

PENSION FUND MANAGEMENT AND CONDITIONAL INDEXATION

BY

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ABSTRACT

Conditional indexation offers a middle way between defined benefit and defined contribution pension schemes. In this paper, we consider a fully-funded pension scheme with conditional indexation. We show how the pension fund can be managed to reduce the risks associated with promised pension benefits when declared benefits are adjusted regularly during the working life. In particular, we derive an investment strategy that provides protection against underfunding at retirement and which is self-financing on average. Our results are illustrated in an extensive simulation study.

KEYWORDS

Conditional Indexation, Fully Funded Pension, Pension Fund Management, Quadratic Hedging.

1. INTRODUCTION

In many developed countries, traditional pension schemes are defined benefit (DB) schemes where pension benefits are linked to final salaries, general wage inflation or other personal and economic variables rather than contributions made during the working life. While these pension schemes provide a high degree of security for the retirement income of workers, they also form a substantial risk for employers or other pension providers. The demographic change in the last few decades due to increased life expectancies and low fertility rates combined with low interest rates has driven up the costs of DB schemes significantly. In the UK, toughening government legislation and new accounting standards have put further pressure on DB schemes. In their “Accounting for Pensions 2009” report, Lane Clark and Peacock estimate that the total pension deficit of the FTSE 100 companies stood at £96 billion in July 2009, which is about 25 percent of the total pension liabilities of £388 billion. The increasing costs of DB schemes have resulted in a large number of employers to close their final salary schemes for new employees as these become too expensive to run. Lane Clark and Peacock report that “Only four FTSE 100

companies reported defined benefit schemes open to new UK employees in their 2008 reports.”

Where DB schemes are closed, new employees are often offered to participate in a defined contribution (DC) scheme where pension benefits are explicitly related to contributions. These schemes have substantial advantages for employers who will usually pay a fixed percentage of salary into a fund. This fund is invested on the financial market and the pension benefits paid at retirement are the final value of the fund. While such a scheme provides a high degree of cost stability for employers, employees are facing a substantial investment risk resulting in a high degree of uncertainty in either pension benefits or the age of retirement as shown by MacDonald and Cairns (2007). For further discussions on the benefits and drawbacks of DB and DC schemes see, for example, Barr and Diamond (2006).

These developments have encouraged a discussion among politicians, pension fund managers, academics and others about the design of pension schemes. On the one hand, this discussion focuses on the funding of pensions: fully-funded pensions versus pay-as-you-go pensions. This aspect of pensions is not considered in this paper. We assume that the proposed pension scheme is fully funded. On the other hand, one aims to find a middle way between DB and DC schemes to achieve a balance between the advantages and disadvantages of these schemes for employers and employees.

One way to keep some of the advantages of DB schemes for employees while linking pension benefits explicitly to contributions is the conditional indexation (CI) of benefits. In such a scheme pension benefits to be received at retirement are increased regularly during the working life according to some formula but only under the condition that sufficient funding is available. CI has been discussed in the academic literature by several authors, see for example de Jong (2008) and Dai and Schumacher (2009a, b). The basic element of a pension plan with CI is that the guaranteed pension benefit is increased regularly until retirement according to some indexation procedure and starting from some initial guaranteed benefit level. In this respect, a CI scheme is similar to a DB pension. The difference between DB and CI is that the increase in guaranteed pension benefits (indexation) is conditional on the availability of sufficient funds. In the case of a single contribution, indexation at any time before retirement is therefore conditional on the performance of the pension fund until that time. CI pension schemes have been implemented, for example, in the Netherlands, see Ponds and van Riel (2009) and Bikker and Vlaar (2007).

The contribution of this paper to the literature on the conditional indexation of pensions is twofold. Firstly, a CI scheme which is slightly different from those considered in the existing literature is suggested. In particular, we propose to adjust declared benefits at the end of every year according to an indexation formula which takes the growth rate of an index and the performance of the pension fund during the preceding year into account rather than the fund value at the end of the year. The considered index might or might not be related to financial markets, for example, an inflation index or

some index related to salaries. This form of CI is similar to a profit-sharing agreement found in With-Profits life insurance policies. These policies have been studied by many authors, see for example Ballotta (2005), Bauer et al. (2006) and Kleinow and Willder (2007).

