1995), and applied to the present problem by Nielsen and Sandmann (1995/96), allows via Itô's Lemma to rewrite the stochastic differential system (3.1) as

$$\begin{aligned} d\left[\frac{P(t,s)}{P(t,T)}\right] &= \frac{P(t,s)}{P(t,T)} \cdot \left[\sigma(t,s) - \sigma(t,T)\right] dW_1^T(t), \\ d\left[\frac{S(t)}{P(t,T)}\right] &= \frac{S(t)}{P(t,T)} \cdot \left(\left[\sigma_S\,\rho - \sigma(t,T)\right] dW_1^T(t) + \sigma_S\sqrt{1-\rho^2} \cdot dW_2^T(t)\right), \end{aligned} \tag{3.3}$$

where $(dW_1^T(t), dW_2^T(t)) = (dW_1(t) - \sigma(t,T)dt, dW_2(t))$ are standard Wiener processes under the Q^T-measure. In this framework the arbitrage price of a financial contract with payoff at maturity date T coincides with the discounted expected value under the Q^T-measure. In particular, the market value of the call-option on the reference portfolio at maturity date is determined by

$$C_0(X(T),G) = P(0,T) \cdot E^T\left[\left(D \cdot \sum_{i=0}^{n-1} \frac{S(T)}{S(t_i)} - G\right)_+\right]. \tag{3.4}$$

From (3.3) one derives further that

$$\begin{aligned} P(t,T) = \frac{P(0,T)}{P(0,t)} \cdot \exp\Big\{ & -\int_0^t [\sigma(u,t) - \sigma(u,T)]\, dW_1^T(u) \\ & + \tfrac{1}{2}\int_0^t [\sigma(u,t) - \sigma(u,T)]^2 du \Big\}, \end{aligned} \tag{3.5}$$

$$\frac{S(T)}{S(t)} = \frac{1}{P(t,T)} \cdot \exp\left\{ \begin{array}{l} -\frac{1}{2}\int_t^T \left([\sigma_S \rho - \sigma(u,T)]^2 + \sigma_S^2(1-\rho^2)\right) du \\ + \int_t^T [\sigma_S \rho - \sigma(u,T)]\, dW_1^T(u) + \int_t^T \sigma_S \sqrt{1-\rho^2}\, dW_2^T(u) \end{array} \right\}, \tag{3.6}$$

and through combination of the expressions (3.5) and (3.6) one obtains

$$\frac{S(T)}{S(t)} = \frac{P(0,t)}{P(0,T)} \cdot \exp\left\{ \begin{array}{l} -\int_0^t [\sigma(u,t) - \sigma(u,T)]\, dW_1^T(u) - \frac{1}{2}\int_0^t [\sigma(u,t) - \sigma(u,T)]^2 du \\ -\frac{1}{2}\int_t^T \left([\sigma_S \rho - \sigma(u,T)]^2 + \sigma_S^2(1-\rho^2)\right) du \\ + \int_t^T [\sigma_S \rho - \sigma(u,T)]\, dW_1^T(u) + \int_t^T \sigma_S \sqrt{1-\rho^2}\, dW_2^T(u) \end{array} \right\}. \tag{3.7}$$

For deterministic volatilities the ratio (3.7) follows a log-normal distribution under the T-forward risk adjusted measure Q^T. In this situation we show in Section 5 how to get analytical approximations for the call-option prices (3.4), where the reference portfolio is given by a sum of correlated log-normal random variables under the Q^T-measure.

4. Fair periodic premium with mortality risk

As already stated we assume that the mortality process is independent of the financial market process and that the insurance company is risk neutral with respect to the mortality risk. Let the infinitesimal measure denoted by $\pi_x(t)\,dt$

represent the probability that the endowment contract terminates in the time interval $(t, t+dt)$. The *fair periodic premium* P is obtained by equating the expected discounted values of costs and benefits, where a proper arbitrage-free pricing justification of this statement is presented in Nielsen and Sandmann (1996), Section 4. It is given by the implicit solution of the equation

$$\begin{aligned} P \cdot \sum_{i=0}^{n-1} P(0,t_i) \cdot \left(1 - \int_0^{t_i} \pi_x(t)\,dt\right) = \\ \int_0^T \left[G \cdot P(0,t) + C_0\big(X(T),G\big)\right] \cdot \pi_x(t)\,dt \\ + \left[G \cdot P(0,T) + C_0\big(X(T),G\big)\right] \cdot \left(1 - \int_0^T \pi_x(t)\,dt\right) \\ C_0\big(X(T),G\big) = P(0,t) \cdot E^t\left[\left(a \cdot P \cdot \sum_{i=0}^{n^*(t)-1} \frac{S(t)}{S(t_i)} - G\right)_+\right], \end{aligned} \tag{4.1}$$

where E^t denotes the t-forward risk adjusted measure Q^t defined similarly to (3.2). Furthermore, the usual simplified assumption that death occurs at the end of a year is made. In this situation, the infinitesimal measure $\pi_x(t)\,dt$ is replaced by the conditional probability ${}_t p_x\, q_{x+t}$ to die in the time interval $[t, t+1)$ given survival at time t and the implicit equation for the fair periodic premium with mortality risk can be rewritten as

$$\begin{aligned} P \cdot \sum_{k=0}^{n-1} P(0,k) \cdot {}_k p_x \\ = \sum_{k=1}^{n} {}_{k-1}p_x\, q_{x+k-1} \cdot \left[G \cdot P(0,k) + C_0\big(X(k),G\big)\right] \\ + {}_n p_x \cdot \left[G \cdot P(0,n) + C_0\big(X(n),G\big)\right], \\ C_0\big(X(k),G\big) = P(0,k) \cdot E^k\left[\left(a \cdot P \cdot \sum_{i=0}^{k-1} \frac{S(k)}{S(i)} - G\right)_+\right], \quad k = 1,\ldots,n. \end{aligned} \tag{4.2}$$

Remark 4.1. Suppose that all call-options in (4.2) are out-of-the-money in such a way that their corresponding call-option prices can be neglected. Then the fair periodic premium coincides with the fair premium of the traditional endowment with sum insured G and it is explicitly given by

$$P = \frac{G \cdot \left[\sum_{k=1}^{n} P(0,k)\, {}_{k-1}p_x\, q_{x+k-1} + P(0,n)\, {}_n p_x\right]}{\sum_{k=0}^{n-1} P(0,k)\, {}_k p_x} \tag{4.3}$$

5. Analytical approximations for the fair periodic premium

In the present Section we show how to determine simple analytical lower and upper approximations for the fair periodic premium in the deterministic case $\sigma(t,s)$ and illustrate the method numerically. For simplicity we assume yearly periodic premiums, which are paid at the time points $t_i = i$, $i = 0, \ldots, n-1$, and $T = n$.

