AMBIGUITY AVERSION: A NEW PERSPECTIVE ON INSURANCE PRICING

BY

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ABSTRACT

This paper intends to develop a feasible framework which incorporates ambiguity aversion into the pricing of insurance products and investigate the implications of ambiguity aversion on the pricing by comparing it with risk aversion. As applications of the framework, we present the closed-form pricing formulae for some insurance products appearing in life insurance and property insurance. Our model confirms that the effects of ambiguity aversion on the pricing of insurance do differ from those of risk aversion. Implications of our model are consistent with some empirical evidences documented in the literature. Our results suggest that taking advantage of natural hedge mechanism can help us control the effects of model uncertainty.

1. INTRODUCTION

Insurance enables consumers to be covered from a large contingent loss by paying a fixed small premium. In order to price risks, many models are developed for calibration. However, there is no doubt that almost all the models cannot predict the future precisely. Model uncertainty exposes insurers to the risk that the prediction of their model might deviate from the actual outcome remarkably. Such risk may lead to severe solvency issues or the failure of insurance markets, especially for the long-term and/or large-quantity insured risk. One example is longevity risk, i.e. unexpected improvements in life expectancies. Life insurers, as well as pension funds, claim that their annuity businesses are losing money due to the unexpected longevity improvements over years, see Cairns et al. (2008) for an elaboration. Another example is catastrophe risk. The loss-related uncertainty associated with a risk appears to be the principal reason for the reluctance of the insurance industry to provide coverage against earthquake damage, see Kunreuther et al. (1993) and Earthquake Project (1990).

Confronted with model uncertainty, the insurer exhibits ambiguity aversion in the sense of Knight (1921) and Ellsberg (1961). Ellsberg (1961) paradox[1]

[1] That is, people prefer to bet on an urn with 50 Red and 50 Blue balls, than in one with 100 total balls but where the number of blue or red balls is unknown.

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describes an attitude of preference for known risks over unknown risks, which is now termed "ambiguity aversion". Ambiguity aversion induces the agent to consider alternative models to protect himself against possible model mis-specifications. In the literature, Segal and Spivak (1988), Gilboa and Schmeidler (1989), Epstein and Wang (1994), Anderson et al. (1999), Hansen and Sargent (2001), Chen and Epstein (2002), Klibanoff et al. (2005), etc. present a series of axiomatic characterizations of ambiguity aversion. Ambiguity aversion has also attracted some attentions in insurance studies. Kunreuther et al. (1993) and Froot (2001) point out that when there is considerable ambiguity and uncertainty, the insurer would prefer to set premiums high or even not to provide insurance against risks.

Our paper explores the impact of ambiguity aversion on the pricing of insurance. Firstly, we develop a feasible framework which incorporates ambiguity aversion into the pricing of insurance products. As applications of the framework, we obtain closed-form pricing formulae for mortality risk and property risk in the dynamic context, which seem new in the literature related to risk pricing[2]. Secondly, our model confirms that the effects of ambiguity aversion on the pricing of insurance do differ from those of risk aversion. In some cases, risk aversion might not yield a sensible price for insurance policy, but the price driven by ambiguity aversion seems quite reasonable. Thirdly, our model illustrates that "natural hedge", if it exists, can alleviate the impact of ambiguity aversion on the pricing. The terminology "natural hedge" means that there are two classes of policies underwritten in opposite directions, such as both life insurance policies and pure endowment policies provided by a life insurance company.

The starting point of our analysis is the utility-equivalence pricing principle, which is pioneered by Hodges and Neuberger (1989) and later extended by Davis et al. (1993). Since then, utility-equivalence pricing has been applied in many different areas of finance and insurance. To list a few, Young and Zariphopoulou (2002), Musiela and Zariphopoulou (2004), Ludkovski and Young (2008), Egami and Young (2008), etc. The utility adopted in this paper follows Anderson et al. (1999), Hansen et al. (1999), Maenhout (2001), Uppal and Wang (2003) and Liu et al. (2005). Compared with Merton (1976) utility, the utility used here incorporates the insurer's pessimism with regard to the reference model. Model uncertainty affects the utility in two different ways. On the one hand, the insurer tends to think through worst-case scenarios so that he underestimates the utility. On the other hand, the insurer knows that the reference model is the best representation of the existing data so that he penalizes his deviation from the model to prevent himself from being too pessimistic. As a whole, the insurer adopts strategies according to the max-minimization principle.

Our formulation of ambiguity aversion falls under the general literature on portfolio decisions that are robust to model mis-specifications. Without knowing

[2] For a review on premium principles, see Young (2004).

a distribution over the multiple priors, a robust decision maker uses rules that work well for a reference model, but that are also insensitive to small perturbations of the reference model. To describe the preference for robustness, the max-min expected utility theory is initiated by Gilboa and Schmeidler (1989) and the robust-control theory is developed in Anderson et al. (1999)[3]. Hasen and Sargent (2001) study links between these two theories and show how to transform the "penalty problem" under the framework of Anderson et al. (1999) into a closely related "constrained problem" under the framework of Gilboa and Schmeidler (1989)[4]. Following Anderson et al. (1999), we model ambiguity aversion by a combination of the max-minimization program and a penalty function for model perturbation. Although we do not make any novel contribution to the mathematical methodology of the robust-control theory, we do make an addition to the related literature by constructing tricky penalty normalization factors and solving the insurer's optimal utilities with ambiguity aversion in closed form when the underlying risk follows some specific stochastic processes.

It is worthwhile mentioning the connections between the methodology of pricing risks in our paper and the theory of coherent (convex) risk measures, which is pioneered by Artzner et al. (1999), and further developed by Frittelli (2000), Delbaen (2002), Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2002), and El Karoui (2009) (to mention only a few)[5]. Firstly, notice that the entropic risk measure corresponds to the certainty equivalence associated with the exponential utility and hence our pricing formulae can be expressed in terms of convex risk measures. Secondly, quite similar to the robust-control theory, the standard dual theorem transforms coherent (convex) risk measures into representations with penalties. Especially, for the convex risk measure defined by an exponential loss function together with model uncertainty, the penalty in its dual representation turns out to be the relative entropy exactly (see Föller and Schied (2002) and Schied (2006)). Thus it seems feasible to study the reservation prices of risks and the hedging strategies with model uncertainty in the context of coherent risk measures along the same line of our arguments.

Our results are closely related to Kunreuther et al. (1993) and Cox and Lin (2007). Kunreuther et al. (1993) empirically examine how model uncertainty affects the premium-setting decisions of actuaries, underwriters, and reinsurers. Their surveys reveal that the recommended premiums increase rapidly as the ambiguity with respect to the probability rises. By introducing a rigorous formulation of ambiguity aversion, our pricing formulae grasp the mechanism that the premium driven by ambiguity aversion can approach infinity. Cox and

[3] A substantial literature in financial economics addresses the robust portfolio-selection problems. For the recent development, see Hansen et al. (1999), Goldfarb and Iyengar (2003), Schied (2005, 2006), Garlappi et al. (2007), and El Karoui (2009) among others.

[4] There is an ongoing discussion about the exact relation between these two theories, we refer the reader to Maenhout (2001), Pathak (2002), Skiadas (2003), Uppal and Wang (2003), etc.

[5] We thank an anonymous referee for pointing out such connections to us.

Lin (2007) find empirical evidence that annuity writing insurers who have more balanced business in life and annuity risks tend to charge lower premiums than otherwise similar insurers. This point is also justified by our pricing formulae.

The difference between the effects of ambiguity aversion and those of risk aversion on the pricing of insurance lies in two aspects. One is that risk aversion does not serve as an incentive for the insurer to adjust the underlying risk model to the safety side. The risk premium driven by risk aversion only reflects the insurer's preference on a given risk distribution. Broadly speaking, the pricing formula driven by risk aversion relies only on the estimated value of the underlying risk reported from a given model, irrespective of which specific model is selected. Two abnormal phenomena may be caused by such pricing mechanism. Firstly, for the policy of pure life insurance, the relative loading driven by risk aversion always becomes larger as long as the maturity decreases, which seems counterintuitive. Secondly, for property insurance, the relative loading driven by risk aversion is very sensitive to the loss magnitude, which suggests that risk aversion might not give rise to a plausible distortion for the pricing of modest losses[6]. In contrast, the risk premium driven by ambiguity aversion highlights the insurer's selection on the models available. With ambiguity aversion, the insurer will adjust the model parameters to the safety side. The impact of such adjustment on risk pricing accumulates when the time extends or the loss increases. Our results demonstrate that the pricing mechanism driven by ambiguity aversion seems much more reasonable.

The other difference is that, although ambiguity aversion leads to a nonlinear increase of premium as the uncertainty accumulates over time or the loss magnitude is enlarged, the increasing pattern is different from that induced by risk aversion. Moreover, the premium driven by ambiguity aversion can be depressed by the existence of natural hedge. Typical examples of long-term policies and large coverage polices can be found in life insurance market and catastrophe insurance market respectively. There is a possibility that extremely high premium is charged by the insurer in both markets. Abnormally high premium and even market failure are observed in catastrophe market, see Kunreuther et al. (1993), Froot (2001) and Ibragimov et al. (2009). However, due to the presence of "natural hedge" in life insurance market, the relevant price approaches the one computed on the reference model. This indicates that taking advantage of natural hedge mechanism, like developing balanced business, can partially eliminate the effects of model uncertainty.

The layout of this paper is as follows. In the second section, we illustrate how to incorporate ambiguity aversion into the utility function. In the third section, we price the risk of continuous fluctuations through an example in life

[6] We concede that risk aversion may be a useful description of the taste for very-large-scale losses, however, for modest losses which the insurer attempts to underwrite, such sensitivity of the pricing to the loss magnitude driven by risk aversion seems abnormal. Rabin (2000) proves this point in a quite general expected utility framework. He points out that "within the expected-utility model, anything but virtual risk neutrality over modest stakes implies manifestly unrealistic risk aversion over large stakes".

insurance and obtain the corresponding closed-form pricing formula. We also compare the impact of ambiguity aversion with that of risk aversion, and analyze the way in which the effects of model uncertainty can be partially offset by "natural hedge". In the fourth section, we price the risk of rare events (jumps) through an example in property insurance and explain why ambiguity aversion can give us a more reasonable pricing mechanism for modest losses. Furthermore, we discuss the effects of ambiguity aversion on the pricing of catastrophe risk and show that ambiguity aversion can lead to a high premium. The final section concludes this paper. All the proofs are relegated to the Appendix.

2. Preliminary

As mentioned in the Introduction, the starting point of our analysis is the utility-equivalence pricing principle. To give a formal definition[7], assume that U and U^L are utility functions for the insurer without and with insurance liabilities respectively. The insurance liability L_T is payable at time T by an insurer who has underwritten the liability. The liability cannot be traded after its transfer from the buyer to the insurer. The reservation price of the insurer $\mathcal{P}$ is defined as the minimum price that satisfies $U(W, t) \leq U^L(W + \mathcal{P}, L, t)$ for all wealth levels W.

As a preparation for the study of $U^L(W + \mathcal{P}, L, t)$, we at first characterize $U(W, t)$ in the presence of model uncertainty and ambiguity aversion in this section. The formulation and derivation of $U(W, t)$ with ambiguity aversion for the power utility is well known in the recent literature, see Anderson et al. (1999), Maenhout (2001), Uppal and Wang (2003) and Liu et al. (2005). Our task here is just to derive $U(W, t)$ for the exponential utility and review some insights on ambiguity aversion reported in the literature for our later use. The reason why we choose exponential utility to describe insurer's preference is that the exponential utility possess very desirable properties for the insurance pricing, see Gerber (1974) and Young and Zarphipoulou (2002).

The insurer invests in the stock market. Dynamics of the stock price S_t is modeled by

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dB_t^S, \tag{1}$$

where B_t^S is a standard Brownian motion under the reference measure $\mathbb{P}^S$, the drift μ_t and the volatility σ_t are positive deterministic functions of t. The agent exhibits ambiguity aversion in the sense of Knight (1921) and Ellsberg (1961) in the way that he considers alternative models to protect himself against possible model mis-specifications. Ambiguity aversion is characterized through a

[7] This definition is borrowed from Young and Zarphipoulou (2002).

