MEASURING COMONOTONICITY IN M-DIMENSIONAL VECTORS

BY

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ABSTRACT

In this contribution, a new measure of comonotonicity for m-dimensional vectors is introduced, with values between zero, representing the independent situation, and one, reflecting a completely comonotonic situation. The main characteristics of this coefficient are examined, and the relations with common dependence measures are analysed. A sample-based version of the comonotonicity coefficient is also derived. Special attention is paid to the explanation of the accuracy of the convex order bound method of Goovaerts, Dhaene et al. in the case of cash flows with Gaussian discounting processes.

KEYWORDS

Comonotonicity, dependence, correlation, concordance, copula, multivariate.

1. INTRODUCTION

One of the recurring issues in applications of multivariate systems in the literature, deals with the dependence in a multivariate system. In the first place it is of course necessary to understand, to quantify and to analyse dependence structures, but for many applications an important question is actually how to manage systems with complex dependence structures.

Broadly speaking, we can distinguish two classes of answers to this question in the current research. Firstly, there is an increasing interest in modelling the real or unknown dependence structure of multivariate vectors, by means of copulas e.g. These models can then be used as starting points toward calculations and estimations for quantities immediately related to the underlying dependence structure. Secondly, another group of researchers make use of extreme types of dependencies so as to derive approximate results for related quantities, without specifying the maybe complex real underlying dependence structure. These two classes of methods both benefit of course from research on the analysis of dependence structures and on dependence measures, see e.g. Lehmann (1966), Schweizer & Wolff (1981), Wolff (1980), Drouet Mari & Kotz (2001), Scarsini (1984), Joe (1990, 1993, 1997), and many others.

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In the financial and actuarial research e.g., both approaches are popular and widespread, each of them of course with its own advantages and disadvantages. On the one hand we have the copula models, most of the time resulting in a rather complete description of the dependence structures under investigation. However, next to the fact that this method can be quite time consuming, one of the difficult problems is how to determine the optimal copula among all alternatives. On the other hand, we can use one of the approximation methods. Such a method does not lead to a complete picture of the dependence structure, but it creates the possibility to derive upper and lower bounds, often by means of easy calculations.

An approximation technique which is frequently used in the current financial and actuarial research, is based on a method of Kaas et al. (2000, 2002) and Dhaene et al. (2002a,b). These authors define bounds in convex order sense, calculated by means of comonotonic vectors. If this technique is applied e.g. to present values of cash flows with stochastic interest rates, it turns out that the approximate results are very accurate. In fact this is not completely unexpected, since compounded rates of return for successive periods are strongly mutually dependent and thus not far away from comonotonicity.

This brings us to the question whether it is possible to measure the degree of comonotonicity in an arbitrary multivariate vector. With such a measure it is possible to predict whether or not an approximation as suggested by Goovaerts, Dhaene et al. would be reasonable for a given vector. If so, it can also be used to quantify how reasonable such an approximation would be.

In this paper, we show how such a measure can be constructed and calculated for arbitrary m-dimensional vectors, and we will use our comonotonicity coefficient to explain why in the case of cash flows the convex bounds reveal such effective approximations. We also present an alternative definition of this coefficient in case the dependence structure can be described by a copula, which makes it possible to link the approximations of the second approach to the modelling techniques of the first approach. Finally, we demonstrate how the comonotonicity coefficient can be estimated based on real data.

In order not to complicate the formulae and the calculations, throughout this paper we will work in a continuous setting. However, an extension to the discrete case is certainly possible.

The paper is organised as follows. In section 2 we recall some definitions and notions on distributions and copulas, and we repeat the most common bivariate dependence measures. Section 3 is the most important part, we introduce our comonotonicity coefficient and we discuss its major properties. Afterwards in section 4, we present several numerical illustrations, including the situation of a cash flow with a Gaussian discounting process. Finally, in section 5 we derive a sample based version of the comonotonicity coefficient, and we present an illustration on empirical data. Conclusions can be found in section 6.

2. Preliminaries

The first topic we briefly describe in this section is the Fréchet space, which is an essential concept when investigating the dependence structure in multivariate vectors.

Definition 1. (Fréchet space) *Let $F_1, \ldots, F_m$ be arbitrary univariate distribution functions. The Fréchet space $\mathcal{R}_m(F_1, \ldots, F_m)$ consists of all the m-dimensional (distribution functions $F_{\boldsymbol{X}}$ of) random vectors $\boldsymbol{X} = (X_1, \ldots, X_m)$ having $F_1, \ldots, F_m$ as marginal distribution functions, or $F_i(x) = \mathrm{Prob}(X_i \leq x)$ for $x \in \mathbb{R}$ and $i = 1, \ldots, m$. If it is clear from the context, we will use the short notation $\mathcal{R}_m$.*

For an unambiguous definition of our comonotonicity coefficient, we will have to restrict ourselves to positive dependent vectors; we will come back to this issue in section 3. A weak form of positive dependence in a multivariate context is *positive lower orthant dependence*.

Definition 2. (PLOD) *The random vector $\boldsymbol{X} = (X_1, \ldots, X_m)$ is said to be positive lower orthant dependent (PLOD) if* $\mathrm{Prob}(X_1 \leq x_1, \ldots, X_m \leq x_m) \geq \prod_{i=1}^{m} \mathrm{Prob}(X_i \leq x_i)$, $\forall (x_1, \ldots, x_m) \in \mathbb{R}^m$.

Starting from this definition we define the qualified Fréchet space as the subset of the Fréchet space with all positive lower orthant dependent vectors.

Definition 3. (qualified Fréchet space) *Let $F_1, \ldots, F_m$ be arbitrary univariate distribution functions. The qualified Fréchet space $\mathcal{R}_m^+(F_1, \ldots, F_m)$ consists of all the m-dimensional (distribution functions $F_{\boldsymbol{X}}$ of) random vectors $\boldsymbol{X} = (X_1, \ldots, X_m)$ having $F_1, \ldots, F_m$ as marginal distribution functions and which are PLOD.*
If it is clear from the context, we will use the short notation $\mathcal{R}_m^+$.

Two important and at the same time extreme elements of a Fréchet space are the comonotonic and the independent vector.

Definition 4. (Comonotonic vector and independent vector) *For every Fréchet space $\mathcal{R}_m(F_1, \ldots, F_m)$, we define the independent vector $\boldsymbol{X}^{\boldsymbol{I}} = (X_1^I, \ldots, X_m^I)$ as the vector with distribution $F_{\boldsymbol{X}}^I(x_1, \ldots, x_m) = \prod_{i=1}^{m} F_i(x_i)$, and the comonotonic vector $\boldsymbol{X}^C = (X_1^C, \ldots, X_m^C)$ as the vector with distribution $F_{\boldsymbol{X}}^C(x_1, \ldots, x_m) = \min_{i=1}^{m} F_i(x_i)$. Both vectors belong to the qualified Fréchet space.*

In Denneberg (1994) and Dhaene et al. (2002a), an alternative characterisation is presented for comonotonicity, by showing that a vector $\boldsymbol{X} = (X_1, \ldots, X_m)$ is comonotonic if for non-decreasing functions $t_1, \ldots, t_m$ and for a random variable Z it is true that $\boldsymbol{X} =_d (t_1(Z), \ldots, t_m(Z))$. Note that with the inverse function defined as $F_i^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_i(x) \geq p\}$, $p \in [0,1]$, the comonotonic vector can be easily constructed as $\boldsymbol{X}^C = (F_1^{-1}(U), \ldots, F_m^{-1}(U))$, with U a random variable, uniformly distributed on [0,1].