5.1. Convex lower and upper bounds for the call-option prices

To find useful analytical approximations for the implicit solution of the equation (4.2) it suffices to bound the call-option prices $C_0(X(t+1), G)$, $t = 0, \ldots, n-1$, where the reference portfolio at time $t+1$ is proportional to the sum of correlated log-normal random variables

$$Y(t+1) = \sum_{i=0}^{t} \frac{S(t+1)}{S(i)}, \quad t = 0, \ldots, n-1. \tag{5.1}$$

For this, we apply the method considered originally in Kaas et al. (2000) and Dhaene et al. (2002). The developments by Vanduffel et al. (2005a/b/2008a) suits exactly our needs. The representation (5.1) shows that it suffices to consider random variables of the general form

$$Y(k) = \sum_{i=0}^{k-1} \frac{S(k)}{S(i)} = \sum_{i=0}^{k-1} \alpha_i(k) e^{Z_i(k)}, \quad k = 1,2,\ldots,n, \tag{5.2}$$

where the $\alpha_i(k)$'s are non-negative and the random vector $(Z_0(k), Z_1(k), \ldots, Z_{k-1}(k))$ follows a multivariate normal distribution with mean vector $(\mu_i(k))_{0 \le i \le k-1}$, $\mu_i(k) = E[Z_i(k)]$, and covariance matrix

$$\begin{aligned}(\rho_{ij}(k)\sigma_i(k)\sigma_j(k))_{0 \le i,j \le k-1}, \ \sigma_i(k)^2 &= \mathrm{Var}[Z_i(k)], \ \rho_{ij}(k)\sigma_i(k)\sigma_j(k) \\ &= \mathrm{Cov}[Z_i(k), Z_j(k)]\end{aligned}$$

For fixed k consider the conditioning random variable defined by

$$\Lambda(k) = \sum_{i=0}^{k-1} \gamma_i(k) Z_i(k) \tag{5.3}$$

for some constants $\gamma_i(k)$. Following Kaas et al. (2000) one defines a random variable

$$Y(k)^{\ell} = E[Y(k) | \Lambda(k)] = \sum_{i=0}^{k-1} \alpha_i(k) e^{\mu_i(k) + \frac{1}{2}(1 - r_i(k)^2)\sigma_i(k)^2 + r_i(k)\sigma_i(k)\frac{\Lambda(k) - E[\Lambda(k)]}{\sigma_{\Lambda(k)}}}, \tag{5.4}$$

where $r_i(k)\sigma_i(k)\sigma_{\Lambda(k)} = \mathrm{Cov}[Z_i(k), \Lambda(k)] = \sum_{j=0}^{k-1} \gamma_j(k)\,\mathrm{Cov}[Z_i(k), Z_j(k)]$, $i = 0, \ldots, k-1$. One has the equality in distribution

$$Y(k)^{\ell} =_d \sum_{i=0}^{k-1} \alpha_i(k) e^{\mu_i(k) + \frac{1}{2}(1 - r_i(k)^2)\sigma_i(k)^2 + r_i(k)\sigma_i(k)\Phi^{-1}(U)}, \tag{5.5}$$

with $\Phi(x)$ the standard normal distribution and U a uniform random variable on (0,1). If all the correlation coefficients $r_i(k)$ defined in (5.4) are non-negative, then $Y(k)^{\ell}$ is a comonotonic sum. In this situation it is well-known that the so-called stop-loss transform with deductible d, $0 < d < \infty$, is determined by

$$E[(Y(k)^{\ell} - d)_+] = \sum_{i=0}^{k-1} \alpha_i(k) e^{\mu_i(k) + \frac{1}{2}\sigma_i(k)^2} \Phi(r_i(k)\sigma_i(k) - \Phi^{-1}(p)) - d(1-p), \tag{5.6}$$

where $\Phi^{-1}(p)$ is the root of the quantile equation

$$Q_p\left[Y(k)^{\ell}\right] = \sum_{i=0}^{k-1} \alpha_i(k) e^{\mu_i(k) + \frac{1}{2}(1 - r_i(k)^2)\sigma_i(k)^2 + r_i(k)\sigma_i(k)\Phi^{-1}(p)} = d. \tag{5.7}$$

From the definitions in (5.4) one sees that a sufficient condition for $r_i(k) \geq 0$ is that all $\gamma_i(k) \geq 0$ and all $\mathrm{Cov}\lfloor Z_i(k), Z_j(k)\rfloor \geq 0$. Using Jensen's inequality it can be proved that $Y(k)^{\ell}$ is a convex lower bound of $Y(k)$, a fact written $Y(k)^{\ell} \leq_{cx} Y(k)$, which means that for any convex function $v(x)$ one has

$$E\left[v\left(Y(k)^{\ell}\right)\right] \leq E\left[v\left(Y(k)\right)\right]. \tag{5.8}$$

In particular, one has for any real number d the inequality

$$E\left[\left(Y(k)^{\ell} - d\right)_+\right] \leq E\left[\left(Y(k) - d\right)_+\right]. \tag{5.9}$$

Note that the idea of using convex lower bounds for Asian option pricing can be traced back to Rogers and Shiu (1995). In Dhaene et al. (2002) the comonotonic convex upper bound, denoted by $Y(k)^u$ and such that $Y(k) \leq_{cx} Y(k)^u$ is proposed. In the lognormal context this random variable can be defined by imposing $r_i(k) = 1$ in (5.4). For this upper bound one has

$$Y(k)^u =_d \sum_{i=0}^{k-1} \alpha_i(k) e^{\mu_i(k) + \sigma_i(k)\Phi^{-1}(U)}, \quad \text{and} \tag{5.10}$$

$$E\left[(Y(k)^u - d)_+\right] = \sum_{i=0}^{k-1} \alpha_i(k) e^{\mu_i(k) + \frac{1}{2}\sigma_i(k)^2} \Phi\left(\sigma_i(k) - \Phi^{-1}(p)\right) - d(1-p), \tag{5.11}$$

where $\Phi^{-1}(p)$ is the root of the quantile equation

$$Q_p\left[Y(k)^u \right] = \sum_{i=0}^{k-1} \alpha_i(k) e^{\mu_i(k) + \sigma_i(k)\Phi^{-1}(p)} = d. \tag{5.12}$$

Since $Y(k)^\ell \leq_{cx} Y(k) \leq_{cx} Y(k)^u$ the following relationships hold:

$$E\left[Y(k)^\ell \right] = E\left[Y(k) \right] = E\left[Y(k)^u \right] = \sum_{i=0}^{k-1} \alpha_i(k) e^{\mu_i(k) + \frac{1}{2}\sigma_i(k)^2}, \tag{5.13}$$

$$\begin{aligned}
&\mathrm{Var}[Y(k)^\ell] = \sum_{i,j=0}^{k-1} \alpha_i(k)\alpha_j(k)\, e^{\mu_i(k)+\mu_j(k)+\frac{1}{2}(\sigma_i(k)^2+\sigma_j(k)^2)}\left(e^{r_i(k)\,r_j(k)\sigma_i(k)\,\sigma_j(k)} - 1\right) \\
&\leq \mathrm{Var}[Y(k)] = \sum_{i,j=0}^{k-1} \alpha_i(k)\alpha_j(k)\, e^{\mu_i(k)+\mu_j(k)+\frac{1}{2}(\sigma_i(k)^2+\sigma_j(k)^2)}\left(e^{\rho_{ij}(k)\sigma_i(k)\,\sigma_j(k)} - 1\right) \\
&\leq \mathrm{Var}[Y(k)^u] = \sum_{i,j=0}^{k-1} \alpha_i(k)\alpha_j(k)\, e^{\mu_i(k)+\mu_j(k)+\frac{1}{2}(\sigma_i(k)^2+\sigma_j(k)^2)}\left(e^{\sigma_i(k)\,\sigma_j(k)} - 1\right),
\end{aligned} \tag{5.14}$$

$$E\left[(Y(k)^\ell - d)_+ \right] \leq E\left[(Y(k) - d)_+ \right] \leq E\left[(Y(k)^u - d)_+ \right],\ d \in R. \tag{5.15}$$