Girsanov transform. Suppose the alternative model is defined by its probability measure $\mathbb{P}^S(\xi^S)$, where $\xi_T^S = \frac{d\mathbb{P}^S(\xi^S)}{d\mathbb{P}^S}$ is its Radon-Nikodym derivative with respect to $\mathbb{P}^S$ subject to

$$\frac{d\xi_t^S}{\xi_t^S} = h_t^S dB_t^S.$$

Notice that h_t^S is a stochastic process adapted to the filtration generated by B_t^S. Under the probability measure $\mathbb{P}^S(\xi^S)$, the process $\hat{B}_t^S$ given by $dB_t^S = h_t^S dt + d\hat{B}_t^S$ is a standard Brownian motion. The drift of $\frac{dS_t}{S_t}$ under the measure $\mathbb{P}^S(\xi^S)$ is $\mu_t + \sigma_t h_t^S$, and then the drift adjustment is $\theta_t^S = \sigma_t h_t^S$. Formally, accepting the alternative probability $\mathbb{P}^S(\xi^S)$ is equivalent to accepting the dynamics

$$\frac{dS_t}{S_t} = (\mu_t + \theta_t^S)dt + \sigma_t d\hat{B}_t^S. \tag{2}$$

Assume the insurer has a source of information about $\hat{B}_t^S$. Let E_t and $E_t^{\xi^S}$ denote the expectation operators under the measures $\mathbb{P}^S$ and $\mathbb{P}^S(\xi^S)$ conditional on the information up to t respectively. We define the index reflecting the information with respect to $\hat{B}_t^S$ in the time period $[t, t+\Delta]$ as

$$I(\xi^S) = E_t^{\xi^S}\left[H\left(\ln\frac{\xi_{t+\Delta}^S}{\xi_t^S}\right)\right].$$

In the Brownian motion setting, the function H is usually chosen as $H(x) = x$ such that

$$I(\xi^S) = \frac{1}{2}\int_t^{t+\Delta} E_t\left[|h_s^S|^2\right]ds \tag{3}$$

is exactly the relative entropy of $\mathbb{P}^S(\xi^S)$ with respect to $\mathbb{P}^S$ from time t to $t+\Delta$.

The utility function $U(W, t)$ for the insurer without insurance liabilities is easy to obtain by the standard optimal control theory. Without insurance liabilities, given the investment decision π_t, the budget equation of the insurer's wealth is

$$dW_t = r_t W_t\, dt + (\mu_t - r_t)\pi_t\, dt + \pi_t \sigma_t\, dB_t^S, \tag{4}$$

where the risk-free rate $r_t > 0$ is assumed to be deterministic. The insurer strives to maximize his utility of wealth at time T. With ambiguity aversion, the insurer adopts strategies according to the max-minimization objective

$$U_t = \sup_{\pi_t}\inf_{\xi^S}\mathcal{F}\left(E_t^{\xi^S}[U_{t+\Delta}]\right), \tag{5}$$

in which

$$\mathcal{F}(x) = \phi^S \psi^S(x) I(\xi^S) + x, \text{ with } I(\xi^S)$$

Here $U_t = U(W_t, t)$ is the indirect utility function conditional on the information up to t. The constant $\phi^S \geq 0$ is a penalty parameter which indicates that accepting the alternative measure $\mathbb{P}(\xi^S)$ will incur a penalty. The term $\psi^S(\cdot)$ is a normalization function that converts the penalty to units of utility and its functional form is often chosen for analytical tractability. The terminal condition of $U(W, t)$ is $U(W, T) = u(W)$ for all wealth levels W, where $u(\cdot)$ is a given utility function. Throughout this paper, we always suppose

$$u(W) = -\frac{1}{\alpha} e^{-\alpha W}, \text{ where } \alpha > 0 \text{ is the risk aversion parameter.}$$

Compared with Merton (1976) utility, the utility defined via (5) incorporates the agent's pessimism with regard to the reference model. The optimization problem (5) involves two decision variables: h_t^S and π_t^S. If the optimal choice of h_t^S is h_t^{S*}, then the agent tends to accept the alternative underlying dynamics (2) with $\theta_t^{S*} = \sigma_t h_t^{S*}$. It affects the utility in opposite directions. On the one hand, accepting the pessimistic model makes the agent underestimate the utility. On the other hand, the penalty on the deviation from the statistically best reference measure $\mathbb{P}^S$ adds a positive contribution to the utility.

We focus on the continuous-time version of the insurer's max-minimization problem (5). With the choice $\psi^S(x) = |x|$, Proposition A.1 gives the closed-form solution, which yields the optimal investment

$$\pi_t^* = \frac{1}{\alpha}\left(\frac{\phi^S}{1+\phi^S}\right)\left(\frac{\mu_t - r_t}{\sigma_t^2}\right) e^{-\int_t^T r_s\, ds}, \tag{6}$$

and the optimal drift adjustment of $\frac{dS_t}{S_t}$

$$\theta_t^{S*} = -\left(\frac{1}{1+\phi^S}\right)(\mu_t - r_t). \tag{7}$$

The functional form $\psi^S(x) = |x|$ can be interpreted intuitively in the way that the penalty of accepting the alternative probability is proportional to the utility under that probability. The strategy (6) is equal to the one adopted by a risk-adverse agent without ambiguity aversion whose risk aversion parameter is $\alpha \times (\frac{1+\phi^S}{\phi^S})$. In other words, ambiguity aversion is not separable from risk aversion, as far as the investment strategy is concerned[8]. In our setup where

[8] This is the observational-equivalence result noted in Anderson et al. (1999) and Maenhout (2001). Uppal and Wang (2003) point out that when there are more than one risky assets, the observational-equivalence result remains valid only if the agent is equally ambiguous about the distributions of returns for all assets.

the value of the optimal utility is concerned, we observe that the optimal utility for the agent who faces the underlying dynamics (1) with the ambiguity aversion parameter ϕ^S is equivalent to the one for the agent who faces

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu_t dt + \sqrt{\frac{\phi^S + 1}{\phi^S}}\,\sigma_t dB_t^S \tag{8}$$

instead of (1) without ambiguity aversion, see Proposition A.1. This phenomenon can also be observed when we consider ambiguity aversion related to the non-tradable insurance liability. The intuition behind (8) is that the possibility of model mis-specification adds another source of uncertainty to the riskiness of the terminal wealth. Due to ambiguity aversion, the agent is prone to overestimating the underlying risk by raising the volatility from σ_t to $\sqrt{\frac{\phi^S+1}{\phi^S}}\,\sigma_t$.

When deriving the utility function U^L of the insurer, we should incorporate the liability or loss process L_t into the dynamic optimization framework. In addition to ambiguity aversion with respect to stock prices, we need also take into account ambiguity aversion with respect to the loss process L_t. Since the insurance liability is non-tradable in the market, the derivation of $U^L(W + \mathcal{P}, L, t)$ is different from that of $U(W, t)$ and the construction of penalty function related to L_t is tricky. Commonly, Brownian motion accounts for a successful description of the risk of continuous fluctuations and Poisson process fits the risk of rare events (jumps) well. Intuitively, the effect of model uncertainty in the context of long-term continuous fluctuations or large-scale rare events is non-ignorable and in turn makes the insurer become conservative with the reference model, aware of the unexpected losses caused by model mis-specifications. In the next two sections, we derive the risk premium of continuous fluctuations and jumps with ambiguity aversion respectively through concrete examples.

Remark 1. For the risk preference of insurers, the assumptions of "risk aversion" and "risk neutral" both have their potentials in explaining insurers' behavior. Our pricing formulae developed below are applicable for both risk-averse and risk-neutral insurers.

3. Pricing the risk of continuous fluctuations

It is widely accepted that continuous fluctuations or Brownian motions constitute a substantial part of the uncertainty appearing in mortality risk. As an illustration of pricing the risk of continuous fluctuations, we study the prices of the pure endowment and the pure life insurance.

To highlight the key point, we assume the evolution of individual mortality λ_t to be an affine process

$$d\lambda_t = (k_{0t} + k_{1t}\lambda_t)dt + (v_{0t} + v_{1t}\lambda_t)^{\frac{1}{2}} dB_t^\lambda \tag{9}$$

under the reference measure $\mathbb{P}^\lambda$, where B_t^λ is another Brownian motion independent of B_t^S. Here k_{0t}, k_{1t}, v_{0t}, v_{1t} are all deterministic functions of t. The model (9) neglects some otherwise important factors, such as the mortality jumps induced by the advances of social medicine. We will return to this at the end of this section. Typical examples of (9) include

Vasicek (VAS) process : $d\lambda_t = \gamma(m_t - \lambda_t)dt + \varsigma dB_t^\lambda$,

Cox-Ingersoll-Ross (CIR) process: $d\lambda_t = \gamma(m_t - \lambda_t)dt + \varsigma\sqrt{\lambda_t}\, dB_t^\lambda$,

where m_t is a deterministic function of the Gompertz type, γ and ς are both positive constants. Biffis (2005), Luciano and Vigna (2005) calibrate the VAS processes with or without jumps to different mortality tables. Dahl (2004), Dahl and Møller (2006) use the CIR process in the pricing and hedging of mortality risk. As what we have done for stock prices, ambiguity aversion with respect to λ_t is also characterized through a Girsanov transform

$$\frac{d\mathbb{P}^\lambda(\xi^\lambda)}{d\mathbb{P}^\lambda} = \xi_T^\lambda = e^{\int_0^T h_t^\lambda dB_t^\lambda - \frac{1}{2}\int_0^T |h_t^\lambda|^2 dt},$$

which implies that the process $\hat{B}_t^\lambda$ given by $dB_t^\lambda = h_t^\lambda dt + d\hat{B}_t^\lambda$ is a standard Brownian motion under the alternative measure $\mathbb{P}^\lambda(\xi^\lambda)$. We define the drift adjustment of mortality as $\theta_t^\lambda = [v_{0t} + v_{1t}\lambda_t]^{\frac{1}{2}} h_t^\lambda$. Thus accepting the alternative probability $\mathbb{P}^\lambda(\xi^\lambda)$ is equivalent to accepting the mortality model

$$d\lambda_t = \left(k_{0t} + k_{1t}\lambda_t + \theta_t^\lambda\right)dt + (v_{0t} + v_{1t}\lambda_t)^{\frac{1}{2}} d\hat{B}_t^\lambda.$$

We use E_t, $E_t^{\xi S,\lambda}$, $E_t^{\xi S}$ and $E_t^{\xi^\lambda}$ to denote the expectation operators under the measures $\mathbb{P}^S \times \mathbb{P}^\lambda$, $\mathbb{P}^S(\xi^S) \times \mathbb{P}^\lambda(\xi^\lambda)$, $\mathbb{P}^S(\xi^S)$ and $\mathbb{P}^\lambda(\xi^\lambda)$ respectively, conditional on the information up to t. Parallel to the previous section, we use the relative entropy of $\mathbb{P}^\lambda(\xi^\lambda)$ with respect to $\mathbb{P}^\lambda$ from time t to $t + \Delta$ to index the information with respect to $\hat{B}_t^\lambda$ in the time period $[t, t + \Delta]$. That is, as in (3), we introduce the information index

$$I(\xi^\lambda) = \frac{1}{2}\int_t^{t+\Delta} E_t\left[|h_s^\lambda|^2\right] ds. \tag{10}$$