Secondly, we show how the pension fund can be managed to reduce the risk of underfunding at retirement. Since the adjustment of guaranteed benefits in CI schemes depends on the performance of the pension fund there exists a feedback effect between the investment strategy used to hedge the liabilities of the fund and the value of these liabilities. Since most pension funds are actively managed, we assume that the fund managers have full discretion about the investment strategy of the fund, and that they use this discretion to minimise the shortfall risk at retirement. To obtain a risk-minimising strategy for the pension fund similar arguments as those developed by Kleinow (2009) for With-Profits policies are used. One of the main results there is that equity plays no role in the optimal With-Profits fund as long as given guarantees are not related to equity-returns. Motivated by these results, we will consider a stochastic interest rate environment, and we will ignore equity in this paper. This is in contrast to the approach by Dai and Schumacher (2009b) who assumed constant interest rates and used a geometric Brownian motion to model equities. To keep our arguments simple we consider a complete market model for interest rates. However, since the index used for the adjustment of pension benefits in our CI scheme might involve non-hedgeable risks, like inflation or changing salaries, the resulting pension fund liabilities cannot be hedged perfectly by a self-financing pension fund portfolio. Instead, the proposed investment strategy is shown to be self-financing between indexation times and mean self-financing overall, compare definition 10.7 in Föllmer and Schied (2004). Loosely speaking, this means, that although positive and negative contributions will be required at indexation times, the expected value of these additional contributions is zero. It should be possible to either re-insure or securitise the risk associated with these contributions, as we further discuss at the end of section 4 after the investment strategy for the pension fund has been developed.

For the remainder of the paper we assume that a worker makes a single contribution now to be entitled to receive a lump-sum pension payment at retirement. Regular contributions are not explicitly considered. However, the design of the fund allows for the sequential purchase of additional pension rights throughout the working life. One could also adjust the pension plan design to consider regular payments of benefits by adding a guaranteed annuity option, as in Kleinow (2009).

The paper is organised as follows. In section 2 the financial market in which the pension fund operates is introduced. The particular pension scheme with CI is proposed in section 3 and relationships to the existing literature are discussed there. In section 4 an investment strategy for the pension fund that minimises the risk of underfunding based on a quadratic criterion is developed. A numerical example is provided in section 5 for illustration. We finish with a short conclusion.

2. THE FINANCIAL MARKET AND NON-HEDGEABLE RISKS

As mentioned in the introduction, a stochastic interest rate model is considered. To keep arguments simple, we use a one-factor model for interest rates with a fixed time horizon $T' \in \mathbb{N}$. As usual in one-factor models, the dynamics of the short interest rate r are specified under the risk-neutral measure \mathbb{Q} . For a comprehensive overview of one-factor models see, for example, Cairns (2004).

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ denote a probability space, and let $W = \{W(t)\}_{t \in [0, T']}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{Q})$. The filtration generated by W is denoted by $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T']}$, and we assume that $\mathcal{F}_{T'} \subset \mathcal{F}$. In the considered model the short term interest rate $r = \{r(t)\}_{t \in [0, T']}$ is the solution of

$$dr(t) = a(r(t))dt + b(r(t))dW(t), \quad r(0) = r_0$$

where a and b are deterministic functions such that a unique strong solution r exists. Note that in this model r is a Markov-process.

The available assets are a bank account and a zero-coupon bond maturing at time T' , called the T' -bond. The value process B of the bank account is given by

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1.$$

We assume that \mathbb{Q} is the risk-neutral measure. The price $Y(s)$ at time s of any \mathcal{F}_t -measurable contingent claim $Y(t)$ is given by the risk-neutral pricing formula

$$Y(s) = \mathbb{E} \left[\frac{B(s)}{B(t)} Y(t) \mid \mathcal{F}_s \right] \quad \forall s, t \in [0, T'].$$

In particular, the price process P of the T' -bond is given by

$$P(t) = \mathbb{E} \left[\frac{B(t)}{B(T')} \mid \mathcal{F}_t \right] \quad \forall t \in [0, T'].$$

We remark that it might be unrealistic to assume that a bond with maturity T' always exists. However, since a one-factor model is considered we could alternatively assume that a “rolling bond” exists as in Boulier et al. (2001). This would not change our results.