For more details on these results we refer to Kaas et al. (2000) and Dhaene et al. (2002). In view of the inequality (5.14), it is clear that the best comonotonic lower bound approximations of $Y(k)$ are the ones for which $\mathrm{Var}\left\lfloor Y(k)^\ell \right\rfloor$ is as close to $\mathrm{Var}[Y(k)]$ as possible. Vanduffel et al. (2005a/b) were the first to propose maximization of the first order approximation of $\mathrm{Var}\left\lfloor Y(k)^\ell \right\rfloor$ obtained by letting $e^{r_i(k)\,r_j(k)\sigma_i(k)\,\sigma_j(k)} - 1 \approx r_i(k)\,r_j(k)\,\sigma_i(k)\,\sigma_j(k)$ to get the following coefficients in (5.3)

$$\gamma_i(k) = \alpha_i(k) e^{\mu_i(k) + \frac{1}{2}\sigma_i(k)^2}, \quad i = 0, \ldots, k-1. \tag{5.16}$$

This most simple choice is retained here and defines the so-called *comonotonic maximum variance approximation* of $Y(k)$. The above analytical specifications immediately imply the following lower and upper bounds for the call-option prices in (4.2)

$$\begin{aligned}
&C_0^\ell(X(k), G) = P(0,k)\cdot a \cdot P \cdot E^k\left[\left(Y(k)^\ell - \frac{G}{a \cdot P}\right)_+\right] \\
&\leq C_0(X(k), G) = P(0,k)\cdot a \cdot P \cdot E^k\left[\left(Y(k) - \frac{G}{a \cdot P}\right)_+\right] \\
&\leq C_0^u(X(k), G) = P(0,k)\cdot a \cdot P \cdot E^k\left[\left(Y(k)^u - \frac{G}{a \cdot P}\right)_+\right],\ k = 1,2,\ldots,n.
\end{aligned} \tag{5.17}$$

It is remarkable that the upper bound in (5.17) is formally identical to the quasi-explicit solution proposed by Kurz (1996) if one sets in her formulas $v_i^2 = \mathrm{Var}[Z_i(k)]$, $i = 0, \ldots, k-1$. To be ready for numerical evaluation, it remains to calculate the model parameters $\alpha_i(k)$, $\sigma_i(k)$, $\rho_{ij}(k)$ under the k-forward risk adjusted measure Q^k for each $k = 1, \ldots, n$. For this, we note that (3.7) implies in case $t = i$, $i = 0, \ldots, k-1$, $T = k$, the representation $\frac{S(k)}{S(i)} = \alpha_i(k) e^{z_i(k)}$, where $Z_i(k) = A_i(k) + B_i(k) + C_i(k)$ with

$$\begin{aligned} A_i(k) &= \int_0^i [\sigma(u,i) - \sigma(u,k)]\, dW_1^k(u), \\ B_i(k) &= \int_i^k [\sigma_S \rho - \sigma(u,k)]\, dW_1^k(u), \quad C_i(k) = \int_i^k \sigma_S \sqrt{1-\rho^2}\, dW_2^k(u), \end{aligned} \tag{5.18}$$

and

$$\alpha_i(k) = \frac{P(0,i)}{P(0,k)} \cdot \exp\left\{ \begin{aligned} &-\tfrac{1}{2}\int_0^i [\sigma(u,i) - \sigma(u,k)]^2 du \\ &-\tfrac{1}{2}\int_i^k \left([\sigma_S \rho - \sigma(u,k)]^2 + \sigma_S^2 (1-p^2) \right) du \end{aligned} \right\} \tag{5.19}$$

Example 5.1 (Nielsen and Sandmann (1995))

In the simplest situation $\sigma(t,s) = \sigma \cdot (s-t)$ the random variables $A_i(k)$, $B_i(k)$, $C_i(k)$ have mean zero, the pairs $(A_i(k), C_j(k))_{0 \le i, j \le k-1}$, $(B_i(k), C_j(k))_{0 \le i, j \le k-1}$ and $(A_i(k), B_j(k))_{0 \le i \le j \le k-1}$ are independent and for $0 \le i \le j \le k-1$ one has

$$\begin{aligned} E^k\left[A_i(k)\, A_j(k)\right] &= E^k\left[A_j(k)^2\right] - (k-j)(j-i)(k-i-j)\sigma^2, \\ E^k\left[B_i(k)\, B_j(k)\right] &= E^k\left[B_j(k)^2\right],\ E^k\left[C_i(k)\, C_j(k)\right] = E^k\left[C_j(k)^2\right], \\ E^k\left[A_j(k) B_i(k)\right] &= \tfrac{1}{2}\sigma(k-j)(j-i)\left[\sigma(2k-i-j) - 2\rho\sigma_S\right]. \end{aligned} \tag{5.20}$$

A calculation shows that for $i, j = 0, \ldots, k-1$, $i \le j$, $k = 1, \ldots, n$ one has (see the Appendix in Nielsen and Sandmann (1995))

$$\begin{aligned} &\mu_i(k) = 0, \quad \gamma_i(k) = \alpha_i(k) e^{\frac{1}{2}\sigma_i(k)^2} = P(0,i)/P(0,k), \\ &\sigma_i(k)^2 = (k-i)\,\sigma_S^2 + \left(i\sigma^2 - \rho\sigma\sigma_S\right)(k-i)^2 + \tfrac{1}{3}\sigma^2 (k-i)^3 \\ &\rho_{ij}(k)\sigma_i(k)\,\sigma_j(k) = \sigma_i(k)^2 + \sigma(k-j)(j-i)\left[\tfrac{1}{2}\sigma(i+j) - \rho\sigma_S\right]. \end{aligned} \tag{5.21}$$

Example 5.2 (Mercurio and Moraleda (2000))

In 1996 Mercurio and Moraleda proposed the following deterministic volatility structure

$$\sigma(t,s) = \sigma \cdot \left[1 + \gamma(s-t)\right] \cdot e^{-\lambda(s-t)}, \quad \sigma, \gamma, \lambda > 0 \tag{5.22}$$

This specification provides a humped volatility structure for any $\gamma > \lambda$ and is stationary, that is it depends only on the difference $s - t$.

Example 5.3 (Moraleda and Vorst (1997))

An alternative humped volatility model that overcomes some drawback of the previous model is the structure (see Mercurio and Moraleda(2001) for discussion and motivation):

$$\sigma(t,s) = \sigma \cdot \frac{1+\gamma s}{1+\gamma t} \cdot e^{-\lambda(s-t)}, \quad \sigma, \gamma, \lambda > 0 \tag{5.23}$$

This function has a humped graph if $\gamma > \lambda$ and $t < (\gamma - \lambda)/\gamma\lambda$.