For the construction of our comonotonicity coefficient, we will rely on the following lemma.

Lemma 1. (Bounds for the Fréchet space) *For any vector $\boldsymbol{X} = (X_1, \ldots, X_m)$ with distribution $F_{\boldsymbol{X}}$ of a given qualified Fréchet space $\mathcal{R}_m^+$, it is true that $F_{\boldsymbol{X}}^I \leq F_{\boldsymbol{X}} \leq F_{\boldsymbol{X}}^C$. For arbitrary vectors of the larger Fréchet space $\mathcal{R}_m$, only the upper bound holds.*

In recent financial research, copulas have grown into a very popular tool for dependence modelling issues. In its essence, a copula is nothing else than a multivariate distribution function with uniform marginal distributions. Or, alternatively, it is the result of the multivariate distribution after rescaling the marginals to uniform distributions. As such, copulas are mostly seen as that part of the joint distribution which captures the dependence structure. In this section, we only provide some of the most important facts on copulas. For a more comprehensive study, we refer to Nelsen (2006) and Cherubini et al. (2004).

Definition 5. (Copula) *An m-dimensional copula C is a function $C : [0,1]^m \to [0,1]$, non-decreasing and right-continuous, with the following properties:*

- $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_m) = 0$
- $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$
- $\Delta_{a_1,b_1}\Delta_{a_2,b_2} \ldots \Delta_{a_m,b_m} C(u_1, \ldots, u_m) \geq 0$ *if* $(a_1, \ldots, a_m) < (b_1, \ldots, b_m) \in [0,1]^m$, *where the differences are defined as*
  $$\Delta_{a_i,b_i}C(u_1, \ldots, u_m) = C(u_1, \ldots, u_{i-1}, b_i, u_{i+1}, \ldots, u_m) - C(u_1, \ldots, u_{i-1}, a_i, u_{i+1}, \ldots, u_m),$$

and this for all $(u_1, \ldots, u_m) \in [0,1]^m$, and for all $i = 1, \ldots, m$.

One of the most fundamental results about copulas is summarised in the following theorem.

Theorem 1 (Sklar's Theorem) *For any m-dimensional distribution function H with marginals $F_1, \ldots, F_m$, there exists an m-dimensional copula C for which $H(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m))$. Conversely, this copula C can be calculated as $C(u_1, \ldots, u_m) = H(F_1^{-1}(u_1), \ldots, F_m^{-1}(u_m))$.*

Remark that the copula C of Theorem 1 is unique if the functions $F_1, \ldots, F_m$ are all continuous; if not, C is only uniquely determined on $\operatorname{Ran} F_1 \times \ldots \times \operatorname{Ran} F_m$.

Note that we don't need copula functions for the definition of our comonotonicity coefficient in the next section, but we will use them for illustration purposes, and also in order to compare our (multivariate) dependence measure with the commonly used (bivariate) measures. In particular, we will use two extreme copulas and three Archimedean copula families, which are commonly used in financial applications:

- Independent copula

$$C^I(\boldsymbol{u}) = \prod_{i=1}^{m} u_i$$

– Comonotonic copula

$$C^C(\boldsymbol{u}) = \min_{i=1}^{m} u_i$$

– Clayton copula family (dependence is concentrated in the lower tails)

$$C_\alpha(\boldsymbol{u}) = \left(\sum_{i=1}^{m} u_i^{-\alpha} - n + 1\right)^{-\frac{1}{\alpha}}, \ \alpha > 0$$

– Gumbel copula family (dependence is concentrated in the upper tails)

$$C_\alpha(\boldsymbol{u}) = \exp\left\{-\left[\sum_{i=1}^{m}\left(-\ln(u_i)^\alpha\right)\right]^{\frac{1}{\alpha}}\right\}, \ \alpha \geq 1$$

– Frank copula family (symmetry between the dependence in lower and upper tails)

$$C_\alpha(\boldsymbol{u}) = -\frac{1}{\alpha}\ln\left(1 + \frac{\prod_{i=1}^{m}\left(e^{-\alpha u_i} - 1\right)}{(e^{-\alpha} - 1)^{m-1}}\right), \ \alpha \in \mathbb{R}$$

with $\boldsymbol{u} = (u_1, \ldots, u_m) \in [0,1]^m$.

A copula family $\{C_\alpha : \alpha \in \mathcal{A}\}$ is called positively ordered, if for all $\boldsymbol{u} \in [0,1]^m$ and for all $\alpha_1 \leq \alpha_2 \in \mathcal{A}$ it is true that $C_{\alpha_1}(\boldsymbol{u}) \leq C_{\alpha_2}(\boldsymbol{u})$.

Finally, we recall here the most common bivariate dependence measures, written by means of copulas.

– Pearson correlation coefficient

$$r_P(X_1, X_2) = \frac{1}{\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)}}\int_0^1\int_0^1 (C(u_1, u_2) - u_1 u_2)\, dF_1^{-1}(u_1) dF_2^{-1}(u_2);$$

– Spearman rank correlation coefficient

$$\rho_s(X_1, X_2) = 12\int_0^1\int_0^1 C(u_1, u_2) du_1 du_2 - 3 = 12\int_0^1\int_0^1 (C(u_1, u_2) - u_1 u_2) du_1 du_2;$$

– Kendall's τ

$$\tau(X_1, X_2) = 4\int_0^1\int_0^1 C(u_1, u_2)\, dC(u_1, u_2) - 1;$$

– Gini's correlation coefficient

$$G(X_1, X_2) = 2\int_0^1\int_0^1 \left(|u_1 + u_2 - 1| - |u_1 - u_2|\right) dC(u_1, u_2);$$

– Blomqvist's correlation coefficient

$$\beta(X_1, X_2) = 4C(1/2, 1/2) - 1$$

– Coefficient of upper tail dependence

$$\lambda_U(X_1, X_2) = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1 - u}.$$

All of these measures were originally limited to two dimensions, but possible extensions to a more-dimensional setting for some of them can be found e.g. in Wolff (1980), Joe (1990, 1997) and Schmidt (2002).

3. The Comonotonicity Coefficient

We will now introduce our measure that accounts for the degree of comonotonicity, the strongest possible positive dependence, in an arbitrary random vector. As mentioned before, copulas receive much attention nowadays. Although there is no doubt about the importance of this concept, we may not forget that not all the information on the multivariate distribution function is incorporated in the copula. Therefore we have chosen to build our measure on the joint distribution function rather than solely on the copula function. As such we use all the available information from the margins and the dependence structure, and not the limited and partial information provided by the copula. This means that our coefficient does not define a concordance measure, but the choice to include information about the margins can be very relevant e.g. in financial questions such as the Value-at-Risk, where a good comprehension of the complete dependence structure, including the marginals, is of great importance.

Note that we will return to copulas and their merits when it turns to properties and illustrations.

3.1. Definition

As it will be clear from the definition below, the comonotonicity coefficient κ will be defined as the ratio of two hypervolumes. The numerator describes the hypervolume between the distribution of the vector under investigation and the distribution of the independent case, while the denominator, inserted to normalise the coefficient, corresponds to the (maximum possible) hypervolume, i.e. between the distributions of the comonotonic and the independent case, which are the two extremes. As we are interested in the ratio of the two hypervolumes and not in the particular values of both hypervolumes, we suggest to work with principal value integrals in case the hypervolumes are diverging when considered separately.