It is well known that this financial market is complete: for any \mathcal{F}_t -measurable contingent claim $Y(t)$ there exists a self-financing portfolio strategy $\xi(t) = (\xi_0(t), \xi_1(t))$ such that $V_\xi(t) = Y(t)$, that is, there exists a previsible process ξ such that for all $t \in [0, T']$ holds

$$\begin{aligned} Y(t) &= V_\xi(t) = \xi_0(t)B(t) + \xi_1(t)P(t) \\ &= Y(0) + \int_0^t \xi_0(u)dB(u) + \int_0^t \xi_1(u)dP(u). \end{aligned} \tag{1}$$

To include non-hedgeable risks, we consider a discrete-time stochastic process $\Lambda = \{\Lambda(t)\}_{t=0, \dots, T'}$. The growth factor of this process during the period $[t-1, t]$ is denoted by

$$\lambda(t) = \Lambda(t) / \Lambda(t-1) \quad \forall t = 1, \dots, T'.$$

We assume that the growth factor in each period depends on the short rate $r(t)$ at the end of the period and a sequence $\gamma(t) = \{\gamma(t)\}_{t=1, \dots, T'}$ of independent and identically distributed random variables with values in \mathbb{R}^n that are defined on (Ω, \mathcal{F}, Q) . Furthermore, we assume that the sequence γ is independent of the Brownian motion W . The model for λ is therefore:

$$\lambda(t) = L(r(t), \gamma(t)) \quad (2)$$

where $L : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a known function. The discrete-time filtration generated by Λ is denoted by $\mathbb{L} = \{\mathcal{L}_t\}_{t \in [0, T']}$. We think of Λ as an index that might be related to the financial market but which also includes non-tradable risks, like mortality, inflation or other demographic and economic quantities that are important for pensions. The growth factor λ of this index will serve as the basis for the indexation of pension benefits as discussed in the following section. Since Λ is driven by some non-tradable risks, contingent claims depending on Λ can, in general, not be hedged, which makes the financial market incomplete.

3. CONDITIONAL INDEXATION

To introduce CI, we consider a worker who wishes to retire at time $T \in \mathbb{N}$ with $T < T'$ (in T years time from now). As mentioned in the introduction, we assume that she decides to make a single contribution now, denoted by $V(0)$, to become a member of a pension fund and to be entitled to a lump-sum payment, denoted by $X(T)$, at retirement. The initial contribution is invested into a portfolio consisting of the available financial assets B and P , and the pension fund is then managed on her behalf.

Depending on the design of the pension plan, the received payments at retirement will vary significantly. On the one hand a defined contribution plan (DC) provides a pension payment $X(T)$ that is equal to the value of the pension fund $V(T)$ at retirement. Since we consider a single contribution now, $V(T)$ will usually be the final value of a self-financing strategy ξ . The pensioner therefore faces the investment risk, that is, the risk of a bad performance of the pension fund resulting in a low value of pension benefits $X(T) = V(T)$.

On the other hand, a defined benefit plan (DB) will provide a defined payment at retirement, which is usually linked to some index. For example, a defined benefit pension linked to an inflation index would provide a fixed real value rather than a fixed nominal value of the benefits $X(T)$ paid at retirement.

However, the pension payment of a DB scheme is usually not linked to the final fund value $V(T)$.

As mentioned in the introduction, CI has been discussed in the literature as an alternative to pure DC or DB schemes. de Jong (2008) considers a CI scheme where indexation is conditional on the asset-liability ratio (funding ratio) of the fund to be larger than 1. He points out: “As the funding ratio depends on the investment policy, we now get a complicated interaction between investment policy and the value of the pension deal”. He does not explicitly address this interaction but considers a number of possible investment policies for the pension fund to study the effect of investment decisions on the value of pension deals.