The determination of the required formulas for the Examples 5.2 and 5.3, which correspond to the specification (5.21), lies beyond the scope of the present paper.

5.2. Iterative algorithm for analytical evaluation of the fair premium bounds

To get the call-option price lower and upper bounds in (5.17) it suffices to use the formulas (5.18) and (5.19) for each $k = 1, \ldots, n$, respectively their counterparts in the Examples 5.1 to 5.3, and insert them in the corresponding formulas (5.6) and (5.11) for the lower and upper stop-loss transform bounds. The specification can be performed using EXCEL spreadsheet calculations, at least for Example 5.1. The only step, which is not straightforward and may require the EXCEL Solver, is the calculation of the roots $\Phi^{-1}(p)$ in (5.7) and (5.12) for each maturity date $k = 1, \ldots, n$. Let us summarize the required calculations for the lower approximation obtained from the lower bound in (5.17). For ease of notation the maturity index n is omitted in the relevant quantities. The lower approximation to the unknown fair periodic premium, denoted by P^{ℓ}, is defined to be the solution P of the modified equation (4.2ℓ), which is obtained from (4.2) by replacing the option prices $C_0(X(k),G)$ by their lower bounds in (5.17). In the defining equation (4.2ℓ) for P^{ℓ} rewrite the deductibles in (5.17) for all indices $k = 1, \ldots, n$ as (the scaling constant $P(0,1)$ is chosen for convenience)

$$\frac{G}{a \cdot P^{\ell}} = \frac{\beta^{\ell}}{P(0,1)} \tag{5.24}$$

for some unknown β^{ℓ}. Consider the roots $x(k)$ of the implicit equations

$$\sum_{i=0}^{k-1} \gamma_i(k) e^{-\frac{1}{2}r_i(k)^2\sigma_i(k)^2 + r_i(k)\sigma_i(k)x(k)} = \frac{\beta^{\ell}}{P(0,1)}, \quad k = 1,\ldots,n, \tag{5.25}$$

which are derived from (5.7). Then, using (4.2ℓ) and (5.6) one sees that β^{ℓ} must also solve the implicit equation

$$\begin{aligned}\frac{1}{a}\cdot\sum_{k=0}^{n-1}P(0,k)\,{}_kp_x &= \frac{\beta^{\ell}}{P(0,1)}\\ &\cdot\left[\sum_{k=1}^{n-1}P(0,k)\,{}_{k-1}p_x\,q_{x+k-1}\Phi(x(k))+P(0,n)\,{}_{n-1}p_x\,\Phi(x(n))\right]\\ &+\sum_{k=1}^{n-1}{}_{k-1}p_x\,q_{x+k-1}\cdot\sum_{i=0}^{k-1}P(0,i)\,\Phi(r_i(k)\sigma_i(k)-x(k))\\ &+{}_{n-1}p_x\cdot\sum_{i=0}^{n-1}P(0,i)\,\Phi(r_i(n)\sigma_i(n)-x(n)).\end{aligned} \tag{5.26}$$

The non-linear system of $n+1$ equations (5.25)-(5.26) in the $n+1$ unknowns $x(k)$, $k=1,\ldots,n$, β^{ℓ}, fully determines the lower premium approximation P^{ℓ}. Replacing β^{ℓ} by β^{u} and all $r_i(k)$'s by $r_i(k)=1$ a similar system of non-linear equations for the upper premium approximation P^{u} is obtained. In practice, the system (5.25)-(5.26) is solved by iteration as follows. Start with an approximation $\beta^{\ell}_{appr}>0$ to the true value β^{ℓ}. Solve (5.25) for the unknowns $x(k)$, $k=1,\ldots,n$ and insert the results into (5.26) to get an approximation a_{appr} to the given proportional share a. Change the approximation β^{ℓ}_{appr} appropriately and repeat the iteration until a_{appr} is sufficiently close to a, which yields a close approximation to β^{ℓ}. The upper unknown value β^{u} is obtained similarly. Then, the true fair periodic premium is approximated by the quantities

$$P^{\ell}=\frac{P(0,1)}{\beta^{\ell}}\cdot\frac{G}{a},\quad P^{u}=\frac{P(0,1)}{\beta^{u}}\cdot\frac{G}{a}. \tag{5.27}$$

Numerical examples suggest that the inequality $P^{\ell}\le P\le P^{u}$ holds true, but a proof is not provided here. The Newton-Raphson method leads to the following simple efficient algorithm to solve (5.25). For $i=0,1,\ldots,k-1$, $k=1,\ldots,n$ set $f_i^k(x)=\gamma_i(x)e^{-\frac{1}{2}r_i(k)^2\sigma_i(k)^2+r_i(k)\sigma_i(k)x}$, where $r_i(k)=1$ for the upper bound specification. Then, for each $k=1,\ldots,n$, solving the equations

$$\sum_{i=0}^{k-1}f_i^k(x)=\frac{\beta^{\ell}}{P(0,1)},\quad k=1,\ldots,n, \tag{5.28}$$

which are equivalent to (5.25), is done as follows. For each $k=1,\ldots,n$ let $x(k)$ be the exact solution of (5.25). For $k=1$ one has (noting that $r_0(1)=1$)

$$x(1)=\frac{1}{2}\sigma_0(1)+\frac{1}{\sigma_0(1)}\ln\{\beta^{\ell}\}. \tag{5.29}$$

For $1 < k \le n$, $m = 0, 1, 2, \ldots$, consider the $(m+1)$-th Newton-Raphson iteration step $x_{m+1}(k)$, which satisfies the recursive relationship

$$x_{m+1}(k) = x_m(k) - \frac{\sum_{i=0}^{k-1} f_i^k\big(x_m(k)\big) - \frac{\beta^\ell}{P(0,1)}}{\sum_{i=0}^{k-1} r_i(k)\sigma_i(k) f_i^k\big(x_m(k)\big)}, \quad m = 1, 2, \ldots \tag{5.30}$$

By appropriate starting point $x_0(k)$ it is known that $\lim_{m \to \infty} x_m(k) = x(k)$. Our numerical experience has shown that the starting points $x_0(2) = x(1)$, $x_0(k) = x_1(k-1)$, $k > 2$, lead after at most five iteration steps to very accurate solutions. In fact, the first order iterates $x_1(k)$, $k > 1$ have been enough accurate to obtain all numerical bounds in Table 5.1.