Definition 6. (Comonotonicity coefficient) *Let F_X^C and F_X^I be the distribution of the comonotonic and independent vector of the qualified Fréchet space $\mathcal{R}_m^+(F_1, \ldots, F_m)$. For any vector $\boldsymbol{X}$ of $\mathcal{R}_m^+(F_1, \ldots, F_m)$ with joint cdf $F_{\boldsymbol{X}}$, the comonotonicity coefficient $\kappa(\boldsymbol{X})$ is defined as:*

$$\kappa(\boldsymbol{X}) = \frac{\int \cdots \int \left(F_{\boldsymbol{X}}(\boldsymbol{x}) - F_{\boldsymbol{X}}^{I}(\boldsymbol{x})\right) d\boldsymbol{x}}{\int \cdots \int \left(F_{\boldsymbol{X}}^{C}(\boldsymbol{x}) - F_{\boldsymbol{X}}^{I}(\boldsymbol{x})\right) d\boldsymbol{x}}, \tag{1}$$

where the integration is performed over the whole domain of $\boldsymbol{X}$*. If the hypervolumes diverge when taken separately,* $\kappa(\boldsymbol{X})$ *is defined as*

$$\kappa(\boldsymbol{X}) = \lim_{a \to +\infty} \frac{\int \cdots \int_{G(a)} \left(F_{\boldsymbol{X}}(\boldsymbol{x}) - F_{\boldsymbol{X}}^{I}(\boldsymbol{x})\right) d\boldsymbol{x}}{\int \cdots \int_{G(a)} \left(F_{\boldsymbol{X}}^{C}(\boldsymbol{x}) - F_{\boldsymbol{X}}^{I}(\boldsymbol{x})\right) d\boldsymbol{x}}, \tag{2}$$

provided the limit exist, and with $G(a)$ *depending on the domain of* $\boldsymbol{X}$ *as follows:*

- *if* $\boldsymbol{X}$ *is defined on* $\mathbb{R}^{m}$*, then* $G(a) = [-a, a]^{m}$;
- *if* $\boldsymbol{X}$ *is defined on* $\mathbb{R}^{+n}$*, then* $G(a) = [1/a, a]^{m}$;
- *if* $\boldsymbol{X}$ *is defined on* $[A_1, +\infty[\times \ldots \times [A_m, +\infty[$*, then* $G(a) = [A_1, a] \times \ldots \times [A_m, a]$;
- *if* $\boldsymbol{X}$ *is defined on a finite domain, then* $G(a)$ *equals the whole domain.*

Note that, when both numerator and denominator converge separately, expression (2) reduces to (1).

In the case of continuous distribution functions, there exists a unique copula C_X for which $F_X(x_1, \ldots, x_m) = C_X(F_1(x_1), \ldots, F_m(x_m))$. Inserting this into the formula for κ in (1) or in (2) results in the following equivalent expression for the comonotonicity coefficient:

$$\kappa(\boldsymbol{X}) = \frac{\int_0^1 \ldots \int_0^1 \left(C_X(u_1, \ldots, u_m) - \prod_{i=1}^{m} u_i\right) dF_1^{-1}(u_1) \ldots dF_m^{-1}(u_m)}{\int_0^1 \ldots \int_0^1 \left(\min_{i=1}^{m} u_i - \prod_{i=1}^{m} u_i\right) dF_1^{-1}(u_1) \ldots dF_m^{-1}(u_m)} \tag{3}$$

or

$$\kappa(\boldsymbol{X}) = \lim_{a \to +\infty} \frac{\int \cdots \int_{H(a)} \left(C_X(u_1, \ldots, u_m) - \prod_{i=1}^{m} u_i\right) dF_1^{-1}(u_1) \ldots dF_m^{-1}(u_m)}{\int \cdots \int_{H(a)} \left(\min_{i=1}^{m} u_i - \prod_{i=1}^{m} u_i\right) dF_1^{-1}(u_1) \ldots dF_m^{-1}(u_m)}, \tag{4}$$

with $H(a) \subset [0,1]^{m}$ depending on the domain of $\boldsymbol{X}$.

E.g. for $G(a) = [-a, a]^{m}$, the integration now must be performed over the area $H(a) = [F_1(-a), F_1(a)] \times \ldots \times [F_m(-a), F_m(a)]$.

3.2. Basic Characteristics

In general, a concordance or dependence measure should satisfy a number of desirable characteristics, as mentioned e.g. in Schweizer & Wolff (1981), Scarsini (1984), or Drouet Mari & Kotz (2001). In the next theorem, we summarise the results for κ with regard to these axioms. A proof is straightforward.

Theorem 2. *The comonotonicity coefficient as defined in Definition 6 satisfies the following properties:*

1. $\kappa(\boldsymbol{X})$ *is defined for any vector* $\boldsymbol{X}$ *of a qualified Fréchet space* $\mathcal{R}_m^+$.
2. $\kappa(\boldsymbol{X})$ *takes values in the range* $[0,1]$.
3. $\kappa(\boldsymbol{X})$ *is symmetrical in the components* X_i, $i = 1, \ldots, m$, *of the vector* $\boldsymbol{X}$.
4. $\kappa(\boldsymbol{X}) = 0$ *if and only if* $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$ *has independent components.*
5. $\kappa(\boldsymbol{X}) = 1$ *if and only if* $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$ *has comonotonic components.*

The second item in this theorem is responsible for the restriction in the definition of our comonotonicity coefficient to vectors that are PLOD, or positive lower orthant dependent. Indeed, if vectors belonging to $\mathcal{R}_m$ minus $\mathcal{R}_m^+$ are allowed, both positive and negative results are possible for κ. This will cause a twofold problem. Firstly, the range of values for κ is not finite anymore at the lower side, since the normalising hypervolume of the denominator no longer corresponds to extreme cases, the independent vector only making up a lower bound for elements of $\mathcal{R}_m^+$. Secondly, and this seems to be much more important, the interpretation of κ is no longer unambiguous. If we allow for vectors outside the qualified Fréchet space, a positive value no longer exclusively belongs to globally positive dependent vectors. If e.g. one of the components of a multivariate vector is negatively dependent on the other components, the overall result could still be positive. Note that also many other dependence measures suffer from this drawback, since they all measure some kind of average dependence.

3.3. Properties of the Comonotonicity Coefficient

Next to the basic characteristics, we now investigate a few other properties of the comonotonicity coefficient κ, related to the performance of the measure. For more details, we refer to Koch & De Schepper (2006).

Proposition 1. *For any qualified Fréchet space* $\mathcal{R}_m^+$ *and for any vector* $\boldsymbol{X} = (X_1, \ldots, X_m)$ *of* $\mathcal{R}_m^+$, *for which the dependence structure can be described by a copula belonging to a positively ordered copula family, the comonotonicity coefficient* $\kappa(\boldsymbol{X})$ *increases with the parameter of the family.*

This follows immediately from equation (3) or (4), and it guarantees that κ is a consistent dependence measure. An illustration can be found in Table 1,

TABLE 1

EVOLUTION OF κ FOR INCREASING VALUES OF α

$n = 2$, Gumbel copula, standard normal marginal distributions								
α	1	1.01	1.1	1.5	2	3	4	∞
$\kappa(\boldsymbol{X})$	0	0.0165	0.148	0.501	0.701	0.858	0.917	1
$n = 3$, Clayton copula, exponential marginal distributions								
α	0	0.1	1	8	10	20	50	∞
$\kappa(\boldsymbol{X})$	0	0.0399	0.290	0.753	0.791	0.881	0.947	1
$n = 4$, Frank copula, uniform[0,1] marginal distributions								
α	0	0.01	1	3	5	10	15	∞
$\kappa(\boldsymbol{X})$	0	0.00121	0.128	0.389	0.593	0.837	0.917	1

where for different dimensions the values of κ are compared with the values of the parameter for three different copulas.