Dai and Schumacher (2009b) consider a Black-Scholes financial market and a stylized CI scheme. In their setting there is no non-hedgeable risk. To build a CI scheme they consider a digital option where the lower payout is adjusted continuously according to the stock market performance. If we assume that the pension fund actually invests into these digital options, then we observe again a feedback effect that is similar to the one that de Jong (2008) mentions, since any change in the stock-market index results simultaneously in a change of the fund value and the liabilities. Rather than adjusting benefits continuously, we assume that indexation takes place at the discrete times $t = 1, \dots, T$ where new information about the index Λ becomes available.

We propose to make the indexation of benefits at the end of any period dependent on the performance of the pension fund during the preceding period rather than the actual fund value to avoid that possible payments into the fund at indexation times affect the calculation of benefits. Furthermore, indexation of benefits conditionally on the performance avoids any effects of possible initial mispricing. If CI is related to the fund value, then benefits could be low because the initial contribution turns out to be too low years after it has been made. We also note that the fund performance and the fund value are perfectly dependent on each other when the pension fund is a self-financing portfolio.

Since we are dealing with non-hedgeable risks we assume that the pension fund provider or sponsor is ready to increase or reduce the fund size at times $t = 1, \dots, T$ when new information about Λ becomes available and guaranteed pension benefits are adjusted. However, since the financial market is complete during each period $[t, t + 1)$ we assume that the portfolio process V is self-financing during each period, and, given the financial market model, V is almost surely continuous during $[t, t + 1)$. This allows us to introduce the notation

$$V_{-}(t) = \lim_{s \rightarrow t, s < t} V(s) \quad \forall t = 1, \dots, T$$

for the value of the fund just before t , that is, the value of the fund at the end of the period $[t - 1, t)$ before any extra payment is made at time t . We then obtain that

$$V(t) = E_t \left[\frac{B(t)}{B(t+1)} V_{-}(t+1) \mid \mathcal{L}_t \right] \quad \forall t = 0, \dots, T-1$$

where E_t denotes the conditional expectation under Q given \mathcal{F}_t , that is, the information generated by the financial market not including the information generated by the index Λ , which is captured by \mathcal{L}_t .

To make the concept of CI precise, we consider the growth factors $\lambda(t)$ of the index process Λ introduced in section 2. As mentioned there, indexation of pension benefits will be based on λ , which might be the growth factor of a consumer price index, wage index, mortality index or a combination of these.

For a given pension fund V and a process λ defined in (2) we now define the process $\{X(t), t = 0, \dots, T\}$ by

$$\begin{aligned} X(0) &= x_0, \\ X(t+1) &= X(t)H\left(\frac{V_-(t+1)}{V(t)}, \lambda(t+1)\right) \quad \forall t = 0, \dots, T-1 \end{aligned} \quad (3)$$

for a positive real valued function $H: \mathbb{R}^2 \mapsto \mathbb{R}_+$ and a constant $x_0 \in \mathbb{R}_+$. We will call H the CI-function and x_0 the nominal initial pension benefit. The actual pension benefit paid at retirement is the final value $X(T)$ of X . Note that the CI-function H is similar to the contract function of a With-Profits contract as used in Kleinow (2009) which we would obtain if λ was constant rather than random.

Our economic interpretation of H is the following: A worker who invests an amount $V(0)$ into a pension fund today will initially be guaranteed to receive a benefit of x_0 at the time of retirement T . For the next T years, this guaranteed benefit is then adjusted at the end of each year according to the CI-function H , and the growth factor λ of the underlying index and the performance of the fund during the preceding year. As mentioned earlier an increase or decrease of the fund size at indexation times $1, \dots, T$ due to payments into or out of the fund has no impact on indexation, since H is a function of the growth factor of the fund calculated before any payments at the end of a period. To illustrate this idea further we provide the following examples.