5.3. Numerical illustration

The illustration of our results is based on Example 5.1 for the following financial market parameters

$$\sigma = 8\%, \quad \sigma_S \rho = 10\%, \quad \sigma_S \sqrt{1-\rho^2} = 15\%. \tag{5.31}$$

We have calculated lower and upper bounds for the fair periodic premiums for three different maturity dates $T = 10, 12, 15$, each with three different specifications of the initial term structure of interest rates (TSIR), namely:

Scenario I : flat initial TSIR $P(0,t) = (1.06)^{-t}, \quad 0 \le t \le T$

Scenario II : normal initial TSIR $P(0,t) = \left[0.04 + (1.02)^{\frac{t}{T}}\right]^{-t}, \quad 0 \le t \le T$

Scenario III: invers initial TSIR $P(0,t) = \left[2.08 - (1.02)^{\frac{t}{T}}\right]^{-t}, \quad 0 \le t \le T$

The mortality risk is assumed to follow Makeham's distribution such that the conditional probability ${}_t p_x q_{x+t}$ of a life aged x to die in the time interval $[t, t+1)$ given survival to time t is given by (values of Nielsen and Sandmann (1995))

$$\begin{gathered} {}_t p_x q_{x+t} = \frac{L_{x+t} - L_{x+t+1}}{L_x}, \quad L_x = b \cdot s^x \cdot g^{c^x}, \quad s = 0.99949255, \\ g = 0.99959845, \quad c = 1.10291509, \quad b = 1000401.71. \end{gathered} \tag{5.32}$$

Besides the three TSIR scenarios we consider three age classes $x = 30, 40, 50$ and three investment strategies $a = 0.4, 0.5, 0.6$. Table 5.1 lists the lower and upper approximations of the fair periodic premiums as well as their average for the fixed guarantee $G = 1000$. The very tightness of the approximations in

Table 5.1 indicates that the sum of correlated lognormals is likely to be "rather" comonotonic. The extent to which this is true and the question whether this phenomenon depends upon the model (here Example 5.1) and/or on the chosen parameters has not yet been analyzed. It is worthwhile to mention that up to now only Monte Carlo simulations have been available to determine the fair periodic premiums. The advantages of simple, fast and very reliable analytical numerical calculations with controlled accuracy are self-evident. Since such an efficient approach, which avoids time consuming Monte Carlo calculations, is almost always preferred, the proposed method can highly be recommended for practical use. Moreover, it might be interesting to analyze more deeply why this method should be preferred to the Monte Carlo one. A referee has suggested that the problem seems to be quadratic in some sense,

TABLE 5.1

LOWER AND UPPER BOUNDS FOR THE FAIR PERIODIC PREMIUM UNDER VARIOUS CONDITIONS

Maturity date T		10			12			15		
Initial TSIR		I	II	III	I	II	III	I	II	III
Age	Share									
		77.45	75.17	79.75	64.50	62.25	66.79	52.04	49.84	54.93
30	0.4	**77.61**	**75.32**	**79.92**	**64.66**	**62.40**	**66.96**	**52.17**	**49.97**	**55.08**
		77.77	75.48	80.08	64.82	62.56	67.12	52.31	50.10	55.23
		81.40	79.00	83.82	69.32	66.91	71.78	57.57	55.15	61.16
	0.5	**81.66**	**79.26**	**84.09**	**69.57**	**67.15**	**72.03**	**57.78**	**55.36**	**61.40**
		81.92	79.51	84.36	69.82	67.38	72.28	58.00	55.56	61.64
		87.26	84.69	89.86	76.18	73.55	78.87	65.30	62.60	70.07
	0.6	**87.66**	**85.09**	**90.27**	**76.56**	**73.91**	**79.26**	**65.64**	**62.92**	**70.45**
		88.06	85.48	90.69	76.93	74.27	79.65	65.97	63.23	70.84
		78.48	76.24	80.75	65.79	63.60	68.02	53.80	51.70	56.64
40	0.4	**78.65**	**76.40**	**80.92**	**65.95**	**63.76**	**68.18**	**53.95**	**51.84**	**56.79**
		78.82	76.57	81.09	66.11	63.91	68.35	54.09	51.98	56.95
		82.55	80.20	84.92	70.80	68.46	73.18	59.68	57.40	63.24
	0.5	**82.82**	**80.47**	**85.20**	**71.05**	**68.70**	**73.44**	**59.91**	**57.61**	**63.49**
		83.10	80.74	85.48	71.30	68.95	73.70	60.14	57.82	63.74
		88.59	86.09	91.13	77.95	75.40	80.55	67.97	65.43	72.72
	0.6	**89.01**	**86.49**	**91.57**	**78.34**	**75.78**	**80.94**	**68.32**	**65.77**	**73.12**
		89.43	86.89	92.01	78.72	76.15	81.34	68.67	66.10	73.53
		81.25	79.12	83.41	69.25	67.22	71.32	58.62	56.79	61.29
50	0.4	**81.44**	**79.31**	**83.60**	**69.43**	**67.39**	**71.49**	**58.77**	**56.95**	**61.46**
		81.63	79.50	83.80	69.60	67.56	71.67	58.93	57.10	61.64
		85.65	83.44	87.90	74.80	72.66	76.98	65.50	63.58	68.92
	0.5	**85.96**	**83.73**	**88.21**	**75.07**	**72.92**	**77.26**	**65.75**	**63.82**	**69.20**
		86.26	84.03	88.51	75.34	73.18	77.53	66.00	64.06	69.48
		92.20	89.86	94.58	82.77	80.49	85.10	75.38	73.37	80.10
	0.6	**92.67**	**90.31**	**95.05**	**83.19**	**80.90**	**85.53**	**75.77**	**73.75**	**80.55**
		93.13	90.77	95.53	83.60	81.30	85.95	76.17	74.14	81.00

which would imply that the analytical solution is of great use. Unfortunately, the author must left this point open for further research.

5.4. Two special closed-form analytical solutions

A more detailed analysis of the non-linear system of equations (5.25)-(5.26) is done in the Appendix. It yields instructive analytical closed-form formulas for the maturity dates $n = 1, 2$.

In case $n = 1$ the lower and upper bound fair premium coincides with the one-year exact fair premium. Setting $c = \frac{1}{2}\sigma_0(1)$ the exact one-year fair premium can be expressed as

$$P = \frac{1}{a\beta} P(0,1)G = \left[\Phi\left(c + \frac{1}{2c}\ln\beta\right) + \frac{1}{\beta}\Phi\left(c - \frac{1}{2c}\ln\beta\right)\right] \cdot P(0,1)G, \quad (5.33)$$

with β solution of the implicit equation

$$\frac{1}{a} = \beta \cdot \Phi\left(c + \frac{1}{2c}\ln\beta\right) + \Phi\left(c - \frac{1}{2c}\ln\beta\right). \quad (5.34)$$

The special closed-form solution for $\beta = 1$ is especially simple and has a nice interpretation. Rewrite (5.34) as $a \cdot P = P(0,1)G$ to see that the amount invested in the one-year reference portfolio is equal to the one-year discounted guaranteed amount by death or survival. The corresponding one-year fair premium is simply equal to $P = 2\Phi(c) \cdot P(0,1) \cdot G$.