Proposition 2. *For any (bivariate) vector* $\boldsymbol{X} = (X_1, X_2)$ *of* $\mathcal{R}_2^+$ *with both marginals Uniform* [0,1] *distributions, the comonotonicity coefficient* $\kappa(X_1, X_2)$ *equals the Spearman's correlation coefficient* $\rho_s(X_1, X_2)$.
For any (m-variate) vector $\boldsymbol{X} = (X_1, \ldots, X_m)$ *of* $\mathcal{R}_m^+$ *with all marginals Uniform* [0,1] *distributions, the comonotonicity coefficient* $\kappa(\boldsymbol{X})$ *equals the multivariate Spearman's correlation coefficient* $\rho_J^m(\boldsymbol{X})$ *as mentioned in Joe (1990).*

Proof: For $m = 2$, with $\int_0^1 \int_0^1 (\min(u_1, u_2) - u_1 u_2)\, du_1\, du_2 = 1/12$, this immediately follows from a comparison of equation (3) with the definition of the Spearman's correlation coefficient in section 2.
For $m > 2$, a combination of $\int_0^1 \cdots \int_0^1 (\min_{i=1}^m u_i - \prod_{i=1}^m u_i)\, du_1 \ldots du_m = \frac{2^m - m - 1}{2^m (m+1)}$ and the coefficient $\rho_J^m(\boldsymbol{X}) = \frac{2^m - m - 1}{2^m (m+1)} \left(\int_0^1 \cdots \int_0^1 C(u_1, \ldots, u_m)\, du_1 \ldots du_m - \frac{1}{2^m} \right)$ leads to the result. Q.E.D.

As a consequence, the coefficient κ can be seen as an extension of the Spearman's correlation coefficient, in the situation where the "true" marginal distributions are not taken into account.

Proposition 3. *For any (bivariate) vector* $\boldsymbol{X} = (X_1, X_2)$ *of* $\mathcal{R}_2^+$ *for which it is true that*

$$\int_0^1 \int_0^1 \left(\mathrm{Min}\{u, v\} - uv \right) dF_1^{-1}(u)\, dF_2^{-1}(v) = \sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)},$$

the comonotonicity coefficient $\kappa(X_1, X_2)$ *equals the Pearson correlation coefficient* $r_p(X_1, X_2)$.

Proof: This result can be demonstrated by comparing the expression for the comonotonicity coefficient in the bivariate case (see equation (3)) with the definition of the Pearson correlation coefficient (see section 2). Q.E.D.

The condition mentioned in Proposition 3 is satisfied e.g. if both marginal distributions are uniform, normal or exponential distributions, not necessarily with the same distribution parameters. The condition is not satisfied e.g. in the case of lognormal marginal distribution functions.

Proposition 4. *For any random variable X and for any set of monotone increasing and invertible real functions $g_1, \ldots, g_m$, the comonotonicity coefficient κ of the vector $(X, g_1(X), \cdots, g_m(X))$ is equal to 1.*

Proof: Starting from the alternative definition of comonotonicity, see section 2, this result is straightforward. Q.E.D.

This property establishes an important difference between the Pearson correlation coefficient and the comonotonicity coefficient κ, the latter being capable of capturing the dependence between a variable X and its transformed value $g(X)$ for a broad class of functions g, and not only for linear transformations as it is the case for the Pearson correlation. A striking example is the situation where X and X^2 are compared. Indeed, while in general $r_p(X_1, X_2) \neq 1$, the desirable result does hold for the comonotonicity coefficient, or $\kappa(X, X^2) = 1$.

Proposition 5. *For any vector $\boldsymbol{X} = (X_1, \ldots, X_m)$ of $\mathcal{R}_m^+$ for which all the marginal distributions are exponential distributions, $X_i \sim (\lambda_i)$, or for which all the marginal distributions are normal distributions, $X_i \sim N[\mu_i, \sigma_i^2]$, and where the copula C as defined in Theorem 1 is functionally independent of the parameters of the marginal distributions, the comonotonicity coefficient $\kappa(\boldsymbol{X})$ is also independent of the parameters λ_i or μ_i and σ_i.*

Proof: This result can be verified by means of a substitution of the marginal distributions into equation (3). Q.E.D.

Combining this proposition with proposition 3, we see that for certain classes of marginal distribution functions, the comonotonicity coefficient κ can also be seen as a kind of multivariate extension of the Pearson correlation.

Proposition 6. *For any set of random vectors $\boldsymbol{X}$ and $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ belonging to the same qualified Fréchet space $\mathcal{R}_m^+(F_1, \ldots, F_m)$, for which the distribution function of $\boldsymbol{X}$ can be written as a convex sum of the distribution functions of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$, $F_{\boldsymbol{X}} = \sum_{i=1}^k \alpha_i F_{\boldsymbol{X}_i}$, $\alpha_i \in [0,1]$ and $\sum_{i=1}^k \alpha_i = 1$, the comonotonicity coefficient of $\boldsymbol{X}$ equals the convex sum of the corresponding comonotonicity coefficients, $\kappa(\boldsymbol{X}) = \sum_{i=1}^k \alpha_i \kappa(\boldsymbol{X}_i)$.*

Proof: The decomposition as mentioned in property 6 can be rewritten as $F_{\boldsymbol{X}} - F_{\boldsymbol{X}}^I = \sum_{i=1}^k \alpha_i [F_{\boldsymbol{X}_i} - F_{\boldsymbol{X}_i}^I]$. A substitution in equation (2) proves the result. Q.E.D.

Corollary 1. *For any vector* $\boldsymbol{X} = (X_1, \ldots, X_m)$ *of* $\mathcal{R}_m^+$ *for which the joint distribution function* $F_{\boldsymbol{X}}$ *can be written as a convex sum of* $F_{\boldsymbol{X}}^C$ *and* $F_{\boldsymbol{X}}^I$, *or* $F_{\boldsymbol{X}} = \alpha F_{\boldsymbol{X}}^C + (1-\alpha) F_{\boldsymbol{X}}^I$, $\alpha \in [0,1]$, *the comonotonicity coefficient* $\kappa(\boldsymbol{X})$ *is equal to* α.

This last corollary states a crucial result, since it demonstrates that our comonotonicity coefficient is completely in harmony with how one would interpret the value of α in the decomposition of an arbitrary distribution into a convex combination of the independent and comonotonic extremes. Note that if in a bivariate case the joint distribution function can be decomposed in this way, the comonotonicity coefficient is also equal to the coefficient of upper tail dependence.

4. Numerical illustrations

4.1. Gaussian Discounting Processes

The first illustration refers to the investigation of present values with stochastic interest rates, which in fact was the immediate cause of the development of our comonotonicity coefficient. Consider a cash flow of future payments with present value

$$V(t) = \sum_{i=1}^{m} \xi(t_i)\, e^{-Y_i},$$

with time points $0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = t$, with $\xi(t_i)$ the deterministic payment at time t_i, and with e^{-Y_i} the stochastic discounting factor over the time period $[0, t_i]$. If the discounting process is modelled by means of Gaussian processes, we start with a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ with independent and normally distributed components X_i, each representing the rate of return for a (small) time period $[t_{i-1}, t_i]$. The compounded rate of return for the whole period $[0, t_i]$ can then be written by $Y_i = X_1 + \ldots + X_i$. Although all components in these sums are independent, the random vector $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ clearly consists of strongly dependent variables.