We consider a contract of endowment insurance. The insurer pays K dollars at time T if the insured survives to that time and $\bar{K}e^{-\int_t^T r_s\, ds}$ dollars to the insured's family if he dies at time t. Under the exponential utility, the insurer is indifferent between paying $\bar{K}e^{-\int_t^T r_s\, ds}$ at the death time t and paying $\bar{K}$ at the maturity T. In this case, the insurer's utility $U_t^L = U^L(W_t, \lambda_t, t)$ conditional on the information up to t satisfies

$$U_t^L = \sup_{\pi_t} \inf_{\xi^S, \xi^\lambda} \mathcal{F}\left(E_t^{\xi S,\lambda}[U_{t+\Delta}^L], E_t^{\xi S}[\bar{U}_{t+\Delta}]\right) \text{ with } U^L(W, \lambda, T) = u(W - K), \tag{11}$$

where

$$\begin{cases} \mathcal{F}(x,y) = [\phi^S \psi^S(x) I(\xi^S) + x]_{\Delta} p_{x+t} + [\phi^S \psi^S(y) I(\xi^S) + y]_{\Delta} q_{x+t} + \phi^\lambda \psi^\lambda(x,y) I(\xi^\lambda), \\ U_t^L = U^L(W_t, \lambda_t, t), \\ \bar{U}_t = U(W_t - \bar{K} e^{-\int_t^T r_s ds}, t). \end{cases}$$

In the above, $I(\xi^\lambda)$ and $I(\xi^\lambda)$ are given by (3) and (10) respectively, ${}_{\Delta}p_{t+x} = E_t^{\xi^\lambda}[e^{-\int_t^{t+\Delta} \lambda_s ds}]$ is the $\mathbb{P}^\lambda(\xi^\lambda)$-probability that the insured survives to time $t + \Delta$ given that he is alive at time t and ${}_{\Delta}q_{x+t} = 1 - {}_{\Delta}p_{t+x}$ is the $\mathbb{P}^\lambda(\xi^\lambda)$-probability that the insured dies before time $t + \Delta$. The constant $\phi^\lambda \geq 0$ is a penalty parameter which indicates that accepting the alternative measure $\mathbb{P}^\lambda(\xi^\lambda)$ will incur a penalty. The term $\psi^\lambda(\cdot,\cdot)$ is again a normalization function that converts the penalty to units of utility and its functional form is also chosen for analytical tractability.

On the continuous-time version of the max-minimization problem (11), after choosing $\psi^S(x) = |x|$ and $\psi^\lambda(x,y) = |x - y|$, Proposition A.3 presents the closed-form solution $U^L(W, \lambda, t) = U(W, t)\, \eta(\lambda, t)$. Intuitively, the functional forms $\psi^S(x) = |x|$ and $\psi^\lambda(x,y) = |x - y|$ mean that the penalty of accepting the alternative probability on the stock price is proportional to the utility under that probability; while the penalty of accepting the alternative probability on mortality is proportional to the difference between the utility without liability and the one with liability under that probability. By definition, the price of the insurer is

$$\mathcal{P} = \frac{1}{\alpha} e^{-\int_t^T r_s ds} \ln \eta(\lambda, t). \tag{12}$$

Similar to (8), the optimal utility for the insurer who faces the underlying dynamics (9) with the ambiguity aversion parameter ϕ^λ is observationally equivalent to the utility for the insurer who faces one of the following two artificial mortalities

$$d\tilde{\lambda}_t^+ = (k_{0t} + k_{1t}\tilde{\lambda}_t^+)dt + \sqrt{\frac{\phi^\lambda + 1}{\phi^\lambda}} (v_{0t} + v_{1t}\tilde{\lambda}_t^+)^{\frac{1}{2}} dB_t^\lambda, \tag{13}$$

$$d\tilde{\lambda}_t^- = (k_{0t} + k_{1t}\tilde{\lambda}_t^-)dt + \sqrt{\frac{\phi^\lambda - 1}{\phi^\lambda}} (v_{0t} + v_{1t}\tilde{\lambda}_t^-)^{\frac{1}{2}} dB_t^\lambda, \tag{14}$$

without ambiguity aversion. To be specific, when $K > \bar{K}$, the insurer takes (13) as the utility-equivalence mortality, while when $K < \bar{K}$ the insurers takes (14) over (13)[9]. With ambiguity aversion, the pricing of the pure endowment is now

[9] For ease of exposition, we assume implicitly that $\phi^\lambda \geq 1$. In the case of $\phi^\lambda < 1$, we can obtain $\mathcal{P}$ in (12) by computing $\eta(\lambda, t)$ directly through Proposition A.3.

based on the artificial mortality (13) or (14) instead of (9). In general, we cannot expect the closed-form expression of $\eta(\lambda, t)$ for arbitrarily given k_{1t} and v_{1t}. However, for both the VAS process and CIR process, we can get $\eta(\lambda, t)$ explicitly. Let ${}_{T-t}\tilde{p}^{+}_{x+t} = E_t[e^{-\int_t^{T+\Delta}\tilde{\lambda}^{+}_s ds}]$ and ${}_{T-t}\tilde{p}^{-}_{x+t} = E_t[e^{-\int_t^{T+\Delta}\tilde{\lambda}^{-}_s ds}]$ be the survival probabilities for the artificial mortalities (13) and (14) respectively. Denote ${}_{T-t}\tilde{q}^{+}_{x+t} = 1 - {}_{T-t}\tilde{p}^{+}_{x+t}$ and ${}_{T-t}\tilde{q}^{-}_{x+t} = 1 - {}_{T-t}\tilde{p}^{-}_{x+t}$.

Theorem 1. *Assume that the dynamic mortality is modeled by the affine process (9). Then the price of the endowment insurance contract under the equivalent utility principle with ambiguity aversion is given by*

$$\mathcal{P}(\alpha, \phi^{\lambda}, K, \bar{K}) = \begin{cases} \frac{1}{\alpha}\ln\left(e^{\alpha\bar{K}}\,{}_{T-t}\tilde{q}^{+}_{x+t} + e^{\alpha K}\,{}_{T-t}\tilde{p}^{+}_{x+t}\right)e^{-\int_t^T r_s\,ds}, & \text{if } \bar{K} < K, \\ Ke^{-\int_t^T r_s\,ds}, & \text{if } \bar{K} = K, \\ \frac{1}{\alpha}\ln\left(e^{\alpha\bar{K}}\,{}_{T-t}\tilde{q}^{-}_{x+t} + e^{\alpha K}\,{}_{T-t}\tilde{p}^{-}_{x+t}\right)e^{-\int_t^T r_s\,ds}, & \text{if } \bar{K} > K. \end{cases} \tag{15}$$

Moreover, the optimal adjustment of the mortality drift for the VAS process is

$$\theta_t^{\lambda *} = \begin{cases} -\frac{\varsigma^2}{\gamma\phi^{\lambda}}\left[1 - e^{-\gamma(T-t)}\right], & \text{if } \bar{K} < K, \\ 0, & \text{if } \bar{K} = K, \\ \frac{\varsigma^2}{\gamma\phi^{\lambda}}\left[1 - e^{-\gamma(T-t)}\right], & \text{if } \bar{K} > K, \end{cases} \tag{16}$$

while the optimal adjustment of the mortality drift for the CIR process is

$$\theta_t^{\lambda *} = \begin{cases} -\frac{2\varsigma^2\left[e^{\kappa_2(T-t)} - 1\right]\lambda_t}{\phi^{\lambda}(\gamma + \kappa_2)\left[e^{\kappa_2(T-t)} - 1\right] + 2\kappa_2}, \ \kappa_2 = \sqrt{\gamma^2 + 2\left(\frac{\phi^{\lambda} + 1}{\phi^{\lambda}}\right)\varsigma^2}, & \text{if } \bar{K} < K, \\ 0, & \text{if } \bar{K} = K, \\ \frac{2\varsigma^2\left[e^{\kappa_2(T-t)} - 1\right]\lambda_t}{\phi^{\lambda}(\gamma + \kappa_2)\left[e^{\kappa_2(T-t)} - 1\right] + 2\kappa_2}, \ \kappa_2 = \sqrt{\gamma^2 + 2\left(\frac{\phi^{\lambda} - 1}{\phi^{\lambda}}\right)\varsigma^2}, & \text{if } \bar{K} > K. \end{cases} \tag{17}$$

Theorem 1 provides a feasible framework which incorporates ambiguity aversion to the pricing of life insurance contracts. To understand how the insurer's ambiguity aversion affects the utility-equivalence price $\mathcal{P}$, we take away the feature of model uncertainty by letting $\phi^{\lambda} = +\infty$. In this situation, the pricing-based mortalities $\tilde{\lambda}^{+}_t$ and $\tilde{\lambda}^{-}_t$ reduce to be the actual one λ_t and thus $\mathcal{P}$ reduces to be the standard price which is driven by risk aversion only. Another notable fact is that with $\phi^{\lambda} = +\infty$, we always have $\theta_t^{\lambda *} = 0$. In other words, without

model uncertainty, risk aversion only does not serve as an incentive for the insurer to adjust the mortality model to the safety side.

When the insurer exhibits ambiguity aversion to the mortality model, i.e. $\phi^{\lambda} < +\infty$, the insurer adjusts the drift of the mortality model to the safety side. Notice from (16) and (17) that the drift adjustment is proportional to $1/\phi^{\lambda}$. Accepting the pessimistic mortality model (13) for the endowment pricing or the model (14) for the life insurance pricing has two conflicting influences on the utility. On the one hand, insurer will underestimate the utility of his wealth. On the other hand, the penalty on the deviation from the statistically best reference measure $\mathbb{P}^{\lambda}$ adds a positive contribution to the utility.

Letting $\bar{K}$ and K be equal to zero respectively, Theorem 1 gives the prices of the pure endowment and the pure life insurance:

$$\mathcal{P}\left(\alpha, \phi^{\lambda}, K, 0\right) = \frac{1}{\alpha}\ln\left[1 + \left(e^{\alpha K} - 1\right)_{T-t}\tilde{p}^{+}_{x+t}\right] e^{-\int_t^T r_s\, ds},$$

$$\mathcal{P}\left(\alpha, \phi^{\lambda}, 0, \bar{K}\right) = \frac{1}{\alpha}\ln\left[1 + \left(e^{\alpha \bar{K}} - 1\right)_{T-t}\tilde{q}^{-}_{x+t}\right] e^{-\int_t^T r_s\, ds}.$$

Without ambiguity aversion, the corresponding prices driven by risk aversion reduce to be[10]

$$\mathcal{P}\left(\alpha, \infty, K, 0\right) = \frac{1}{\alpha}\ln\left[1 + \left(e^{\alpha K} - 1\right)_{T-t}p_{x+t}\right] e^{-\int_t^T r_s\, ds}, \tag{18}$$

$$\mathcal{P}\left(\alpha, \infty, 0, \bar{K}\right) = \frac{1}{\alpha}\ln\left[1 + \left(e^{\alpha \bar{K}} - 1\right)_{T-t}q_{x+t}\right] e^{-\int_t^T r_s\, ds}. \tag{19}$$

where ${}_{T-t}p_{x+t} = E_t[e^{-\int_t^T \lambda_s\, ds}]$ is the survival probability under the reference mortality model and ${}_{T-t}q_{x+t} = 1 - {}_{T-t}p_{x+t}$. The facts

$${}_{T-t}\tilde{p}^{+}_{x+t} > {}_{T-t}p_{x+t} \quad \text{and} \quad {}_{T-t}\tilde{q}^{-}_{x+t} > {}_{T-t}q_{x+t}$$

imply

$$\mathcal{P}\left(\alpha, \phi^{\lambda}, K, 0\right) > \mathcal{P}\left(\alpha, \infty, K, 0\right) \quad \text{and} \quad \mathcal{P}\left(\alpha, \phi^{\lambda}, 0, \bar{K}\right) > \mathcal{P}\left(\alpha, \infty, 0, \bar{K}\right).$$

The intuition is that in addition to risk aversion, ambiguity aversion provides another motivation to charge a positive premium. For a risk-neutral insurer, the pricing formulae driven by ambiguity aversion turn out to be

$$\mathcal{P}\left(0, \phi^{\lambda}, K, 0\right) = K \times {}_{T-t}\tilde{p}^{+}_{x+t} \quad \text{and} \quad \mathcal{P}\left(0, \phi^{\lambda}, 0, \bar{K}\right) = \bar{K} \times {}_{T-t}\tilde{q}^{-}_{x+t},$$

which seem new in the literature related to mortality risk pricing[11].