For $n = 2$ we obtain a special solution, which can be expressed explicitly in term of the model parameters. The lower approximation for the fair periodic premium is specified by

$$x(1) = \frac{1}{2}\sigma_0(1) \frac{1}{\sigma_0(1)} \ln\{\beta^\ell\}, \; x(2) = \tfrac{1}{2} r_0(2)\sigma_0(2),$$

$$\frac{\beta^\ell}{P(0,1)} = \frac{1}{P(0,2)}\left[1 + P(0,1)\, e^{-\frac{1}{2} r_1(2)\sigma_1(2)\{r_0(2)\sigma_0(2) - r_1(2)\sigma_1(2)\}}\right],$$

$$\frac{1}{a} \cdot [1 + P(0,1)p_x] = \frac{\beta^\ell}{P(0,1)} \cdot [P(0,1)q_x\Phi(x(1)) + P(0,2)p_x\Phi(x(2))]$$

$$+ q_x\Phi(\sigma_0(1) - x(1)) + p_x[\Phi(r_0(2)\sigma_0(2) - x(2)) + P(0,1)\Phi(r_1(2)\sigma_1(2) - x(2))],$$

$$P^\ell = \frac{1}{a \cdot \beta^\ell} \cdot P(0,1)G. \quad (5.35)$$

The upper approximation for the fair periodic premium $P^u = \dfrac{1}{a \cdot \beta^u} \cdot P(0,1)G$ is obtained from (5.35) setting $r_0(2) = r_1(2) = 1$ and replacing β^ℓ by β^u.

6. A CONTRACT WITH VARIABLE INCREASING GUARANTEES

Instead of constant guaranteed minimum benefits in case of death or survival it appears more appealing from a policyholder's point of view to profit from variable guaranteed minimum benefits over the contract duration. In this context guaranteed minimum benefits that increase over time are especially attractive. For example, AXA Winterthur has been the first Swiss life insurer to launch such a private pension product in May 2006.

To analyze the unit-linked endowment contract with variable guarantees the fixed guarantee G in the previous development is changed to a variable guarantee defined and denoted as follows:

$G_{i+1}(n)$: variable guaranteed minimum death benefit (GMDB) at time t_{i+1}, $i = 0, \ldots, n-1$, respectively guaranteed minimum accumulation benefit (GMAB) at maturity date $t_n = T$, $i = n-1$

For simplicity we restrict ourselves to the deterministic case $\sigma(t,s) = \sigma \cdot (s-t)$ and to the yearly periodic case $t_i = i$, $i = 0, \ldots, n-1$, $T = n$ and illustrate the method numerically. Let us fix the amount D invested in the reference portfolio. The comonotonic approximations of Section 5 will yield lower and upper approximations for the variable guaranteed minimum benefits $G_{i+1}(n)^\ell$, $G_{i+1}(n)^u$, fair periodic premiums $P(n)^\ell$, $P(n)^u$, as well as corresponding fractions $a(n)^\ell$, $a(n)^u$ of the premiums such that $D = a(n)^\ell \cdot P(n)^\ell$, $D = a(n)^u \cdot P(n)^u$. From (5.25)-(5.26) it is straightforward to see that the lower approximations are determined by the following explicit and closed-form analytical formulas in the n variables $x(k)$, $k = 1, \ldots, n$:

$$G_k(n)^\ell = D \cdot \sum_{i=0}^{k-1} \gamma_i(k) e^{-\frac{1}{2} r_i(k)^2 \sigma_i(k)^2 + r_i(k)\sigma_i(k)x(k)}, \quad k = 1, \ldots, n. \tag{6.1}$$

$$\begin{aligned} P(n)^\ell \cdot \sum_{k=0}^{n-1} P(0,k)\,{}_k p_x = \\ G_k(n)^\ell \cdot \left[\sum_{k=1}^{n-1} P(0,k)\,{}_{k-1}p_x\, q_{x+k-1} \Phi(x(k)) + P(0,n)\,{}_{n-1}p_x \Phi(x(n)) \right] \\ + D \cdot \left[\begin{array}{l} \sum_{k=1}^{n-1} {}_{k-1}p_x\, q_{x+k-1} \cdot \sum_{i=0}^{k-1} P(0,i)\Phi(r_i(k)\sigma_i(k) - x(k)) \\ + {}_{n-1}p_x \cdot \sum_{i=0}^{n-1} P(0,i)\Phi(r_i(n)\sigma_i(n) - x(n)) \end{array} \right]. \end{aligned} \tag{6.2}$$

In our numerical examples below the following simple specification of the variables is made:

$$x(k) = \tfrac{1}{2} r_0(k)\sigma_0(k), \quad k = 1, \ldots, n. \tag{6.3}$$

Replacing everywhere all $r_i(k)$'s by $r_i(k) = 1$ similar formulas for the upper approximations are obtained.

Let us illustrate these results quantitatively. For a constant yearly invested amount $D = 1000$ in the reference portfolio, the same financial market parameters,

TABLE 6.1

LOWER BOUND, AVERAGE AND UPPER BOUNDS FOR THE INCREASING GUARANTEES

Year	Initial TSIR								
	I			II			III		
1	1060	1060	1060	1060	1060	1060	1060	1060	1060
2	2188	2188	2188	2198	2198	2198	2178	2178	2178
3	3384	3385	3386	3412	3413	3414	3357	3358	3359
4	4659	4662	4664	4713	4716	4719	4606	4609	4611
5	6026	6031	6036	6116	6121	6126	5938	5943	5948
6	7503	7512	7521	7639	7649	7658	7371	7380	7388
7	9116	9131	9146	9312	9327	9343	8928	8942	8956
8	10899	10922	10945	11169	11193	11217	10640	10662	10684
9	12895	12930	12965	13258	13294	13331	12549	12582	12615
10	**15163**	**15214**	**15264**	**15641**	**15694**	**15748**	**14708**	**14756**	**14804**
11	17779	17852	17924	18402	18478	18555	17189	17257	17325
12	**20847**	**20949**	**21051**	**21652**	**21761**	**21869**	**20085**	**20182**	**20278**
13	24506	24650	24793	25544	25698	25851	23528	23662	23797
14	28952	29153	29354	30292	30508	30723	27695	27882	28069
15	**34461**	**34741**	**35022**	**36196**	**36499**	**36802**	**32838**	**33098**	**33358**

TABLE 6.2

LOWER BOUND, AVERAGE AND UPPER BOUNDS FOR THE FAIR PERIODIC PREMIUM

Maturity date T	10			12			15		
Initial TSIR	I	II	III	I	II	III	I	II	III
Age									
30	1435	1436	1434	1602	1604	1601	1953	1958	1947
	1443	**1443**	**1442**	**1612**	**1614**	**1610**	**1968**	**1973**	**1962**
	1450	1451	1449	1621	1623	1619	1982	1988	1976
40	1437	1438	1436	1605	1608	1603	1956	1963	1948
	1445	**1446**	**1443**	**1615**	**1618**	**1612**	**1970**	**1978**	**1962**
	1452	1453	1451	1624	1627	1621	1985	1993	1977
50	1442	1445	1440	1613	1619	1608	1963	1977	1950
	1450	**1453**	**1447**	**1623**	**1628**	**1617**	**1977**	**1991**	**1964**
	1457	1460	1455	1632	1637	1626	1991	2005	1978

initial TSIR and mortality assumptions as in Section 5, we obtain the results of Tables 6.1 and 6.2. One observes that the approximations for both the increasing guaranteed minimum benefits and the fair periodic premiums are rather close together. It is remarkable that the dependence on both the age at entry and the TSIR scenario is rather limited. This numerical stability could not be expected a priori and is somewhat surprising. However, there is of course an important dependence on the level of interest rates at insurance issue. For example, assume a 3% interest rate level instead of a 6% level with the following initial TSIR:

Scenario I : flat initial TSIR $P(0,t) = (1.03)^{-t}, \quad 0 \le t \le T$

Scenario II : normal initial TSIR $P(0,t) = \left[0.02 + (1.01)^{\frac{t}{T}}\right]^{-t}, \quad 0 \le t \le T$

Scenario III: invers initial TSIR $P(0,t) = \left[2.04 - (1.01)^{\frac{t}{T}}\right]^{-t}, \quad 0 \le t \le T$

With this initial TSIR one obtains the results summarized in the Tables 6.3 and 6.4. The same observations as before can be made. The lower interest rate level results in a significant decrease in the guaranteed minimum benefits, which can be offered, however, at a similar fair periodic premium level. One notes a somewhat counterintuitive fact, namely that the premium for the longer maturity date decreases with the entry age (though very slightly).