The calculation of the exact distribution of the (stochastic) present value above is very hard, and in most cases even impossible. Goovaerts et al. (2000),

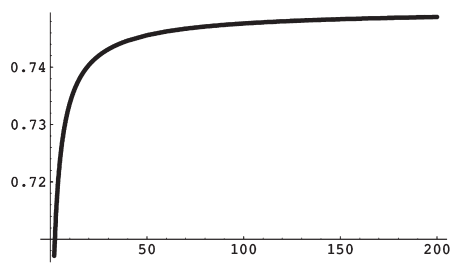


FIGURE 1: Evolution of $\kappa(\boldsymbol{Y})$ with increasing dimension n.

Kaas et al. (2000), Dhaene et al. (2002a) suggested to work with an upper bound in convex order sense, replacing the vector $\boldsymbol{Y}$ by its comonotonic counterpart, which corresponds to the strongest possible dependence structure. The calculations needed for this upper bound are much easier than they are for the exact distribution, and yet the upper bound turns out to be a very accurate approximation in case the payments $\xi(t_i)$ are all non-negative (see Kaas et al. (2000) and Goovaerts et al. (2000)). We will show now that this good result can be explained by quantifying the dependence in the vector $\boldsymbol{Y}$ by means of the comonotonicity coefficient κ.

In the case where all the components X_i are (independent and) standard normally distributed, the components Y_i correspond to normal distributions with zero mean and with variance equal to i, and analytical results for the comonotonicity coefficient are possible. Note that in this case Proposition 5 is not applicable since the joint distribution function is not independent of the parameters of the marginal distributions.

When n is equal to 2, the comonotonicity coefficient of the vector $\boldsymbol{Y} = (Y_1, Y_2)$ can be calculated as:

$$\kappa(Y_1, Y_2) = \frac{1}{\sqrt{2}} \approx 0.707107.$$

For $n = 3$, the calculations are much more lengthy, finally resulting in

$$\kappa(Y_1, Y_2, Y_3) = \frac{4}{\sqrt{6} \cdot \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\right)} \approx 0.714828.$$

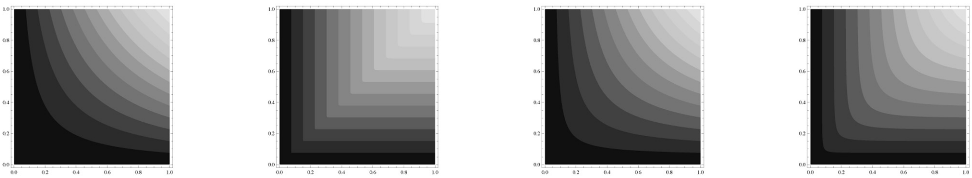
Figure 1 indicates how $\kappa(\boldsymbol{Y})$ increases with the dimension of the vector $\boldsymbol{Y}$.

In the case where the components X_i are (independent and) identically but more generally normally distributed, $X_i \sim N[\mu, \sigma^2]$, the analytical calculations become extremely complex. However, numerical calculations suggest that the values of the normal parameters do not have any effect on the resulting comonotonicity coefficient of the vector $\boldsymbol{Y}$.

In any case, the high value of κ for the vector $\boldsymbol{Y}$ confirms that due to the construction of the cumulative discounting process, the components of the vector $\boldsymbol{Y}$ indeed are very strongly positive dependent. This explains why in such discounting issues an approximation of the original vector by its comonotonic counterpart indeed makes up a very good approach.

4.2. Contourplots

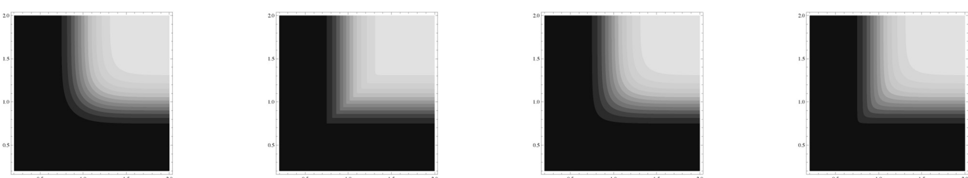
For the interpretation or visualisation of κ, contourplots constitute a nice and efficient tool. A first advantage is that they can be used to get an idea about the meaning of the absolute numerical value of κ, just by comparing the contourplots for the vector under investigation with the contourplots for the independent and comonotonic counterparts.



(a) $\boldsymbol{X}^I$ $\kappa(\boldsymbol{X}) = 0$ (b) $\boldsymbol{X}^C$ $\kappa(\boldsymbol{X}) = 1$ (c) $\boldsymbol{X} \sim \text{Clayton}(0.5)$ $\kappa(\boldsymbol{X}) = 0.295$ (d) $\boldsymbol{X} \sim \text{Clayton}(3)$ $\kappa(\boldsymbol{X}) = 0.786$

FIGURE 2: Contourplots for $n = 2$ – uniform [0,1] marginal distribution.

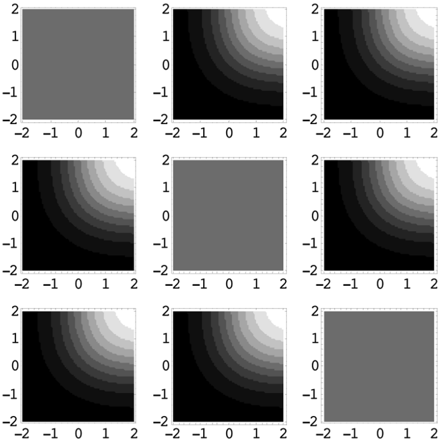
In Figure 2, we present contourplots for a bivariate example. Consider a vector $\boldsymbol{X} = (X_1, X_2)$, with uniform marginal distributions. Part (a) and (b) of the figure refer to the extreme situations ($\kappa = 0$ and $\kappa = 1$), while in part (c) and (d) we show the contourplot in case the dependence structure is modelled by means of a Clayton copula with different parameters. Comparing the different pictures, we see that the situation with a rather low value of κ is clearly more similar to the independent situation, whereas the situation with a rather high value of κ much more resembles the comonotonic situation. In order to show the influence of the marginals, Figure 3 repeats the same type of contourplots, but now for a vector with lognormal $[0, (0.2)^2]$ marginals. As in Figure 2, for a large value of κ, the picture tends to the comonotonic picture, while for a small value of κ, the similarity with the independent picture is apparent.



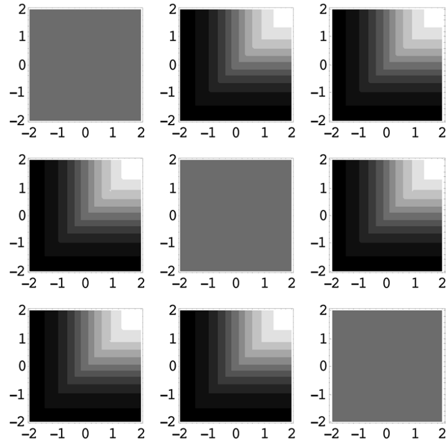
(a) $\boldsymbol{X}^I$ $\kappa(\boldsymbol{X}) = 0$ (b) $\boldsymbol{X}^C$ $\kappa(\boldsymbol{X}) = 1$ (c) $\boldsymbol{X} \sim \text{Clayton}(0.5)$ $\kappa(\boldsymbol{X}) = 0.272$ (d) $\boldsymbol{X} \sim \text{Clayton}(3)$ $\kappa(\boldsymbol{X}) = 0.709$

FIGURE 3: Contourplots for $n = 2$ – Lognormal $[0, (0.2)^2]$ distributions.