[10] See also Young and Zariphopoulou (2002).

[11] For a review of the existing premium principles for insurance contracts, see Young (2004).

The effect of ambiguity aversion differs from the effect of risk aversion on insurance pricing. The risk premium driven by risk aversion only reflects the insurer's preference on a given risk distribution. Generally speaking, the risk premium driven by risk aversion relies only on the estimated value of the underlying risk reported from a given model, irrespective of which specific model is selected. In contrast, the risk premium driven by ambiguity aversion highlights the insurer's selection on the models available. With ambiguity aversion, the insurer will adjust the model parameters to the safety side. The impact of such adjustment on risk pricing will accumulate when the time extends or the loss increases. In what follows, we compare the patterns of influence of risk aversion and ambiguity aversion through an illustrative example. Suppose the insurer underwrites policies of pure endowment and life insurance. Remember that $\mathcal{P}(0,\infty,K,\bar{K})$ is the actuarial value of an endowment insurance policy under the reference mortality model. We define

$$\mathcal{R}\left(\alpha,\phi^{\lambda},K,\bar{K}\right)=\frac{\mathcal{P}\left(\alpha,\phi^{\lambda},K,\bar{K}\right)}{\mathcal{P}\left(0,\infty,K,\bar{K}\right)}$$

to measure the relative loading charged by the insurer, and make the comparisons of

$$\begin{cases}\mathcal{R}\left(\alpha,\infty,K,0\right) \text{ versus } \mathcal{R}\left(0,\phi^{\lambda},K,0\right),\\ \mathcal{R}\left(\alpha,\infty,0,\bar{K}\right) \text{ versus } \mathcal{R}\left(0,\phi^{\lambda},0,\bar{K}\right),\end{cases} \tag{20}$$

where the left hand side involves risk aversion only and yet the right hand side involves ambiguity aversion only. Figure 1 displays the comparative results of (20) by using a set of artificial parameters.

We observe that the pattern of influence of ambiguity aversion is different from that of risk aversion on the pricing. In view of (18)-(19), the fact

$$\begin{cases}\lim_{x\to 0}\dfrac{1}{\alpha K x}\ln\left(1+(e^{\alpha K}-1)\,x\right)=\dfrac{1}{\alpha K}\,(e^{\alpha K}-1),\\ \lim_{x\to 1}\dfrac{1}{\alpha K x}\ln\left(1+(e^{\alpha K}-1)\,x\right)=1,\end{cases}$$

illustrates that risk aversion always imposes a large relative loading to a risk with small probability. For the pure endowment, the underlying event becomes a small probability event when the maturity is long enough. Consequently, for the pure endowment the relative loading charged by a risk-averse insurer is always higher when T is larger, see $\mathcal{R}(0.8,\infty,1,0)$ in Panels A and C of Figure 1. But for the life insurance, the small probability event occurs when the maturity is short and hence the relative loading is always higher when T is smaller, see $\mathcal{R}(0.8,\infty,0,1)$ in Panels B and D of Figure 1, which seems somewhat counter-intuitive.

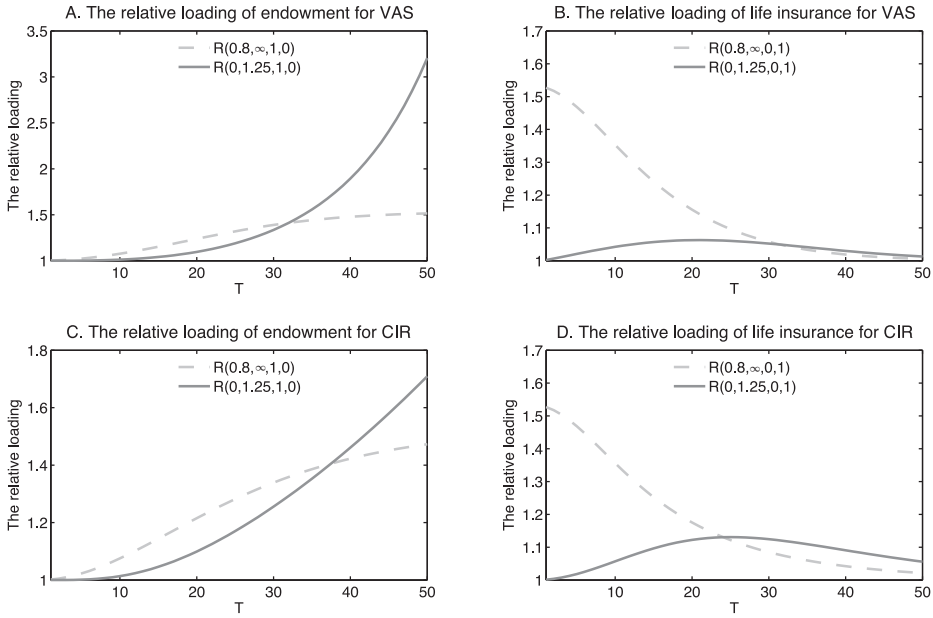

FIGURE 1: $\mathcal{R}(0.8, \infty, 1, 0)$ v.s. $\mathcal{R}(0, 1.25, 1, 0)$ and $\mathcal{R}(0.8, \infty, 0, 1)$ v.s. $\mathcal{R}(0, 1.25, 0, 1)$. In the VAS model, we choose $\gamma = 0.01$, $m_t = 0.5$ and $\varsigma = 0.01$. In the CIR model, we choose $\gamma = 0.01$, $m_t = 0.5$ and $\varsigma = 0.1$. The initial mortality value is taken as $\lambda_0 = 0.003$.

In contrast, the relative loading imposed by ambiguity aversion is always monotonously increasing with T. For both the VAS and the CIR models, it is easy to deduce

$$\lim_{T \to \infty} \mathcal{R}\left(0, \phi^{\lambda}, K, 0\right) = \infty.$$

As the endowment maturity extends, the relative loading driven by risk aversion is bounded from above, while the relative loading driven by ambiguity aversion can approach infinity, see $\mathcal{R}(0, 1.25, 1, 0)$ in Panels A and C of Figure 1. This indicates that model uncertainty has a substantial impact on the pricing, especially when the uncertainty persists in a long time period. For the life insurance, ambiguity aversion delivers an inverted-U pattern of the relative loading, as shown by $\mathcal{R}(0, 1.25, 0, 1)$ in Panels B and D of Figure 1. Compared with the decreasing pattern of relative loading with T driven by risk aversion, the inverted-U pattern indicates an intuitively reasonable result in the sense that the relative loading of the life insurance should be small when the maturity T is short.

Remark 2. The inverted-U pattern in Figure F1 can be explained as follows. When T is small, e.g., $T < 20$ (years), the relative loading driven by ambiguity

aversion increases with T, due to the fact that model uncertainty accumulates over time. When T is large, e.g., $T > 40$ (years), both the referred and the adjusted probabilities of death approach 1, which makes the relative loading decrease to be 1.

With ambiguity aversion, the insurer tends to take for granted that the expected potential loss is larger than the one estimated from the reference model. When the insurer underwrites two classes of policies in opposite directions, he will suffer a smaller loss in one direction in case that the loss in the other direction becomes larger. This is the so-called "natural hedge" mechanism, see Cox and Lin (2007). In life insurance market, there exists natural hedge between life insurance and endowment. Suppose that the insurer underwrites a total amount of K dollars of endowment and $\bar{K}$ dollars of life insurance. Assume that the purchaser of life insurance policies and the purchaser of endowment policies come from a homogeneous cohort, the persons in which have the same age and the same health status such that the mortality of persons in the cohort can be described by the same reference model. Then the choice of mortality model depends on whether K or $\bar{K}$ is larger. If $K > \bar{K}$, then the effect of model uncertainty on $\bar{K}$ dollar endowment can be completely offset by the effect of model uncertainty on life insurance[12]. Due to the presence of $(K - \bar{K})$ dollar endowment, the insurer will overestimate the insured's survival probability to protect himself from suffering the loss arising from the unexpected improvement in mortality. As a result, the insurer tends to charge lower premiums for life insurance policies. The situation of $K < \bar{K}$ can be analyzed in a similar way. On the whole, developing balanced business can partially eliminate the effects of model uncertainty and in turn decrease the premium caused by ambiguity aversion.

Specially, the risk of model uncertainty can be partially hedged by designing hybrid insurance products. An example is the endowment insurance contract. It is easy to verify that for $\phi^\lambda > 0$, $K > 0$, $\bar{K} > 0$, there is

$$\mathcal{P}(0, \phi^\lambda, K, 0) + \mathcal{P}(0, \phi^\lambda, 0, \bar{K}) > \mathcal{P}(0, \phi^\lambda, K, \bar{K}). \tag{21}$$

This amounts to say that buying an endowment insurance with payoff $(K, \bar{K})$ is cheaper than buying the endowment with payoff K and the life insurance with payoff $\bar{K}$ separately. The equality $\mathcal{R}(0, \phi^\lambda, K, \bar{K}) = 1$ achieves if and only if $K = \bar{K}$, irrespective of ϕ^λ. Likewise, it can be proved that

$$\mathcal{R}(0, \phi^\lambda, K, \varepsilon) < \mathcal{R}(0, \phi^\lambda, K, 0) \text{ and } \mathcal{R}(0, \phi^\lambda, \varepsilon, \bar{K}) < \mathcal{R}(0, \phi^\lambda, 0, \bar{K}), \text{ for all } \varepsilon > 0.$$

Figure 2 visualizes the comparisons between $\mathcal{R}(0, 1.25, 2, 0.1)$ and $\mathcal{R}(0, 1.25, 2, 0)$, $\mathcal{R}(0, 1.25, 0.1, 2)$ and $\mathcal{R}(0, 1.25, 0, 2)$ respectively. It shows that even if the

[12] In practice, there is only partial hedge between life insurance and endowment due to the existence of basis risk. In our context, the perfect hedge is employed just for a brief statement of our point.

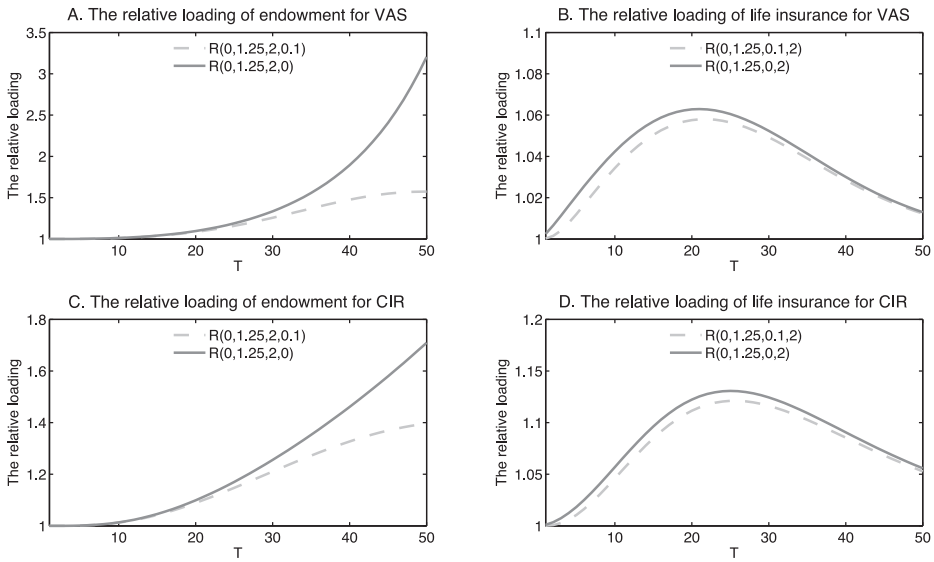

FIGURE 2: $\mathcal{R}(0, 1.25, 2, 0.1)$ v.s. $\mathcal{R}(0, 1.25, 2, 0)$ and $\mathcal{R}(0, 1.25, 0.1, 2)$ v.s. $\mathcal{R}(0, 1.25, 0, 2)$. The parameters used for this figure are the same as those used for Figure 1.

balanced amount ε accounts merely for 5% (=0.1/2), the relative loading can be depressed obviously. In particular, panels A and C in Figure 2 illustrate that $\mathcal{R}(0, 1.25, 2, 0.1)$ is significantly smaller than $\mathcal{R}(0, 1.25, 2, 0)$ when $T > 40$ (years).