TABLE 6.3

LOWER BOUND, AVERAGE AND UPPER BOUNDS FOR THE INCREASING GUARANTEES

Year	Initial TSIR								
	I			II			III		
1	1030	1030	1030	1030	1030	1030	1030	1030	1030
2	2095	2095	2095	2100	2100	2100	2090	2090	2090
3	3193	3194	3195	3207	3208	3208	3179	3180	3181
4	4330	4333	4335	4356	4359	4361	4304	4307	4309
5	5516	5521	5526	5559	5564	5569	5474	5479	5484
6	6765	6774	6783	6828	6838	6847	6702	6711	6720
7	8096	8111	8126	8186	8201	8216	8008	8022	8037
8	9535	9558	9581	9656	9680	9704	9415	9438	9461
9	11115	11149	11184	11276	11311	11347	10957	10991	11025
10	**12880**	**12931**	**12981**	**13089**	**13141**	**13193**	**12676**	**12725**	**12774**
11	14890	14962	15034	15158	15232	15307	14628	14698	14768
12	**17221**	**17323**	**17425**	**17564**	**17669**	**17774**	**16889**	**16988**	**17087**
13	19982	20125	20268	20417	20565	20713	19560	19698	19837
14	23318	23519	23719	23872	24080	24288	22782	22976	23170
15	**27439**	**27720**	**28000**	**28147**	**28438**	**28730**	**26755**	**27025**	**27296**

TABLE 6.4

LOWER BOUND, AVERAGE AND UPPER BOUNDS FOR THE FAIR PERIODIC PREMIUM

Maturity date T	10			12			15		
Initial TSIR	I	II	III	I	II	III	I	II	III
Age									
30	1428 **1436** 1445	1428 **1437** 1445	1428 **1436** 1444	1595 **1606** 1616	1596 **1606** 1617	1594 **1605** 1615	1955 **1973** 1991	1958 **1976** 1994	1953 **1970** 1988
40	1429 **1437** 1446	1430 **1438** 1446	1428 **1437** 1445	1596 **1607** 1617	1597 **1608** 1619	1595 **1605** 1616	1954 **1971** 1989	1958 **1975** 1993	1950 **1968** 1985
50	1432 **1440** 1448	1433 **1441** 1449	1430 **1439** 1447	1599 **1609** 1619	1601 **1611** 1622	1596 **1607** 1617	1950 **1967** 1984	1956 **1973** 1991	1944 **1961** 1978

ACKNOWLEGEMENTS

The author is grateful to the referees for careful reading, critical comments, additional references and helpful suggestions for improved presentation.

APPENDIX

Derivation of the special closed-form solutions of Section 5.4

One-year maturity date

In case $n = 1$ one gets from (5.4) and (5.21) immediately the following model parameters

$$\gamma_0(1) = \frac{1}{P(0,1)}, \quad \sigma_0(1) = \sqrt{\sigma_S^2 - \rho\sigma\sigma_S + \tfrac{1}{3}\sigma^2}, \quad r_0(1) = 1. \qquad \text{(A.1)}$$

Setting $c = \frac{1}{2}\sigma_0(1)$, $\beta = \beta^\ell = \beta^u$, the solution to (5.25) equals $x(1) = c + \frac{1}{2c}\ln\beta$. Inserted into (5.26) one obtains that β solves the implicit equation (5.34) and the exact one-year fair premium is then given by (5.33).

Two-year maturity date

In case $n = 2$ the required model parameter from (5.4) and (5.21) are given by

$$
\begin{aligned}
&\gamma_0(2) = \frac{1}{P(0,2)}, \quad \gamma_1(2) = \frac{P(0,1)}{P(0,2)}, \\
&\sigma_0(2)^2 = 2\sigma_S^2 - 4\rho\sigma\sigma_S + \tfrac{8}{3}\sigma^2, \quad \sigma_1(2)^2 = \sigma_S^2 - (\sigma^2 - \rho\sigma\sigma_S) + \tfrac{1}{3}\sigma^2, \\
&\rho_{01}(2)\sigma_0(2)\sigma_1(2) = \sigma_0(2)^2 + \sigma\left[\tfrac{1}{2}\sigma - \rho\sigma_S\right], \\
&\sigma_{\Lambda(2)}^2 = \left[\gamma_0(2)\sigma_0(2)\right]^2 + 2\gamma_0(2)\gamma_1(2)\rho_{01}(2)\sigma_0(2)\sigma_1(2) + \left[\gamma_1(2)\sigma_1(2)\right]^2 \\
&r_0(2) = \frac{\gamma_0(2)\sigma_0(2) + \gamma_1(2)\rho_{01}(2)\sigma_1(2)}{\sigma_{\Lambda(2)}}, \quad r_1(2) = \frac{\gamma_0(2)\rho_{01}(2)\sigma_0(2) + \gamma_1(2)\sigma_1(2)}{\sigma_{\Lambda(2)}}.
\end{aligned}
\tag{A.2}
$$

The system (5.25)-(5.26) is equivalent to the system with unknowns $x(1)$, $x(2)$, β^{ℓ}:

$$
\begin{aligned}
&\frac{\beta^{\ell}}{P(0,1)} = \frac{1}{P(0,1)} e^{-\frac{1}{2}\sigma_0(1)^2 + \sigma_0(1)x(1)}, \\
&\frac{\beta^{\ell}}{P(0,1)} = \frac{1}{P(0,2)} e^{-\frac{1}{2}r_0(2)^2\sigma_0(2)^2 + r_0(2)\sigma_0(2)x(2)} + \frac{P(0,1)}{P(0,2)} e^{-\frac{1}{2}r_1(2)^2\sigma_1(2)^2 + r_1(2)\sigma_1(2)x(2)}, \\
&\frac{1}{a}\cdot\left[1 + P(0,1)p_x\right] = \frac{\beta^{\ell}}{P(0,1)}\cdot\left[P(0,1)q_x\Phi(x(1)) + P(0,2)p_x\Phi(x(2))\right] \\
&+ q_x\Phi(\sigma_0(1) - x(1)) + p_x\left[\Phi(r_0(2)\sigma_0(2) - x(2)) + P(0,1)\Phi(r_1(2)\sigma_1(2) - x(2))\right].
\end{aligned}
\tag{A.3}
$$

This system implies without difficulty the special solution (5.35).

References

AASE, K.K. and Persson, S.-A. (1994) Pricing of unit-linked life insurance policies. *Scandinavian Actuarial Journal*, **1**, 26-52.