Example 1. *In a DB scheme, the benefit paid at retirement is independent of the performance of the pension fund. Full indexation would mean that the CI-function is of the form $H_1(v, l) = l$ and the amount paid at time T is $X(T) = x_0 \Lambda(T) / \Lambda(0)$.*

Example 2. *In a pure DC scheme, the benefit paid at retirement is independent of the index Λ . In this case we have $H_2(v, l) = v$ and $X(T) = V(T)$ where V is a self-financing portfolio.*

Example 3. *Another example for a CI-function is*

$$H_3(v, l) = \min\{v^\delta, l\} \quad \text{with } \delta \in (0, 1).$$

In such a CI scheme benefits grow with the same rate as the index Λ if the performance of the fund in the preceding year was sufficiently good. Choosing $\delta < 1$

means on one hand that the growth rate of guaranteed benefits is less than the return on the fund if this return was positive. On the other hand a negative return on the fund reduces the value of guaranteed benefits but that reduction is less than in the case of $\delta = 1$.

Example 4. In our framework we can also allow for an indexation scheme where full indexation is guaranteed until retirement (DB pension), and in addition the pension fund member participates to some extent in the success of the investment strategy applied by the pension fund. This arrangement is obtained by choosing

$$H_4(v, l) = \max\{v^\delta, l\} \text{ for } \delta \in (0, 1).$$

These examples are not exclusive and the CI-function could have many different forms. However, we have to restrict the set of possible CI-functions to make them economically meaningful and to be able to prove the results in the remainder to this paper. Therefore, we make the following two assumptions about the CI-function H .

- (A1) $H(v, l)$ is continuous and non-decreasing in v for all $l \in \mathbb{R}$,
- (A2) $v/H(v, l)$ is strictly increasing, $\lim_{v \rightarrow 0} v/H(v, l) = 0$ and $\lim_{v \rightarrow \infty} v/H(v, l) = \infty$ for all $l > 0$.

The first assumption, (A1), says that a relatively large return on the pension fund during one period will result in a higher growth rate of guaranteed pension benefits at the end of that period than a lower return.

The methods we apply later require the second assumption, (A2), to hold. For an economic interpretation, we argue that $v/H(v, \lambda)$ should not be decreasing since relatively high returns on the pension fund assets should improve the solvency of the fund rather than resulting in an over-proportional increase in the liabilities which would weaken the solvency of the fund. Note, that the CI-function H_2 in example 2 does not fulfil assumption (A2). However, this particular CI-function is not relevant for this paper since the sponsor or the management of a DC pension scheme faces no shortfall risk. Assumption (A2) is also the reason for restricting δ to $(0, 1)$ in examples 3 and 4.

Since we focus on the financial risks related to final pension payments, we assume that the only liability of the pension fund is the benefit payment $X(T)$ at retirement. We therefore ignore surrender options or other options and guarantees embedded in some pension products. Note, however, that various risk factors affecting pension funds, in particular mortality and inflation, can be considered by choosing an appropriate model for λ in (2) as mentioned in section 2.

4. MANAGING THE FUND

From the definition of the pension benefits $X(T)$ in (3) it is obvious that $X(T)$ is the payoff of a path-dependent contingent claim with underlying risk processes

V and Λ . A market-consistent value of $X(T)$ at any time $t < T$ could therefore be obtained by applying risk-neutral valuation. This would, however, require to know the probability law of the pension fund process V under Q . We could proceed by assuming a particular form for V , for example a geometric Brownian motion. However, this approach is not consistent with our assumption that the pension fund is actively managed. Also, pricing is not the only issue here. The pension fund management is also interested in hedging their risks. Assuming a particular model for V and then using risk neutral valuation would imply that pension fund members are charged a risk premium which is not invested into the fund but in a separate hedge portfolio. We want to avoid an additional hedge portfolio and assume that the full contribution $V(0)$ is invested into the pension fund. In the following we aim to find an investment strategy for the pension fund such that the shortfall risk of the fund is reduced according to a quadratic criterion. We therefore consider risk management objectives for making investment decisions. There are alternative methods of finding an optimal strategy for a pension fund. These are mainly based on utility maximization, for example Cairns et al. (2006) for a DC pension, or prospect theory, for example Dai and Schumacher (2009b), rather than risk management.