Remark 3. This section provides an illustration of how to price the risk of continuous fluctuations with ambiguity aversion. We concede that this example does not accommodate all important factors in the underlying dynamics. However, the above pricing framework is feasible enough to admit one to incorporate some other important elements for insurance pricing. Possible extensions of the pricing formulae in Theorem 1 include:

(I). For the long-term life insurance, the ambiguity with respect to interest rate has become oral tradition among some subsets of practitioners. It is possible to incorporate such ambiguity into the pricing formulae. The outline of this idea is given below. At first, we model the interest rate as

$$dr_t = (a_{0t} + a_{1t} r_t)dt + (b_{0t} + b_{1t} r_t)^{\frac{1}{2}} dB_t^r,$$

where B_t^r is an independent Brownian motion under the reference measure $\mathbb{P}^r$. Following what we have done for (9) and (10), define

$$\frac{d\mathbb{P}^r(\xi^r)}{d\mathbb{P}^r} = \xi_T^r = e^{\int_0^T h_t^r dB_t^r - \frac{1}{2}\int_0^T |h_t^r|^2 dt} \text{ and } I(\xi^r) = \frac{1}{2}\int_t^{t+\Delta} E_t^{\mathbb{P}^r}\left[|h_s^r|^2\right] ds.$$

Then, we introduce a default-free zero-coupon bond that pays \$1 at time T, whose t-time price is given by

$$F(r,t,T) = E_t^{\mathbb{Q}^r(\xi^r)}\Big[\, e^{-\int_t^T r_s\, ds}\,\Big],$$

where $\mathbb{Q}^r(\xi^r)$ is the risk-neutral measure corresponding to $\mathbb{P}^r(\xi^r)$ with an unambiguous loading[13]. The default-free zero-coupon bond is available as an investment instrument. By constructing an appropriate penalty on the deviation from $\mathbb{P}^r$, we can follow Ludkovski and Young (2008) to derive the utilities $U(W,t)$ and $U^L(W,L,t)$. One can expect that with ambiguity aversion to interest rate, the insurer will adjust the model of interest rate to the safety side, which gives rise to a positive risk premium.

(II). As mentioned before, mortality jumps maybe appear. For instance, if there is unexpected breakthrough in the treatment of heart disease, cancer or Aids, a great increase in people's lifetime can be expected. Such improvements in mortality are regarded to be too significant to be continuous, and can be captured by jumps, see Cox et al. (2006). We will gain experience in pricing the risk of rare event (jumps) in the following section. By integrating these techniques, one is able to price the (mortality) risk which is modeled by the diffusion process with jumps.

Remark 4. Various pricing methods have been put forward to price the mortality risk: Milevsky et al. (2005), Bayraktar and Young (2007) and Young (2008) propose a Sharpe ratio rule; Young and Zariphopoulou (2002) apply the principle of equivalent utility; Milevsky and Promislow (2001), Dahl (2004); Biffis (2005), Cairns et al. (2006), etc. use the risk-neutral theory. As far as robustness is concerned, Chen et al. (2010) give a detailed comparison of the above methods. Our pricing formulae in Theorem 1 can be seen as additions to the literature on life insurance pricing by incorporating uncertainty to mortality risk modeling.

4. Pricing the risk of jumps

The frequency of insurance claims is usually modeled by a Poisson process[14]. To exhibit how one can price the risk of jumps with ambiguity aversion, we give the pricing formula for a simple contract in property insurance.

In property insurance market, the insurer provides coverage against potential property losses. The arrival rate of loss events is usually described as a Poisson process in the literature, see for instance Cummins and Geman (1995),

[13] To be specific, we write $\frac{d\mathbb{Q}^r(\xi^r)}{d\mathbb{P}^r(\xi^r)} = e^{-\int_t^T q_s d\hat{B}_s^r - \frac{1}{2}\int_t^T q_s^2\, ds}$, where $\hat{B}_t^r$ given by $dB_t^r = h_t^r dt + d\hat{B}_t^r$ is the Brownian motion under the measure $\mathbb{P}^r(\xi^r)$. The loading q_t is assumed to be an explicit adapted process.

[14] See Klugman et al. (2004) and the references therein.

Young and Zarphipoulou (2002), Cox et al. (2004), etc. Without loss of generality, we assume that the property loss is modeled by

$$dL_t = YdN_t, \quad \text{with} \quad Y = f(Z), \tag{22}$$

where N_t is a Poisson process indicating the occurrence of loss events, Z is a random variable, $f(\cdot)$ is a nonnegative deterministic function, and $Y = f(Z)$ characterizes the loss magnitude. Under the statistically best reference measure $\mathbb{P}^L$, we assume that the jump intensity is κ and the mean of the jump size Y is $y = E[f(Z)]$. The alternative model is defined via its probability measure $P^L(\xi^L)$, where $\xi_T^L = \frac{d\mathbb{P}^L(\xi^L)}{d\mathbb{P}^L}$ is the Radon-Nikodym derivative subject to

$$d\xi_t^L = \left(e^{h_1^L} \times \frac{f(h_2^L Z)}{E[f(h_2^L Z)]} - 1\right)\xi_{t-}^L dN_t - \left(e^{h_1^L} - 1\right)\kappa\xi_t^L dt.$$

Here h_1^L and h_2^L are stochastic processes adapted to the filtration generated by L_t. Under the alternative measure $\mathbb{P}^L(\xi^L)$, the jump intensity and the mean jump size are changed to be

$$\kappa^{\xi^L} = \kappa e^{h_1^L} \quad \text{and} \quad y^{\xi^L} = y \times \frac{E[f(h_2^L Z)f(Z)]}{E[f(h_2^L Z)]\,E[f(Z)]}. \tag{23}$$

It is obvious that $\frac{E[f(h_2^L Z)f(Z)]}{E[f(h_2^L Z)]E[f(Z)]} > 1$ provided f is monotonic and $h_2^L \neq 0$. The formal derivation of (23) can be found in the Appendix. Therefore accepting the alternative probability $\mathbb{P}^L(\xi^L)$ is equivalent to accepting the adjusted model

$$dL_t = f(\hat{Z})d\hat{N}_t, \tag{24}$$

where $\hat{N}_t$ is a Poisson process with the jump intensity κ^{ξ^L} and $\hat{Z}$ is a random variable such that $E[f(\hat{Z})] = y^{\xi^L}$. As in the former section, accepting (24) will incur a penalty. To formulate the penalty, we use E_t, $E_t^{\xi^{S,\lambda}}$, $E_t^{\xi^S}$ and $E_t^{\xi^\lambda}$ to denote the expectation operators under the measures $\mathbb{P}^S \times \mathbb{P}^\lambda$, $\mathbb{P}^S(\xi^S) \times \mathbb{P}^\lambda(\xi^\lambda)$, $\mathbb{P}^S(\xi^S)$ and $\mathbb{P}^\lambda(\xi^\lambda)$ respectively, conditional on the information up to t. Following Liu et al. (2005), we define the index describing the information with respect to $\hat{N}_t$ and $\hat{Z}$ in the time period $[t, t+\Delta]$ as

$$I(\xi^L) = E_t^{\xi^L}\left[H\left(\ln\frac{\xi_{t+\Delta}^L}{\xi_t^L}\right)\right], \quad \text{where } H(x) = x + \beta(e^x - 1) \text{ for some } \beta > 0. \tag{25}$$

Notice that when β approaches zero, the information measure in (25) reduces to be the relative entropy. A positive β in (25) is important in handling ambiguity

aversion toward the jump component, since it can prevent the insurer from going overboard to the extremely bad case of $y^{\xi^L} = \infty$.

For the insurer, with the liability to pay L_t if the property loss occurs and zero otherwise, the utility $U_t^L = U^L(W_t, L_t, t)$ conditional on the information up to t satisfies

$$U_t^L = \sup_{\pi_t} \inf_{\xi^S, \xi^L} \mathcal{F}\Big(E_t^{\xi^{S,L}}[U_{t+\Delta}^L]\Big) \text{ with } U^L(W, L, T) = u(W - L), \tag{26}$$

where

$$\mathcal{F}(x, y) = \phi^S \psi^S(x) I(\xi^S) + \phi^L \psi^L(x) I(\xi^L) + x.$$

In the above, $I(\xi^S)$ and $I(\xi^L)$ are given by (3) and (25) respectively. The constant $\phi^L \geq 0$ is a penalty parameter and the term $\psi^L(\cdot)$ is again a normalization function that converts the penalty to units of utility and its functional form is also chosen for analytical tractability. In the current setup, there is a one-to-one correspondence between the pair of adjustment parameters (h_1^L, h_2^L) and the pair of penalty parameters (ϕ^L, β).

On the continuous-time version of the max-minimization problem (26), after choosing $\psi^S(x) = |x|$ and $\psi^L(x) = \alpha|x|$, Proposition A.4 gives the closed-form solution $U^L(W, L, t) = U(W, t)e^{\alpha L}\eta(t)$. Intuitively, the function $\psi^L(U^L) = \frac{\partial}{\partial L}|U^L| = \alpha|U^L|$ suggests that the penalty of accepting the alternative probability on the loss is proportional to the marginal utility with respect to the loss under that probability. By definition, when losses have never occurred before time t, the price of the property insurance charged by the insurer is

$$\mathcal{P} = \frac{1}{\alpha} e^{-\int_t^T r_s\, ds} \ln \eta(t).$$

More explicitly, we have the following theorem.

Theorem 2. *Assume that the property loss is modeled by the jump process (22). Then the price of the property insurance contract under the equivalent utility principle with ambiguity aversion is given by*

$$\begin{aligned}\mathcal{P}(\alpha, \phi^L, \beta) = {} & \underbrace{\kappa e^{h_1^{L*}}(T-t)E\left[\frac{f(h_2^{L*}Z)}{E[f(h_2^{L*}Z)]}\left(\frac{e^{\alpha f(Z)}-1}{\alpha}\right)\right]e^{-\int_t^T r_s\, ds}}_{\textit{pricing based on a conservative model}} \\ & - \underbrace{\phi^L(T-t)\Upsilon\left(h_1^{L*}, h_2^{L*}\right)e^{-\int_t^T r_s\, ds}}_{\textit{incurring a penalty}},\end{aligned} \tag{27}$$

where h_1^{L*} and h_2^{L*} are the solution of the optimization problem (A.4) and the function $\Upsilon(\cdot,\cdot)$ is defined in Proposition A.4 in the Appendix.