BACINELLO, A. (2007) *Equity linked life insurance*. Encyclopedia of Quantitative Risk Assessment. J. Wiley.

BACINELLO, A. and ORTU, F. (1993a) Pricing equity-linked life insurance with endogenous minimum guarantees. *Insurance: Mathematics and Economics* **12**, 245-257. Corrigendum. *Insurance: Mathematics and Economics* **13**, 303-304.

BACINELLO, A. and ORTU, F. (1993b) Pricing guaranteed securities-linked life insurance under interest rate risk. 3*rd International AFIR Colloquium*, **4**, 35-55. Available as http://www.actuaries.org/AFIR/colloquia/Rome/Bacinello_Ortu.pdf

BACINELLO, A. and ORTU, F. (1994) *Single and periodic premiums for guaranteed equity-linked life insurance under interest-rate risk: the lognormal + Vasicek case.* In: L. Peccati, M. Viren (Eds.). Financial Modelling: Recent Research. Physica-Verlag, Heidelberg, 1-25.

BACINELLO, A., BIFFIS, E. and MILLOSSOVICH, P. (2009a) Pricing life insurance contracts with early exercise features. *Journal Computational Applied Mathematics* **233(1)**, 27-35.

BACINELLO, A., BIFFIS, E. and MILLOSSOVICH, P. (2009b) Regression-based algorithms for life insurance contracts with surrender guarantees. Appears in *Quantitative Finance*.

BOYLE, P.P. and SCHWARTZ, E.S. (1977) Equilibrium prices of guarantees under equity-linked contracts. *Journal of Risk and Insurance* **44**, 639-680.

Brennan, M.J. and Schwartz, E.S. (1976) The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics* **3**, 195-213.

Brennan, M.J. and Schwartz, E.S. (1979a) Alternative investment strategies for the issuers of equity linked life insurance with an asset value guarantee. *Journal of Business* **52**, 63-93.

Brennan, M.J. and Schwartz, E.S. (1979b) *Pricing and investment strategies for equity-linked life insurance*. Monograph no. 7, Huebner Foundation for Insurance Education. Wharton School, University of Pennsylvania, Philadelphia.

Corby, F.B. (1977) Reserves for maturity guarantees under unit-linked policies. *Journal of the Institute of Actuaries* **104**, 259-273.

Costabile, M., Gaudenzi, M., Massabo, I. and Zanette, A. (2009) Evaluating fair premiums of equity-linked policies with surrender option in a bivariate model. *Insurance: Mathematics and Economics* **45(2)**, 286-295.

De Felice, M. and Moriconi, F. (2005) Market based tools for managing the life insurance company. *ASTIN Bulletin* **35**, 79-111.

Delbaen, F. (1990) Equity linked policies. *Bulletin Association des Actuaires Belges*, 33-52.

Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D. (2002) The concept of comonotonicity in actuarial science and finance: applications. *Insurance: Mathematics and Economics* **31(2)**, 133-161.

Geman, H., El Karoui, N. and Rochet, J.-C. (1995) Changes of numeraire, changes of probability measure and option pricing. *Journal of Applied Probability* **32**, 443-458.

Hardy, M. (2003) Investment Guarantees: Modelling and Risk Management for Equity-Linked Life Insurance. *Wiley Finance* **168**.

Ho, T.S.Y. and Lee, S. (1986) Term structure movements and pricing interest rate contingent claims. *Journal of Finance* **41**, 1011-1029.

Hull, J. and White, A. (1990) Pricing interest rate derivative securities. *The Review of Financial Studies* **3**, 573-592.

Jamshidian, F. (1989) An exact bond option formula. *Journal of Finance* **44(1)**, 205-209.

Jamshidian, F. (1991) Bond and option evaluation in the Gaussian interest rate model. *Research in Finance* **9**, 131-170.

Kaas, R., Dhaene, J. and Goovaerts, M.J. (2000) Upper and lower bounds for sums of random variables. *Insurance: Mathematics and Economics* **27(2)**, 151-168.

Kurz, A. (1996) *Pricing of equity-linked life insurance policies with an asset value guarantee and periodic premiums*. 6th International AFIR Colloquium, Nürnberg, Germany. Available as http://www.actuaries.org/AFIR/colloquia/Nuernberg/Kurz.pdf

Mercurio, F. and Moraleda, J. (2000) An analytically tractable interest rate model with humped volatility. *European Journal of Operational Research* **120**, 205-214.

Mercurio, F. and Moraleda, J. (2001) A family of humped volatility models. *The European Journal of Finance* **7**, 93-116.

Moraleda, J. and Vorst, T. (1997) Pricing American interest rate claims with humped volatility models. *Journal of Banking and Finance* **21(8)**, 1131-1157.

Nielsen, J.A. and Sandmann, K. (1995) Equity-linked life insurance – a model with stochastic interest rates. *Insurance: Mathematics and Economics* **16(3)**, 225-253.

Nielsen, J.A. and Sandmann, K. (1996) Uniqueness of the fair premium for equity-linked life insurance contracts. *The GENEVA Papers on Risk and Insurance Theory* **21**, 65-102.

Nielsen, J.A. and Sandmann, K. (2002) *The fair premium of an equity-linked life and pension insurance*. In K. Sandman, P.J. Schönbucher (Eds.). Advances in Finance and Stochastics – Essays in Honor of Dieter Sondermann, 219-256. Springer Verlag.

Persson, S.-A. (1994) *Pricing life insurance contracts under financial uncertainty*. PhD-thesis, Norwegian School of Economics and Business Administration, Bergen.

Ritchken, P. and Chuang, I. (1999) Interest rate option pricing with volatility humps. *Review of Derivative Research* **3**, 237-262.

Rogers, L.C.G. and Shi, Z. (1995) The value of an Asian option. *Journal of Applied Probability* **32(4)**, 1077-1088.

Vanduffel, S., Hoedemakers, T. and Dhaene, J. (2005a) Comparing approximations for sums of non-independent lognormal random variables. *North American Actuarial Journal* **9(4)**, 71-82.

VANDUFFEL, S., DHAENE, J. and GOOVAERTS, M. (2005b) On the evaluation of saving-consumption plans. *Journal of Pension Economics and Finance* **4(1)**, 17-30.

VANDUFFEL, S., CHEN, X., DHAENE, J., GOOVAERTS, M., HENRARD, L. and KAAS, R. (2008a) Optimal approximations for risk measures of sums of lognormals based on conditional expectations. *Journal of Computational and Applied Mathematics* **221(1)**, 202-218.

VANDUFFEL, S., SHANG, Z., HENRARD, L., DHAENE, J. and VALDEZ, E. (2008b) Analytical bounds and approximations for annuities and Asian options. *Insurance: Mathematics and Economics* **42(3)**, 1109-1117.

VASICEK, O.A. (1977) An equilibrium characterization of the term structure. *Journal of Financial Economics* **5**, 177-188.

YOLCU, Y. (2005) *One-factor interest rate models: analytical solutions and approximations*. Master Thesis, Department of Financial Mathematics. The Middle East Technical University. http://www3.iam.metu.edu.tr/iam/images/9/9e/Yelizyolcuthesis.pdf

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