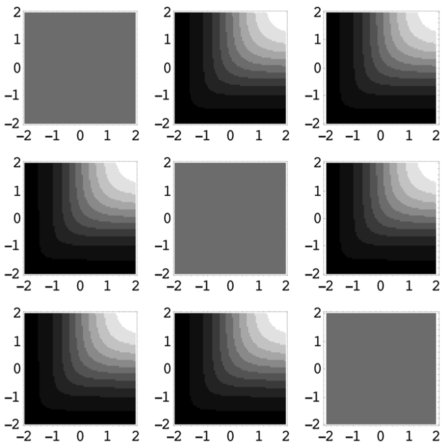
This technique is also workable in more than 2 dimensions, by looking at all possible two-dimensional projections. This is illustrated in Figure 4, for a vector $\boldsymbol{X} = (X_1, X_2, X_3)$, with standard normal marginal distributions. In this second example, the extremes are compared to a Clayton copula dependence structure. Note that with a Clayton copula, the dependence is concentrated in the lower tails. Graph (c) indeed reveals that in the lower tails the situation is tending towards a more comonotonic situation, while for the upper tails the contours are more similar to the independent case.



(a) $\boldsymbol{X}$ independent, $\kappa(\boldsymbol{X}) = 0$



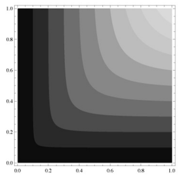
(b) $\boldsymbol{X}$ comonotonic, $\kappa(\boldsymbol{X}) = 1$



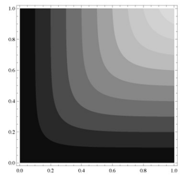
(c) $\boldsymbol{X} \sim$ Clayton(2), $\kappa(\boldsymbol{X}) = 0.684$

FIGURE 4: Contourplots for $n = 3$ – standard normal marginal distributions.

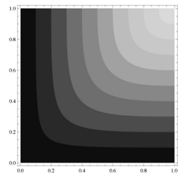
Secondly, contourplots can also be used to analyse different dependence structures ending up with the same value of the comonotonicity coefficient κ. An elaborated illustration is provided in Figure 5, where contourplots are depicted for 15 different joint distributions, with five different marginal distributions (uniform on [0,1], standard normal, exponential[1], lognormal[0,(0.2)^2], gamma[5,1]), and three copula families (Clayton, Frank and Gumbel), all corresponding to a situation with κ equal to 0.80. It can be observed that there is a large degree of similarity in these plots, although they are created by means of rather different models.



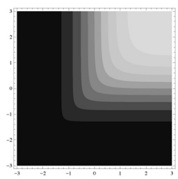
(a) uniform marginals
Clayton copula



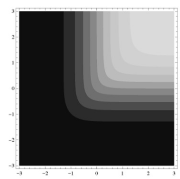
(b) uniform marginals
Frank copula



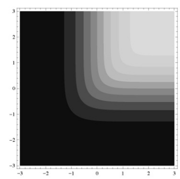
(c) uniform marginals
Gumbel copula



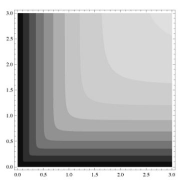
(d) normal marginals
Clayton copula



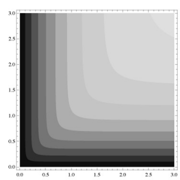
(e) normal marginals
Frank copula



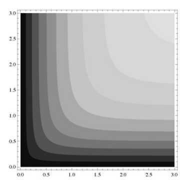
(f) normal marginals
Gumbel copula



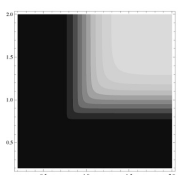
(g) exponential marginals
Clayton copula



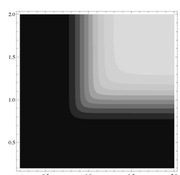
(h) exponential marginals
Frank copula



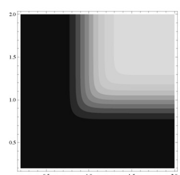
(i) exponential marginals
Gumbel copula



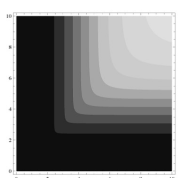
(j) lognormal marginals
Clayton copula



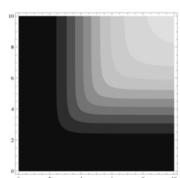
(k) lognormal marginals
Frank copula



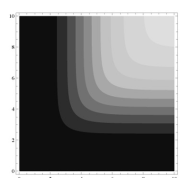
(l) lognormal marginals
Gumbel copula



(m) gamma marginals
Clayton copula



(n) gamma marginals
Frank copula



(o) gamma marginals
Gumbel copula

FIGURE 5: Contourplots for different bivariate dependence structures with $\kappa(\boldsymbol{X}) = 0.80$.

4.3. Comparison of the Comonotonicity Coefficient with Classic Bivariate Measures

Assume now that we are working in a bivariate environment, investigating the dependence structure in a vector $\boldsymbol{X} = (X_1, X_2)$. We already mentioned some connections between our comonotonicity coefficient and classic association measures in the propositions in section 3. If the marginal distributions of X_1 and X_2 are both uniform [0,1], κ is equal to the Spearman rank coefficient ρ_s (see Proposition 2); if the marginal distributions satisfy a special condition, then κ is equal to the Pearson correlation r_p (see Proposition 3).

In Table 2 we compare κ with the classic dependence measures in a particular bivariate example not satisfying the previous specifications. The first variable is assumed to be lognormally distributed, $X_1 \sim \mathrm{LogN}[0.04, (0.2)^2]$, for the second one we have chosen a normal distribution, $X_2 \sim \mathrm{N}[0.04, (0.2)^2]$; the dependence is modelled by means of a Frank copula with parameter α. As expected, all the dependence measures increase with the parameter value of the copula α, but for each of them this occurs in a distinct way. This is of course a consequence of the definitions of each of these dependence measures, all measuring another aspect of the actual dependence.

Let us examine in particular the relation between the comonotonicity coefficient κ and Kendall's τ on the one hand and between κ and Spearman's rank coefficient on the other hand.

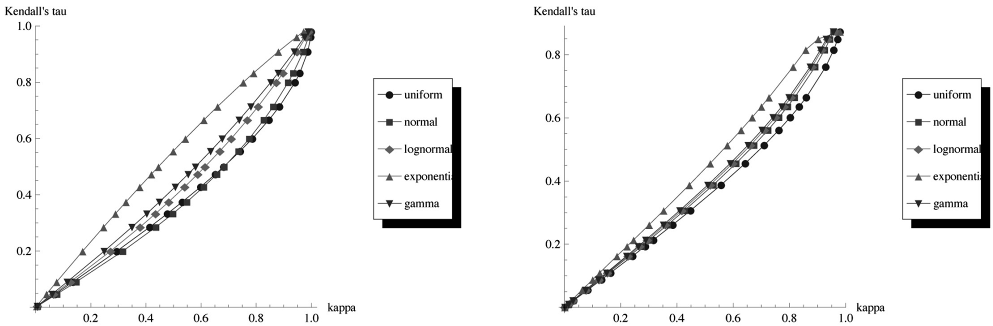
The popularity of Archimedean copulas (e.g. Clayton, Frank, Gumbel) causes Kendall's τ to be the dominant dependence measure in a copula setting. This is due to the nice mathematical connection that exists between the generator of the Archimedean copula and τ (see Genest & MacKay (1986)). Although τ is undeniable mathematical convenient, τ is not necessary suitable for measuring comonotonicity. Furthermore, Kendall's tau is a concordance measure and κ is not, since we take the marginal distributions into account. Clearly both measures are not equal. In the example used in Table 2, we see that κ is smaller than τ for small values of α, corresponding to situations with