When the insurer exhibits ambiguity aversion to the reference model, the insurer selects the conservative model (24) with the adjustment parameters (h_1^{L*}, h_2^{L*}) so that he will overestimate the potential risk relative to the reference model. This can be observed in the first term of (27). Giving up the statistically best reference model will lead to a penalty on the pricing, which is captured by the second term of (27). When $\phi^L = \infty$ and $\beta = \infty$, the impact of ambiguity aversion vanishes and (27) reduces to be the price driven by risk aversion only:

$$\mathcal{P}(\alpha, \infty, \infty) = \kappa(T-t)E\left[\left(\frac{e^{\alpha f(Z)}-1}{\alpha}\right)\right] e^{-\int_t^T r_s\, ds}. \tag{28}$$

When the insurer is risk-neutral, the effect of risk aversion disappears and (27) becomes

$$\mathcal{P}(0, \phi^L, \beta) = (T-t)\left[\kappa e^{h_1^{L*}} \frac{E[f(h_2^{L*}Z)f(Z)]}{E[f(h_2^{L*}Z)]} - \phi^L \Upsilon(h_1^{L*}, h_2^{L*})\right] e^{-\int_t^T r_s\, ds}, \tag{29}$$

where h_1^{L*} and h_2^{L*} are now the solution to the optimization problem below

$$\sup_{h_1^L, h_2^L}\left[\kappa e^{h_1^L} \frac{E[f(h_2^L Z)f(Z)]}{E[f(h_2^L Z)]} - \phi^L \Upsilon\left(h_1^L, h_2^L\right)\right]. \tag{30}$$

The pricing formula (29) driven by ambiguity aversion seems new in the literature related to property insurance pricing. Remember that the actuarial value of this property insurance policy under the reference model is

$$\mathcal{A} = \kappa(T-t)E[f(Z)]\, e^{-\int_t^T r_s\, ds}.$$

We define

$$\mathcal{R}(\alpha, \phi^L, \beta) = \frac{\mathcal{P}(\alpha, \phi^L, \beta)}{\mathcal{A}}$$

to measure the relative loading charged by the insurer whose risk aversion parameter and ambiguity aversion parameters are α and (ϕ^L, β) respectively. We will compare $\mathcal{R}(\alpha, \infty, \infty)$ with $\mathcal{R}(0, \phi^L, \beta)$ in the sequel.

The exponential family distributions are commonly used for the calibration of property losses, see for example Klugman et al. (2004). To get a broad perception on the pricing formula (29), we display $\mathcal{R}(\alpha, \infty, \infty)$ and $\mathcal{R}(0, \phi^L, \beta)$ by assuming that $f(x) = x$ and Z obeys the exponential distribution in Figure 3. From Panel A, we see that $\mathcal{R}(\alpha, \infty, \infty)$ increases explosively with $E[Z]$. Especially,

when $E[Z] > 1/\alpha$, the pricing formula (28) driven by risk aversion even cannot yield a well-defined (finite) value. This observation shows that the relative loading driven by risk aversion is very sensitive to the loss magnitude. If we evaluate a relative large risky loss by using the risk aversion parameter α which is calibrated well to the price of a modest risky loss, we will encounter an abnormally high price. For example, in Panel A of Figure 3, one sees clearly that $\mathcal{R}(0.1341, \infty, \infty)$ for $E[Z] = 6$ does not exceed 10, but $\mathcal{R}(0.1341, \infty, \infty)$ for $E[Z] = 8$ approaches 70 abruptly. This result suggests that risk aversion might not provide a plausible account of risk preference over modest stakes[15]. One may suspect that our observation relies heavily on the specification of the negative exponential utility function. However, the intuition behind our observation is verified in a quite general expected utility framework. Under the very weak condition that the utility is increasing and strictly concave, Rabin (2000) proves that "within the expected-utility model, anything but virtual risk neutrality over modest stakes implies manifestly unrealistic risk aversion over large stakes"[16].

On the contrary, it seems that ambiguity aversion can help us achieve a reasonable pricing mechanism in the sense that the relative loading should increase moderately with the magnitude for modest losses. In Figure 3, we match the parameters (ϕ^L, β) and α such that the relative loading driven by risk aversion equals the one driven by ambiguity aversion at the point $E[Z] = 0.5$. By comparison of Panel A and Panel B in Figure 3, we see that $\mathcal{R}(0, 1.25, 1)$ increases much slower than $\mathcal{R}(0.1341, \infty, \infty)$. In fact, the first-order condition of the optimization problem (30) in the current setup is

$$\frac{E[Z^2]}{\phi^L E[Z]} = h_1^L + \frac{E[Z \ln Z]}{E[Z]} - \ln E[Z] + 2\beta\left(e^{h_1^L}\frac{E[Z^2]}{E[Z]^2} - 1\right),$$

following which we have

$$h_1^L = \ln E[Z] - \ln(2\beta\phi^L) + o(1) \text{ and } \mathcal{R}(0, \phi^L, \beta) = \frac{1}{2\beta\phi^L} E[Z][1 + o(1)],$$

[15] Though risk aversion may be a useful description of the taste for very-large-scale losses, for modest losses which the insurer attempts to underwrite, such sensitivity of the pricing to the loss magnitude driven by risk aversion seems abnormal. An example from financial markets is given by Rabin (2000), who writes "an expected-utility maximizer with CARA preferences who turns down 50/50 lose \$1,000/gain \$1,200 gambles will only be willing to keep \$8,875 of her portfolio in the stock market, no matter how large her total investments in stocks and bonds. If she turns down a 50/50 lose \$100/gain \$110 bet, she will be willing to keep only \$1,600 of her portfolio in the stock market-keeping the rest in bonds (which average 6% lower annual return). While it is widely believed that investors are too cautious in their investment behavior, no one believes they are this risk averse".

[16] The argument in this paragraph is not to say risk aversion should not be taken into account in explaining the risk premium. It just reminds us to be careful of the use of risk aversion in the pricing of modest losses.

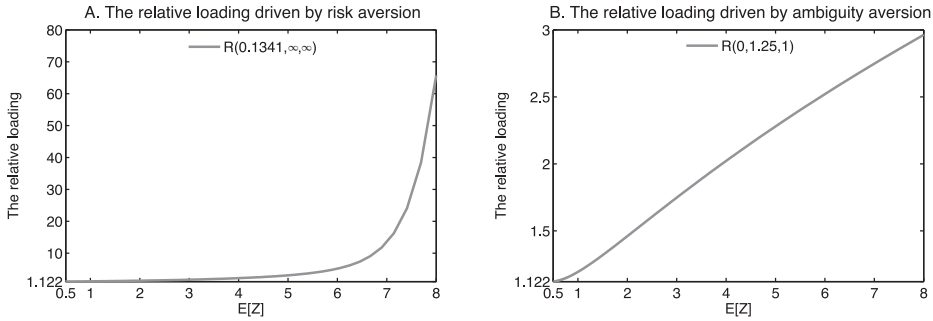

FIGURE 3: $\mathcal{R}(0.1341, \infty, \infty)$ v.s. $\mathcal{R}(0, 1.25, 1)$.
This figure is based on the assumptions that $f(x) = x$ and Z obeys the exponential distribution.

where $o(1)$ is the infinitesimal residual satisfying $\lim_{E[Z]\to\infty} o(1) = 0$. This confirms that ambiguity aversion leads to a linear relative loading of $E[Z]$, which does increase much slower than $\mathcal{R}(\alpha, \infty, \infty)$.

At last, we discuss the possible application of (29) to the pricing of catastrophe insurance. The magnitude of catastrophe loss is widely accepted to be heavy-tailed in the sense that its tail probability is not exponentially bounded. One typical calibration for catastrophe loss is

$$Y = f(Z) = e^{Z}, \text{ where } Z \text{ is a random variable with normal distribution,} \tag{31}$$

see Lee and Yu (2002), Egami and Young (2008), Lin et al. (2009), etc. For catastrophe loss which is modeled by (31), it is notable that the pricing formula (28) driven by risk aversion cannot produce a finite value. However, the pricing formula (29) driven by ambiguity aversion can lead us to a sensible result. Assume the mean and the variance of Z are m and ς^2 respectively, and then

$$\frac{E[f(h_2^L Z)\, f(Z)]}{E[f(h_2^L Z)]} = e^{m + \frac{1}{2}\varsigma^2 + h_2^L \varsigma^2}, \tag{32}$$

$$\Upsilon\left(h_1^L, h_2^L\right) = \kappa\left[1 + \beta + \left(h_1^L + \frac{1}{2}(h_2^L)^2 \varsigma^2 - 1\right) e^{h_1^L} + \beta\left(e^{h_1^L + (h_2^L)^2 \varsigma^2} - 2\right) e^{h_1^L}\right], \tag{33}$$

and (29) turns out to be

$$\mathcal{P} = e^{-\int_t^T r_s\, ds}\,(T - t)\left[\kappa e^{m + \frac{1}{2}\varsigma^2 + h_2^{L*}\varsigma^2 + h_1^{L*}} - \phi^L \Upsilon\left(h_1^{L*}, h_2^{L*}\right)\right],$$

where h_1^{L*} and h_2^{L*} are subject to the first-order condition of the optimization problem (30)

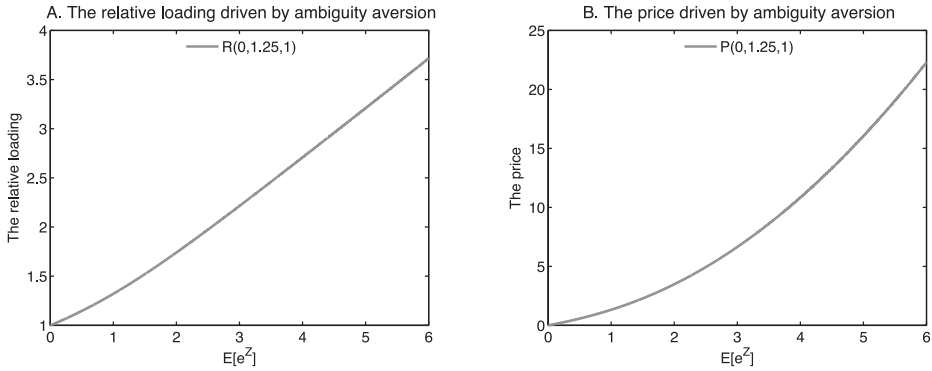

FIGURE 4: $\mathcal{R}(0, 1.25, 1)$ and $\mathcal{P}(0, 1.25, 1)$.
This figure is based on the assumptions that $f(x) = e^x$ and Z obeys the normal distribution with $\varsigma = 1$.

$$\begin{cases} \dfrac{1}{\phi^L}\, e^{m+\frac{1}{2}\varsigma^2+h_2^L\varsigma^2} = h_1^L + \frac{1}{2}(h_2^L)^2\varsigma^2 + 2\beta\left(e^{h_1^L+(h_2^L)^2\varsigma^2} - 1\right), \\ \dfrac{1}{\phi^L}\, e^{m+\frac{1}{2}\varsigma^2+h_2^L\varsigma^2} = h_2^L\left(1 + 2\beta e^{h_1^L+(h_2^L)^2\varsigma^2}\right). \end{cases}$$

We remark that by inspection of (32)-(33) and (30), it is clear that a positive β does prevent the occurrence of the extremely bad case $h_2^{L*} = \infty$. With ς and κ being fixed, when m approaches infinity, it is easy to derive the asymptotics

$$\begin{aligned} h_1^{L*} &= m + \frac{1}{2}\varsigma^2 - \ln\left(2\beta\phi^L\right) + o(1), \\ h_2^{L*} &= 1 + o(1), \\ \mathcal{R}(0, \phi^L, \beta) &= \frac{1}{4\beta\phi^L}\, e^{m+\frac{3}{2}\varsigma^2}[1 + o(1)] = \frac{1}{4\beta\phi^L}\, e^{\varsigma^2} E[e^Z][1 + o(1)]. \end{aligned} \tag{34}$$

where $o(1)$ is the infinitesimal residual satisfying $\lim_{m\to\infty} o(1) = 0$. The equation (34) implies that when the insurer is risk neutral, ambiguity aversion can result in a linear increase of the relative loading with catastrophe losses. We illustrate this tendency in Figure 4. Panel B in Figure 4 exhibits that the linear relative loading will induce a nonlinear (square) increase of $\mathcal{P}(0, \phi^L, \beta)$ with respect to the related actuarial value under the reference model. This result implies that ambiguity aversion can lead to a high premium of catastrophe insurance, which is commonplace in practice.

5. CONCLUSION

This paper develops a feasible framework which incorporates ambiguity aversion into the pricing of insurance products. Under the utility-equivalence principle,

we obtain closed-form pricing formulae for the risk of continuous fluctuations and the risk of rare events.