Instead of assuming a particular model for V we will proceed by constructing the pension fund V such that the shortfall risk associated with the liabilities $X(T)$ is reduced as far as possible. To eliminate the risk completely we would need to find a self-financing portfolio ξ with value process V_ξ , such that V_ξ/B is a Q -martingale and

$$Q[V_\xi(T) = X(T)] = 1. \quad (4)$$

Such a portfolio ξ exists if Λ is deterministic as shown in Kleinow (2009). However, since we assume that the index Λ depends on some non-tradable risks, such a self-financing portfolio will, in general, not exist in the situation we consider here. We choose instead to apply a quadratic hedging approach in each period $[t, t + 1)$, starting with the last period and then working backwards. This leads to the following definition.

Definition 1. *We call a process V an optimal, or risk-minimizing, pension fund if*

- (i) $Q[V(T) = X(T)] = 1$,
- (ii) $V(t)$ is $\mathcal{F}_t \otimes \mathcal{L}_t$ -measurable for all $t = 0, \dots, T$,
and for all $t = 1, \dots, T$ holds
- (iii) $V_-(t) = \arg \min_{Y \in \mathcal{F}_t \otimes \mathcal{L}_{t-1}} E_t[\{Y - V(t)\}^2 \mid \mathcal{L}_{t-1}]$.
- (iv) $V(s) = E_s \left[\frac{B(s)}{B(t)} V_-(t) \mid \mathcal{L}_{t-1} \right]$ for all $s \in [t-1, t)$

Part (i) in this definition means that the pension fund provider wishes to hedge the risks associated with the pension benefits $X(T)$ by choosing an appropriate

investment strategy for the fund with $V(T) = X(T)$ rather than setting up a separate hedge portfolio. Since r is Markovian and due to our model for λ in (2), the second part (ii) of the above definition says that the required value of the pension fund at time t is a deterministic function of the index value $\Lambda(t)$ and the short rate $r(t)$ at that time. Part (iii) says that the investment strategy in each period is chosen such that the hedging error at the end of the period is minimized given the information about the index at the beginning of the period. Note that $V_-(t)$ is by definition $\mathcal{F}_t \otimes \mathcal{L}_{t-1}$ measurable. Again, since r is Markovian and due to (2) we obtain that $V_-(t)$ is a function of $r(t)$ and $\Lambda(t-1)$. Also note that (iii) is equivalent to

$$V_-(t) = E_t[V(t) | \mathcal{L}_{t-1}] \quad (5)$$

since for every $Y \in \mathcal{F}_t \otimes \mathcal{L}_{t-1}$ we have

$$\begin{aligned} E_t[\{Y - V(t)\}^2 | \mathcal{L}_{t-1}] &= E_t[\{Y - E_t[V(t) | \mathcal{L}_{t-1}]\}^2 | \mathcal{L}_{t-1}] \\ &\quad + E_t[\{E_t[V(t) | \mathcal{L}_{t-1}] - V(t)\}^2 | \mathcal{L}_{t-1}] \end{aligned}$$

Since the first term in the above equation is non-negative, the minimum in part (iii) of definition 1 is obtained by the conditional expectation in (5). Since the financial market is complete during each period, $V_-(t)$ is attainable. Finally, part (iv) reflects the fact that the investment strategy should be self-financing during each period and, therefore, the available funds $V(t-1)$ at the beginning of a period should be the risk-neutral price of the fund value $V_-(t)$ at the end of that period before any extra payments are made.

Definition 1 motivates the following economic interpretation of $V(t)$ and $V_-(t)$. We think of $V(t)$ as the required funding at time t , and $V_-(t)$ represents the available funding at that time. In particular, at retirement the required funding is $V(T) = X(T)$. However, $X(T)$ depends on the available funds $V_-(T)$ through equation (3). This feedback effect is not only observed at retirement but at any indexation time. The aim of the fund management is now to minimize the squared difference between required and available funds by choosing an appropriate investment strategy.