TABLE 2

COMPARISON OF $\kappa(\boldsymbol{X})$ WITH CLASSIC DEPENDENCE MEASURES

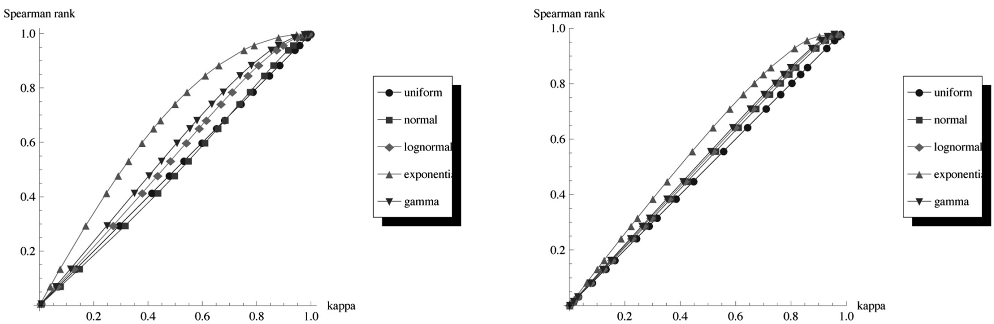
α	$\kappa(X_1, X_2)$	Pearson	Spearman	Kendall	Blomqvist
0.01	0.00167	0.00144	0.00746	0.00498	0.00125
0.5	0.0830	0.0719	0.295	0.200	0.0623
1	0.164	0.142	0.478	0.333	0.124
2	0.316	0.274	0.682	0.500	0.240
3	0.447	0.387	0.786	0.600	0.344
10	0.850	0.736	0.958	0.833	0.725
20	0.949	0.822	0.987	0.909	0.861

a weak dependence between X_1 and X_2, and that κ dominates τ for higher values of α and thus for a stronger dependent vector.



(a) Clayton copula (b) Frank copula

FIGURE 6: Relation between comonotonicity coefficient κ and Kendall's τ.



(a) Clayton copula (b) Frank copula

FIGURE 7: Relation between comonotonicity coefficient κ and Spearman's rank.

In Figure 6, the relation between κ (on the horizontal axis) and τ (on the vertical axis) is displayed for a few different situations. Five types of marginal distributions are combined with the Frank copula, which is symmetric in the upper and lower tail, and with the Clayton copula, for which the dependence is concentrated in the lower tails. As it can be observed from both pictures, there is no general relationship between both measures.

As mentioned earlier, κ can be seen as an extension of Spearman's rank taking into account the effect of the marginals. In Figure 7, the relation between κ (on the horizontal axis) and Spearman's rank (on the vertical axis) is depicted, for the same set of models as in figure 6. Note that for uniform marginals, both measures are identical (cf. Proposition 2).

5. The Comonotonicity Coefficient – Sample Version

5.1. Calculation

Suppose we want to estimate the comonotonicity coefficient for m variables, for which we have N coupled observations, not necessarily independent, summarized in a data matrix Y (dimension $N \times m$).

The empirical distribution can be written as

$$\hat{F}_{\boldsymbol{X}}(x_1,\ldots,x_m) = \frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{m}\mathbb{I}(Y_{ij} \leq x_j);$$

for the empirical versions of the distribution for the independent and comonotonic vectors of the same Fréchet space, we have

$$\begin{aligned}\hat{F}^{C}_{\boldsymbol{X}}(x_1,\ldots,x_m) &= \min\left\{\hat{F}_{X_1}(x_1),\ldots,\hat{F}_{X_m}(x_m)\right\}\\ \hat{F}^{I}_{\boldsymbol{X}}(x_1,\ldots,x_m) &= \hat{F}_{X_1}(x_1)\cdot\ldots\cdot\hat{F}_{X_m}(x_m).\end{aligned}$$

Now, define $m_j := \min_{i=1}^{N} Y_{ij}$ and $M_j := \max_{i=1}^{N} Y_{ij}$, and denote $Y_j^{(k)}$ for the k-th ordered observation for the j-th variable. As a consequence, $Y_j^{(1)} = m_j$ and $Y_j^{(N)} = M_j$ for any $j = 1, \ldots, m$, and $\int_{m_j}^{M_j}\mathbb{I}(Y_{ij} \leq x_j)\,dx_j = M_j - Y_{ij}$ for any $i = 1, \ldots, N$, $j = 1, \ldots, m$.

If we replace the distribution functions in Definition 6 by their empirical versions, we get a sample-based version for the comonotonicity coefficient.

Definition 7. (Sample-based Comonotonicity coefficient) *For a given data matrix Y with N observations for m variables, and with the notations above, the sample-based comonotonicity coefficient $\hat{\kappa}(\boldsymbol{X})$* is defined as:

$$\hat{\kappa}(\boldsymbol{X}) = \frac{\frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{m}\left(M_j - Y_{ij}\right) - \frac{1}{N^m}\prod_{j=1}^{m}\left(NM_j - \sum_{i=1}^{N} Y_{ij}\right)}{\frac{1}{N}\sum_{i=1}^{N}\prod_{j=1}^{m}\left(M_j - Y_j^{(i)}\right) - \frac{1}{N^m}\prod_{j=1}^{m}\left(NM_j - \sum_{i=1}^{N} Y_{ij}\right)}. \tag{5}$$

Note that, as the empirical distribution function is a consistent estimator of the real cumulative distribution function, the sample-based comonotonicity coefficient $\hat{\kappa}$ is also a consistent estimator of the comonotonicity coefficient κ.

In practical applications, it is necessary to test whether the data are positive lower orthant dependent, before the calculation of the estimate for the comonotonicity coefficient is performed. We suggest to rely on the results of Denuit & Scaillet (2004) or Scaillet (2005) for this purpose. A bootstrap procedure can be used in order to estimate the error on the sample-based comonotonicity coefficient.

5.2. Empirical illustration

In order to illustrate the possibilities of the sample-based comonotonicity coefficient in more dimensions, we consider seven world market indices:

– S&P 500 (U.S.);
– Dow Jones Industrial Average (U.S.);
– Nikkei index (Japan);
– FTSE 100 index (U.K.);
– CAC index (France);
– DAX index (Germany);
– SMI index (Switzerland).

We included several European indices so as to create a meaningful subset. We look at the monthly values, particularly at the closing prices at the last trading day of each month, for the period January 2000 to November 2009 (119 data points). In Figure 8, the evolution of time is shown for the 7 rescaled indices; Table 3 summarizes some descriptive statistics for these indices.

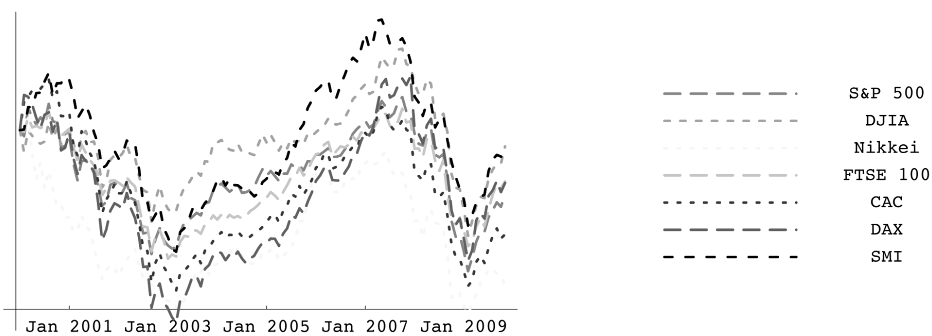


FIGURE 8: Rescaled values of the 7 investigated indices.