We compare the effects of ambiguity aversion with those of risk aversion on the pricing. Conceptually, the risk premium driven by risk aversion relies only on the estimated value of the underlying risk reported from a given model, irrespective of which specific model is selected. Two abnormal phenomena may arise from such pricing mechanism. First, for the policy of pure life insurance, the relative loading driven by risk aversion becomes larger as the maturity decreases, which seems counterintuitive. Second, for property insurance, the relative loading driven by risk aversion is very sensitive to the loss magnitude, which seems odd for modest losses. However, the risk premium driven by ambiguity aversion highlights the insurer's selection on the models available. With ambiguity aversion, the insurer will adjust the model parameters to the safety side. The impact of such adjustment on risk pricing accumulates when the time extends or the loss increases, which is reasonable for the pricing of pure life insurance and modest losses.

Our pricing formulae grasp the mechanism that the premium driven by ambiguity aversion can approach infinity as well as can be depressed by the existence of natural hedge. Taking advantage of natural hedge mechanism can help us control the effects of model uncertainty. This provides a theoretic explanation for the empirical findings of Kunreuther et al. (1993) and Cox and Lin (2007).

We concede that some quantitative results of this paper depend on the specific formulation of the penalty functions, however, the intuition behind our results is quite general. Our work sheds light on the effects of ambiguity aversion on insurance pricing from the perspective of insurance supply. Further empirical investigation of the pricing with ambiguity, including the calibration of our pricing model, would be very useful.

Appendix

A. Technical propositions

Define two differential operators

$$\mathcal{D}^{\pi_t, h_t^S} = r_t W \frac{\partial}{\partial W} + (\mu_t - r_t)\pi_t \frac{\partial}{\partial W} + h_t^S \sigma_t \pi_t \frac{\partial}{\partial W} + \frac{1}{2}\sigma_t^2 \pi_t^2 \frac{\partial^2}{\partial W^2},$$

$$\mathcal{D}^{h_t^\lambda} = (k_{0t} + k_{1t}\lambda_t)\frac{\partial}{\partial \lambda} + h_t^\lambda (v_{0t} + v_{1t}\lambda_t)^{\frac{1}{2}} \frac{\partial}{\partial \lambda} + \frac{1}{2}(v_{0t} + v_{1t}\lambda_t)\frac{\partial^2}{\partial \lambda^2}.$$

In the following, we always choose $\psi^S(x) = |x|$ and $\psi^\lambda(x, y) = |x - y|$.

Proposition A.1. *The HJB equation for the max-minimization problem (5) is*

$$0 = \sup_{\pi_t} \inf_{h_t^S} \left[\frac{\partial}{\partial t} U + \mathcal{D}^{\pi_t, h_t^S} U + \frac{1}{2}\phi^S \psi^S(U)(h_t^S)^2 \right], \quad U(W, T) = u(W). \quad \text{(A.1)}$$

The closed-form solution of U is $U(W,t) = -\frac{1}{\alpha} e^{-a_t W - b_t}$, where a_t and b_t are in (B.2).

Proposition A.2. *For the affine process (9), under the reference probability $\mathbb{P}^\lambda$, we have ${}_{T-t}p_{x+t} = E_t[e^{-\int_t^T \lambda_s ds}] = e^{a_t \lambda_t + b_t}$, where a_t and b_t are determined by (B.4)-(B.5).*

Proposition A.3. *The HJB equation for the max-minimization problem (11) is*

$$0 = \sup_{\pi_t} \inf_{h_t^S, h_t^\lambda} \Big[\frac{\partial}{\partial t} U^L + \mathcal{D}^{\pi_t, h_t^S} U^L + \mathcal{D}^{h_t^\lambda} U^L + \lambda_t(\bar{U} - U^L) + \frac{1}{2}\phi^S \psi^S(U^L)(h_t^S)^2 + \frac{1}{2}\phi^\lambda \psi^\lambda(U^L, \bar{U})(h_t^\lambda)^2 \Big], \quad U^L(W, \lambda, T) = u(W - K). \tag{A.2}$$

The closed-form solution of U^L is $U^L(W,\lambda,t) = U(W,t)\,\eta(\lambda,t)$, where $U(W,t)$ is as in Proposition A.1 and

$$\eta(\lambda,t) = \begin{cases} e^{\alpha \bar{K}} + e^{a_t \lambda + b_t}, & \text{if } \bar{K} < K, \text{where } a_t \text{ and } b_t \text{ are in (B.8)-(B.9)}; \\ e^{\alpha K}, & \text{if } \bar{K} = K; \\ e^{\alpha \bar{K}} - e^{a_t \lambda + b_t}, & \text{if } \bar{K} > K, \text{where } a_t \text{ and } b_t \text{ are in (B.10)-(B.11)}. \end{cases}$$

Proposition A.4. *The HJB equation for the max-minimization problem (26) is*

$$0 = \sup_{\pi_t} \inf_{h_t^S, h_1^L, h_2^L} \Big[\frac{\partial}{\partial t} U^L + \mathcal{D}^{\pi_t, h_t^S} U^L + \kappa e^{h_1^L} E_t^{\xi^L}\big[U^L(W, L + f(Z), t) - U^L(W, L, t)\big] + \frac{1}{2}\phi^S \psi^S(U^L)(h_t^S)^2 + \phi^L \psi^L(U^L)\,\Upsilon(h_1^L, h_2^L) \Big] \tag{A.3}$$

with the terminal condition $U^L(W,L,T) = u(W - L)$. In the above,

$$\Upsilon(h_1^L, h_2^L) = \kappa(1+\beta) + \kappa e^{h_1^L}\left(h_1^L + E\left[\frac{f(h_2^L Z)}{E[f(h_2^L Z)]} \ln\left(\frac{f(h_2^L Z)}{E[f(h_2^L Z)]} \right) \right] - 1 \right) + \beta \kappa e^{h_1^L}\left(e^{h_1^L} \frac{E[f(h_2^L Z)^2]}{E[f(h_2^L Z)]^2} - 2 \right).$$

We introduce $\Lambda(h_1^L, h_2^L) = \kappa e^{h_1^L} E\left[\frac{f(h_2^L Z)}{E[f(h_2^L Z)]} \left(\frac{e^{\alpha f(Z)} - 1}{\alpha} \right) \right] - \phi^L \Upsilon(h_1^L, h_2^L)$ and denote

$$\Lambda(h_1^{L^*}, h_2^{L^*}) = \sup_{h_1^L, h_2^L} \Lambda(h_1^L, h_2^L). \tag{A.4}$$

Then the closed-form solution of U^L *is* $U^L(W,L,t) = U(W,t)e^{\alpha L}\eta(t)$, *where* $U(W,t)$ *is as in Proposition A.1 and* $\eta(t) = e^{\alpha\Lambda(h_1^{L*}, h_2^{L*})(T-t)}$.

B. Detailed proofs

The proof of Proposition A.1. The formal derivation of the HJB equation (A.1) is standard. In fact, rewrite (5) as

$$0 = \sup_{\pi_t} \inf_{\xi^S} \left[\phi^S \psi^S \left(E_t^{\xi^S}[U_{t+\Delta}] \right) \frac{I(\xi^S)}{\Delta} + \frac{1}{\Delta} \left(E_t^{\xi^S}[U_{t+\Delta}] - U_t \right) \right].$$

Letting $\Delta \to 0$, using the facts

$$\lim_{\Delta\to 0} \frac{I(\xi^S)}{\Delta} = \frac{1}{2}(h_t^S)^2 \quad \text{and} \quad \lim_{\Delta\to 0} \frac{1}{\Delta}\left(E_t^{\xi^S}[U_{t+\Delta}] - U_t \right) = \frac{\partial}{\partial t} U + \mathcal{D}^{\pi_t, h_t^S} U,$$

we can easily arrive at (A.1). Under the negative exponential utility, the indirect utility U is negative. If we fit the solution $U(W,t) = -\frac{1}{\alpha} e^{-a_t W - b_t}$, then we have

$$\frac{\partial}{\partial t} U = -U(Wa_t' + b_t'), \ \frac{\partial}{\partial W} U = -Ua_t, \ \frac{\partial^2}{\partial W^2} U = Ua_t^2,$$

and the HJB equation (A.1) turns out to be

$$0 = \sup_{\pi_t} \inf_{h_t^S} \left[(a_t' + r_t a_t) W + b_t' + \pi_t(\mu_t - r_t) a_t + \pi_t \sigma_t h_t^S a_t - \frac{\pi_t^2}{2} \sigma_t^2 a_t^2 \right.$$
$$\left. - \frac{\phi^S}{2U} \psi^S(U)(h_t^S)^2 \right].$$

When we choose $\psi^S(U) = |U|$, the optimal choices of h_t^S and π_t are

$$h_t^{S*} = -\frac{1}{\phi^S} \pi_t \sigma_t a_t, \quad \text{and} \quad \pi_t^* = \frac{1}{a_t} \left(\frac{\mu_t - r_t}{\sigma_t^2} \right) \left(\frac{\phi^S}{1 + \phi^S} \right). \tag{B.1}$$

After inserting (B.1) into the above HJB equation, we are led to the ordinary differential equations (ODE's)

$$a_t' + r_t a_t = 0 \quad \text{and} \quad b_t' + \frac{1}{2} \left(\frac{\phi^S}{1 + \phi^S} \right) \left(\frac{\mu_t - r_t}{\sigma_t} \right)^2 = 0$$

with $a_T = \alpha$ and $b_T = 0$. It is then easy to obtain

$$a_t = \alpha e^{\int_t^T r_s\,ds} \quad \text{and} \quad b_t = \frac{1}{2}\left(\frac{\phi^S}{1+\phi^S}\right)\int_t^T \left(\frac{\mu_s - r_s}{\sigma_s}\right)^2 ds. \tag{B.2}$$

The expressions of a_t and b_t together with (B.1) give (6) and (7). □

The proof of Proposition A.2. Write ${}_{T-t}p_{x+t} = G(\lambda_t, t)$ and then $e^{-\int_0^t \lambda_s\,ds}\,G(\lambda_t, t)$ is a martingale under $\mathbb{P}^\lambda$. By Itô formula, the drift of $e^{-\int_0^t \lambda_s\,ds}\,G(\lambda_t, t)$ must vanish, i.e.,

$$0 = \frac{\partial}{\partial t}G - \lambda G + (k_{0t} + k_{1t}\lambda)\frac{\partial}{\partial \lambda}G + \frac{1}{2}(v_{0t} + v_{1t}\lambda)\frac{\partial^2}{\partial \lambda^2}G. \tag{B.3}$$

We fit the solution $G(\lambda, t) = e^{a_t\lambda + b_t}$ into (B.3) and obtain

$$0 = \left(a_t' - 1 + k_{1t}a_t + \frac{1}{2}v_{1t}a_t^2\right)\lambda + \left(b_t' + k_{0t}a_t + \frac{1}{2}v_{0t}a_t^2\right),$$

which is equivalent to

$$0 = a_t' - 1 + k_{1t}a_t + \frac{1}{2}v_{1t}a_t^2, \quad a_T = 0, \tag{B.4}$$

$$0 = b_t' + k_{0t}a_t + \frac{1}{2}v_{0t}a_t^2, \quad b_T = 0. \tag{B.5}$$

The ODE (B.4) is of the Riccati type and (B.5) can be solved directly once we know a_t. □

The proof of Proposition A.3. The formal derivation of the HJB equation (A.2) is standard. In fact, rewrite (11) as

$$\begin{aligned} 0 = \sup_{\pi_t}\inf_{\xi^S,\xi^\lambda}\Bigg[& \phi^S\psi^S\left(E_t^{\xi^S,\lambda}[U_{t+\Delta}^L]\right)\frac{I(\xi^S)}{\Delta} + \frac{1}{\Delta}\left(E_t^{\xi^S,\lambda}[U_{t+\Delta}^L] - U_t^L\right)\Bigg]\,{}_\Delta p_{x+t} \\ & + \left[\phi^S\psi^S\left(E_t^{\xi^S}[U_{t+\Delta}]\right)I(\xi^S) + \left(E_t^{\xi^S}[\bar{U}_{t+\Delta}] - U_t^L\right)\right]\frac{{}_\Delta q_{x+t}}{\Delta} \\ & + \phi^\lambda\psi^\lambda\left(E_t^{\xi^S,\lambda}[U_{t+\Delta}^L], E_t^{\xi^S}[\bar{U}_{t+\Delta}]\right)\frac{I(\xi^\lambda)}{\Delta}. \end{aligned}$$