Before we can show that an optimal pension fund exists and how it can be obtained we need to introduce some notation. The notation here is similar to the notation used in Kleinow (2009). We first define the functions

$$h(v, r(t)) = E_t[H(v, \lambda(t))] = E_t[H(v, \lambda(t)) | \mathcal{L}_{t-1}] \quad \forall v \in \mathbb{R} \quad (6)$$

and

$$g(x, v_0, \rho, v) = \frac{v}{xh(v/v_0, \rho)} \quad (7)$$

where the second equality in (6) holds due to our model for λ in (2), and since γ is a sequence of independent and identically distributed random variables. Note that g has an economic interpretation:

$$g(X(t-1), V(t-1), r(t), V_-(t)) = \frac{V_-(t)}{E_t[X(t) | \mathcal{L}_{t-1}]}$$

is the ratio of available funds at time t (before extra payments) and the expected guaranteed benefits $X(t)$ at t , where the expectation is conditional on $X(t-1)$ and the information generated by the financial market up to time t . It now follows from assumption (A2) that g is invertible with respect to its last argument:

$$c = g(x, v_0, \rho, v) \iff v = g^{-1}(x, v_0, \rho, c) \quad \forall x, v_0, \rho, v, c > 0. \quad (8)$$

Note that $g(\alpha x, \alpha v_0, \rho, \alpha v) = g(x, v_0, \rho, v)$ for all $\alpha \neq 0$ and therefore g^{-1} as defined in (8) has the property

$$g^{-1}\left(x, v_0, \rho, c\right) = x g^{-1}\left(1, \frac{v_0}{x}, \rho, c\right) \quad \forall x, v_0, \rho, c > 0. \quad (9)$$

With these definitions we can now formulate our main result about hedging CI pension schemes.

Theorem 1. *1. There exists a \mathbb{F} -adapted process C which is independent of the sequence γ such that*

(C1) $C(T) = 1$ a.s. and

(C2) $C(t) = E_t\left[\frac{B(t)}{B(t+1)} g^{-1}(1, C(t), r(t+1), C(t+1))\right]$ for all $t = 0, \dots, T-1$.

2. Given this process C an optimal pension fund as defined in definition 1 is given by

a) $V(0) = x_0 C(0)$,

b) for all $t = 1, \dots, T$:

$$V_-(t) = g^{-1}(X(t-1), V(t-1), r(t), C(t)) \quad (10)$$

$$X(t) = X(t-1) H(V_-(t) / V(t-1), \lambda(t)) \quad (11)$$

$$V(t) = X(t) C(t) \text{ and} \quad (12)$$

c) $V(s) = E_s\left[\frac{B(s)}{B(t)} V_-(t)\right]$ for all $s \in [t-1, t)$ and all $t = 1, \dots, T$.

This theorem and its proof are similar to results obtained in Kleionow (2009). The main difference is that we now face some non-hedgable risks. The proof is provided in the appendix.

Similar to our economic interpretation of g and motivated by (12), we think of the process $C(t)$ as the required “asset-liability ratio” at time t although $X(t)$ does not represent the full liabilities of the fund at that time. Alternatively, we can consider $C(t)$ to be the relative value of the pension scheme. Similarly, $V(t)$ can be thought of as the required value or target value of the fund at t . A particular consequence of theorem 1 is that the initial contribution paid into the pension fund should be $V(0) = x_0 C(0)$. Furthermore, given this initial contribution the fund managers will invest $V(0)$ into the replicating (self-financing) portfolio of a contingent claim with maturity 1 and payoff function

$$V_-(1) = g^{-1}(x_0, V(0), r(1), C(1)).$$

At time 1 the management will then declare $X(1)$ according to the CI-function, adjust the fund size from $V_-(1)$ to $V(1) = X(1)C(1)$ and invest it into a self-financing portfolio with value function

$$V_-(2) = g^{-1}(X(1), V(1), r(2), C(2))$$

at time 2. They will then follow this procedure until retirement at time T .

It should be noted that for a given financial market model, the investment strategy is completely determined by the CI-function and the objective to invest into an optimal pension fund as defined in definition 1. This can be compared to a strategy which is determined by a utility function and the objective to maximize expected utility. In that sense the CI-function describes preferences of pension fund members in terms of guarantees or profit-participation rules rather than utility. The link between the choice of a CI-scheme and preferences has been discussed by Dai and Schumacher (2009b) to some extent.