TABLE 3

DESCRIPTIVE STATISTICS OF MONTHLY INDEX VALUES

	S&P	DJIA	Nikkei	FTSE	CAC	DAX	SMI
Jan 2000	1394.46	10940.5	19539.7	6228.5	5659.81	6835.6	6894.7
Nov 2009	1095.63	10344.8	9345.55	5190.7	3680.15	5625.95	6261.0
minimum	735.09 (Feb 2009)	7062.93 (Feb 2009)	7568.42 (Feb 2009)	3567.4 (Jan 2003)	2618.46 (Mar 2003)	2423.87 (Mar 2003)	4085.6 (Mar 2003)
maximum	1549.38 (Oct 2007)	13930.0 (Oct 2007)	20337.3 (Mar 2000)	6721.6 (Oct 2007)	6625.42 (Aug 2000)	8067.32 (Dec 2007)	9450.8 May 2007)

Figure 8 nicely illustrates the (well-known) finding that there is a high level of dependence between the different market indices. This is confirmed by the correlation matrix, which is given in 4 below. Testing for positive lower orthant dependence (see Scaillet (2005)) results in a p-value of 1., reassuring that the calculation of the comonotonicity coefficient is allowed.

TABLE 4

CORRELATION MATRIX OF SELECTED INDICES

	S&P	DJIA	Nikkei	FTSE	CAC	DAX	SMI
S&P	1	0.91482	0.91409	0.94679	0.92462	0.87668	0.90716
DJIA	0.91482	1	0.78688	0.83333	0.74581	0.80178	0.87109
Nikkei	0.91409	0.78688	1	0.90553	0.89684	0.83294	0.88158
FTSE	0.94679	0.83333	0.90553	1	0.96923	0.94816	0.95479
CAC	0.92462	0.74581	0.89684	0.96923	1	0.89914	0.90455
DAX	0.87668	0.80178	0.83294	0.94816	0.89914	1	0.91584
SMI	0.90716	0.87109	0.88158	0.95479	0.90455	0.91584	1

This correlation matrix refers to pair-wise correlations, and it does not give a global measure for the group of indices; it also measures solely the linear correlations. The advantage of calculating the sample-based comonotonicity coefficient for the whole group or for a subset, is the fact that this approach makes it possible to get a much completer idea about the real dependence between the seven indices.

The calculation of $\hat{\kappa}$ results in the following values:

– for the whole group: $\hat{\kappa} = 0.87578$;
– for the US selection (S&P, DJIA): $\hat{\kappa} = 0.93629$;
– for the European selection (FTSE, CAC, DAX, SMI): $\hat{\kappa} = 0.95334$.

Note that the value for $\hat{\kappa}$(S&P, DJIA) is slightly higher than the correlation between the two indices. This is due to the fact that the comonotonicity coefficient not only measures the *linear* interdependence, but that it measures the degree of (overall) comonotonicity, see also Proposition 4.

It is also interesting to compare the estimated value for the comonotonicity coefficient for the whole period 2000-2009, with the results for subperiods, e.g. the periods before and after the 2007 financial crisis:

– for the whole period: $\hat{\kappa} = 0.87578$;
– for the period until June 2007: $\hat{\kappa} = 0.93982$;
– for the period starting from July 2007: $\hat{\kappa} = 0.97982$.

It can be observed that the estimated values for the comonotonicity coefficients are rather high in all cases, illustrating again the high degree of (positive) dependence between the different market indices.

An estimate for the error of the sample-based comonotonicity coefficients can be found through a bootstrap procedure. As we are faced with time series data with serial correlation and possibly also with heteroskedasticity, we use a block bootstrap in order to improve the performance. We illustrate the reliability of the $\hat{\kappa}$ estimate for the 7 market indices over the period 2000-2009 by means of a block bootstrap with 5000 subsamples, where each of the subsamples is composed of 24 overlapping blocks with block length 5 for the first simulation, and of 40 overlapping blocks with block length 3 for the second simulation[1]. Figure 9 shows the histogram of the bootstrapped $\hat{\kappa}$ estimates, together with the normal approximation. The left panel corresponds to a block length of 5, the right panel to a block length of 3. Results for the mean and standard deviation and for 90% confidence intervals can be found in Table 5.

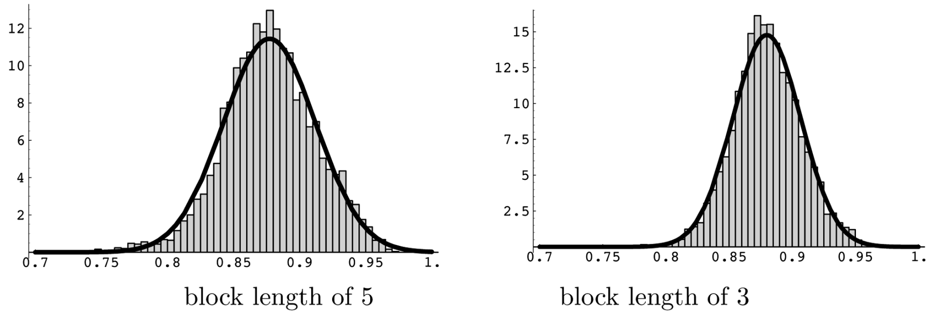


FIGURE 9: Histogram of bootstrap estimates for $\hat{\kappa}$ for the index data.

TABLE 5

DESCRIPTIVE STATISTICS FOR THE BOOTSTRAP SUBSAMPLES TO BE COMPARED TO A VALUE FOR THE SAMPLE-BASED COMONOTONICITY COEFFICIENT $\hat{\kappa}$ = 0.87578.

	block length of 5	block length of 3
mean value	$\mu_k = 0.87716$	$\mu_k = 0.87988$
standard deviation	$\sigma_k = 0.03488$	$\sigma_k = 0.02702$
non parametric C.I.	(0.82070; 0.93498)	(0.83594; 0.92435)
normality based C.I.	(0.80879; 0.94553)	(0.82693; 0.93284)

Note that a smaller block length results in a lower standard deviation (but higher bias) and that a larger block length results in a lower bias (but higher standard deviation).

[1] The choice of the block length is in line with the numerical examples in Hall et al. (1995); as our sample consists of 119 data points, a block length of 5 implies a number of blocks equal to 24, and an block length of 3 implies a number of blocks equal to 40.

6. Conclusion

In this paper we introduced a new measure of positive dependence, the comonotonicity coefficient κ. We proved that this coefficient is well-defined in any dimension, we derived some properties, and we showed that an interpretation of the result is unambiguous. Applying this comonotonicity coefficient to stochastic interest rates in discounting processes, we were able to give quantitative arguments why the approximation method with convex bounds, developed by Goovaerts et al. (2000); Kaas et al. (2000); Dhaene et al. (2002a) performs so well in the case of present values of cash-flows with Gaussian discounting processes. We also showed how to construct a sample-based comonotonicity coefficient, which gives an estimate for the comonotonicity coefficient based on empirical data. The calculation of the suggested estimator turns out to be straightforward, even for high-dimensional data, which is demonstrated for a 7-dimensional example.

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