Letting $\Delta \to 0$, using the facts

$$\lim_{\Delta\to 0}\frac{I(\xi^S)}{\Delta} = \frac{1}{2}(h_t^S)^2, \quad \lim_{\Delta\to 0}\frac{I(\xi^\lambda)}{\Delta} = \frac{1}{2}(h_t^\lambda)^2, \quad \lim_{\Delta\to 0}\frac{{}_\Delta q_{x+t}}{\Delta} = \lambda_t,$$

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\left(E_t^{\xi^S,\lambda}[U_{t+\Delta}^L] - U_t^L\right) = \frac{\partial}{\partial t}U^L + \mathcal{D}^{\pi_t, h_t^S}U^L + \mathcal{D}^{h_t^\lambda}U^L,$$

we can easily arrive at (A.2). If we choose $\psi^S(U) = |U|$, $\psi^\lambda(U^L, \bar{U}) = |U^L - \bar{U}|$ and fit the solution $U^L(W, \lambda, t) = U(W, t)\,\eta(\lambda, t)$, the optimal choice of h_t^λ is

$$h_t^{\lambda *} = -\frac{\eta_\lambda}{\phi^\lambda \left|\eta - e^{\alpha \bar{K}}\right|}\,(v_{0t} + v_{1t}\lambda_t)^{\frac{1}{2}}. \tag{B.6}$$

Inserting (B.6) into (A.2), we find that η must satisfy

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\eta + \lambda\left(e^{\alpha\bar{K}} - \eta\right) + (k_{0t} + k_{1t}\lambda)\,\frac{\partial}{\partial \lambda}\eta + \frac{1}{2}(v_{0t} + v_{1t}\lambda)\,\frac{\partial^2}{\partial \lambda^2}\eta \\ &\quad + \frac{(v_{0t} + v_{1t}\lambda)}{2\phi^\lambda\left|\eta - e^{\alpha K}\right|}\left(\frac{\partial}{\partial \lambda}\eta\right)^2, \ \eta(\lambda, T) = e^{\alpha K}. \end{aligned} \tag{B.7}$$

To solve (B.7), we differentiate three cases. Case 1 is $\bar{K} < K$. In this case, we assume $\eta(\lambda, t) = e^{\alpha\bar{K}} + e^{a_t\lambda + b_t}$. It follows from (B.7) that

$$0 = \left[a_t' - 1 + k_{1t}a_t + \frac{1}{2}\left(\frac{\phi^\lambda + 1}{\phi^\lambda}\right)v_{1t}a_t^2\right]\lambda + \left[b_t' + k_{0t}a_t + \frac{1}{2}\left(\frac{\phi^\lambda + 1}{\phi^\lambda}\right)v_{0t}a_t^2\right].$$

We solve the ODE's

$$0 = a_t' - 1 + k_{1t}a_t + \frac{1}{2}\left(\frac{\phi^\lambda + 1}{\phi^\lambda}\right)v_{1t}a_t^2, \ a_T = 0, \tag{B.8}$$

$$0 = b_t' + k_{0t}a_t + \frac{1}{2}\left(\frac{\phi^\lambda + 1}{\phi^\lambda}\right)v_{0t}a_t^2, \ \ b_T = \ln\left(e^{\alpha K} - e^{\alpha\bar{K}}\right), \tag{B.9}$$

and then obtain the desired results. Case 2 is $\bar{K} > K$. In this case, we assume $\eta(\lambda, t) = e^{\alpha\bar{K}} - e^{a_t\lambda + b_t}$. It follows from (B.7) that

$$0 = \left[a_t' - 1 + k_{1t}a_t + \frac{1}{2}\left(\frac{\phi^\lambda - 1}{\phi^\lambda}\right)v_{1t}a_t^2\right]\lambda + \left[b_t' + k_{0t}a_t + \frac{1}{2}\left(\frac{\phi^\lambda - 1}{\phi^\lambda}\right)v_{0t}a_t^2\right].$$

We solve the ODE's

$$0 = a_t' - 1 + k_{1t}a_t + \frac{1}{2}\left(\frac{\phi^\lambda - 1}{\phi^\lambda}\right)v_{1t}a_t^2, \ \ a_T = 0, \tag{B.10}$$

$$0 = b_t' + k_{0t}a_t + \frac{1}{2}\left(\frac{\phi^\lambda - 1}{\phi^\lambda}\right)v_{0t}a_t^2, \ \ b_T = \ln\left(e^{\alpha\bar{K}} - e^{\alpha K}\right), \tag{B.11}$$

and then obtain the desired results. Case 3 is $\bar{K} = K$, which can be approximated by the case $\bar{K} > K$ and the case $\bar{K} < K$. □

The proof of Theorem 1. By comparison of (B.4)-(B.5) with (B.8)-(B.9) and (B.10)-(B.11), it is easy to find that $\eta(\lambda, t) = e^{\alpha\bar{K}} + (e^{\alpha K} - e^{\alpha\bar{K}})_{T-t}\tilde{p}^{+}_{x+t}$ when $\bar{K} < K$ and $\eta(\lambda, t) = e^{\alpha\bar{K}} - (e^{\alpha\bar{K}} - e^{\alpha K})_{T-t}\tilde{p}^{-}_{x+t}$ when $\bar{K} > K$. Hence (15) follows directly from (12). For the VAS process, we solve the ODE's (B.4)-(B.5) and obtain

$$a_t = -\frac{1}{\gamma}\left[1 - e^{-\gamma(T-t)}\right] \text{ and } b_t = \gamma\int_t^T m_s a_s\, ds + \frac{\varsigma^2}{2}\int_t^T a_s^2\, ds.$$

Equation (16) follows from (B.6) and the expression of a_t. For the CIR process, we solve the ODE's (B.4)-(B.5) and get

$$a_t = -\frac{2\left[e^{\kappa_1(T-t)} - 1\right]}{(\gamma + \kappa_1)\left[e^{\kappa_1(T-t)} - 1\right] + 2\kappa_1}, \quad \kappa_1 = \sqrt{\gamma^2 + 2\varsigma^2}, \quad b_t = \gamma\int_t^T m_s a_s\, ds.$$

Equation (17) follows from (B.6) and the expression of a_t. □

The proof of Proposition A.4. The formal derivation of the HJB equation (A.3) is standard. In fact, rewrite (26) as

$$\begin{aligned} 0 = \sup_{\pi_t} \inf_{\xi^S, \xi^L} &\left[\phi^S \psi^S\left(E_t^{\xi^{S,L}}[U^L_{t+\Delta}]\right)\frac{I(\xi^S)}{\Delta} + \phi^L \psi^L\left(E_t^{\xi^{S,L}}[U^L_{t+\Delta}]\right)\frac{I(\xi^L)}{\Delta}\right] \\ &+ \frac{1}{\Delta}\left(E_t^{\xi^{S,L}}[U^L_{t+\Delta}] - U^L_t\right). \end{aligned}$$

Letting $\Delta \to 0$, we have $\lim_{\Delta\to 0}\frac{I(\xi^S)}{\Delta} = \frac{1}{2}(h_t^S)^2$ and

$$\begin{aligned} \lim_{\Delta\to 0}\frac{1}{\Delta}\left(E_t^{\xi^{S,L}}[U^L_{t+\Delta}] - U^L_t\right) = &\frac{\partial}{\partial t}U^L + \kappa e^{h_1^L} E_t^{\xi^L}\left[U^L(W, L+Y, t) - U^L(W, L, t)\right] \\ &+ \mathcal{D}^{\pi_t, h_t^S} U^L. \end{aligned}$$

To obtain $\lim_{\Delta\to 0}\frac{I(\xi^L)}{\Delta}$, we restructure $I(\xi^L)$ as

$$I(\xi^L) = \frac{1}{\xi_t^L}E_t\left[\xi^L_{t+\Delta}\ln\xi^L_{t+\Delta} - \xi^L_t\ln\xi^L_t\right] + \frac{\beta}{(\xi_t^L)^2}E_t\left[(\xi^L_{t+\Delta})^2 - (\xi^L_t)^2\right].$$

Applying Itô lemma for jump-diffusion to $\xi_t^L\ln\xi_t^L$ and $(\xi_t^L)^2$, we achieve

$$\lim_{\Delta\to 0}\frac{1}{\Delta\xi_t^L}E_t\Big[\xi_{t+\Delta}^L\ln\xi_{t+\Delta}^L-\xi_t^L\ln\xi_t^L\Big]$$
$$=\kappa\left[1+e^{h_1^L}\left(h_1^L+E\left[\frac{f(h_2^LZ)}{E[f(h_2^LZ)]}\ln\left(\frac{f(h_2^LZ)}{E[f(h_2^LZ)]}\right)\right]-1\right)\right],$$
$$\lim_{\Delta\to 0}\frac{1}{\Delta(\xi_t^L)^2}E_t\Big[(\xi_{t+\Delta}^L)^2-(\xi_t^L)^2\Big]$$
$$=\kappa\left[1+e^{h_1^L}\left(e^{h_1^L}\frac{E[f(h_2^LZ)^2]}{E[f(h_2^LZ)]^2}-2\right)\right],$$

which yield $\lim_{\Delta\to 0}\frac{I(\xi^L)}{\Delta}=\Upsilon(h_1^L,h_2^L)$. Based on the above limits, we can easily arrive at (A.2). We choose $\psi^S(U^L)=|U^L|$, $\psi^L(U^L)=\frac{\partial}{\partial L}U^L=\alpha\psi^L(U^L)$, fit the solution $U^L(W,L,t)=U(W,t)e^{\alpha L}\eta(t)$ and get

$$\eta_t'+\alpha\sup_{h_1^L,h_2^L}\left(\kappa e^{h_1^L}E^{\xi^L}\left[\frac{e^{\alpha Y}-1}{\alpha}\right]-\phi^L\Upsilon\Big(h_1^L,h_2^L\Big)\right)\eta_t=0,\quad \eta_T=1.$$

The proof is concluded with the observation $E^{\xi^L}\left[\frac{e^{\alpha Y}-1}{\alpha}\right]=E\left[\frac{f(h_2^LZ)}{E[f(h_2^LZ)]}\left(\frac{e^{\alpha f(Z)}-1}{\alpha}\right)\right]$.

□

The derivation of (23). Applying the Girsanov transform for point process to the compensated Poisson process $M_t=N_t-\kappa t$, we have under $\mathbb{P}(\xi^L)$ that the process $\hat{M}_t$ satisfying

$$d\hat{M}_t=dM_t-\kappa E\left[e^{h_1^L}\frac{f(h_2^LZ)}{E[f(h_2^LZ)]}-1\right]dt=dN_t-\kappa e^{h_1^L}dt$$

is a martingale. Hence the intensity of N_t under $\mathbb{P}(\xi^L)$ is $\kappa e^{h_1^L}$. Next, applying the Girsanov transform to $M_t=L_t-\kappa E[f(Z)]dt$, we have under $\mathbb{P}(\xi^L)$ that the process $\hat{M}_t$ satisfying

$$d\hat{M}_t=dM_t-\kappa E\left[\left(e^{h_1^L}\frac{f(h_2^LZ)}{E[f(h_2^LZ)]}-1\right)f(Z)\right]dt$$
$$=dL_t-\kappa e^{h_1^L}\frac{E[f(h_2^LZ)f(Z)]}{E[f(h_2^LZ)]}dt$$

is a martingale. Hence the intensity of L_t under $\mathbb{P}(\xi^L)$ is $\kappa e^{h_1^L}\frac{E[f(h_2^LZ)f(Z)]}{E[f(h_2^LZ)]}$.

□

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