The payment to be made into the fund at any indexation time $t = 1, \dots, T$ is $V(t) - V_-(t)$, which could be negative (a surplus) or positive (a deficit). The conditional expected value of the payment at the next indexation time given the information one year earlier is

$$\mathbb{E}_{t-1}[V(t) - V_-(t) | \mathcal{L}_{t-1}] = \mathbb{E}_{t-1}[\mathbb{E}_t[V(t) - V_-(t) | \mathcal{L}_{t-1}]] = 0.$$

Although the proposed strategy is not self-financing, it is therefore self-financing on average. The economic question arising here is then: Who should pay in case of a deficit or a surplus at any time t ? This could be the company managing the pension fund, a sponsor (employer) or a pension fund member (employee). It is not the aim of this paper to provide a detailed discussion of this issue. However, the risk associated with a potentially high deficit arising at some point should have implications on the initial contribution made by the pension fund member, as there might be extra charges necessary to compensate for this risk. It should be noted that this issue reflects the incompleteness

of the market, since risks associated with the index Λ cannot be hedged completely.

Alternatively, the risk of large additional contributions can also be re-insured or securitized as mentioned in the introduction. As these additional contributions are the result of non-hedgable risks, they are not related to other risks in the financial market. This would mean that the premium for a re-insurance contract covering any additional contributions should be approximately the expected value of these contributions, which is zero. The proposed strategy therefore reduces the risk associated with CI to the “residual risk” that cannot be hedged.

Let us also mention that in the particular case in which a perfect hedge is possible, that is, $V(t) = V_-(t)$ for all indexation times t , we recover the results obtained by Kleinow (2009) for With-Profits life insurance policies.

5. EXAMPLE

We consider two examples for CI, namely examples 3 and 4 in section 3. The CI-functions are therefore

$$H_3(v, l) = \min \{v^\delta, l\} \text{ and } H_4(v, l) = \max \{v^\delta, l\} \quad (13)$$

with $\delta \in (0, 1)$.

To model the index Λ we assume a simple relationship between the short term interest rate and the growth rate of the index, given by

$$\lambda(t) = \exp(r(t) - \gamma(t)) \text{ with } \gamma(t) \sim N(\mu, \sigma_\gamma^2) \quad (14)$$

where we think of $\gamma(t)$ as the real interest rate, and of $\log \lambda(t)$ as the inflation rate. Note that we always assume that γ is a sequence of independent random variables, and that this might not be a good model for the real interest rate. However, it is our aim in this section to illustrate our main results, rather than to provide a realistic model for interest rates.

Our model for the short interest rate r is a Vasicek-model:

$$dr(t) = a(b - r(t))dt + \sigma dW(t) \quad (15)$$

where a , b and σ are positive constants and W is a Brownian motion under Q as in section 2.

To obtain the optimal pension fund we have to find the process C in theorem 1 first. Therefore, we have to solve equation (C2) in that theorem numerically. To this end we consider a discrete state space by defining a sequence of equidistant points $\tilde{r}_1, \dots, \tilde{r}_K \in \mathbb{R}$ and $\varepsilon = (\tilde{r}_{i+1} - \tilde{r}_i)/2$ as in Kleinow (2009). We then define the intervals $R_1 = (-\infty, \tilde{r}_1 + \varepsilon]$, $R_k = (\tilde{r}_k - \varepsilon, \infty)$ and $R_i = (\tilde{r}_i - \varepsilon, \tilde{r}_i + \varepsilon]$ for $i = 2, \dots, K - 1$.

Since $C(T) = 1$ and the interest rate model is Markovian, $C(t)$ will be a deterministic function of $r(t)$ for all $t = 0, \dots, T-1$. We are therefore using the notation $C(t) = C(t, r(t))$.

Using the sequence \tilde{r} we now approximate $C(t, r(t))$:

$$C(t, r(t)) \approx \tilde{C}(t, r(t)) = \sum_{i=1}^K C(t, \tilde{r}_i) 1_{\{r(t) \in R_i\}}.$$

For $t = T$ we set $C(T, \tilde{r}_i) = 1$ for all \tilde{r}_i , and we use the following approximation for (C2) in theorem 1

$$\begin{aligned} C(t, \tilde{r}_i) &= \mathbb{E} \left[\exp \left(- \int_t^{t+1} r(s) ds \right) \right. \\ &\quad \left. g^{-1} \left(1, C(t, \tilde{r}_i), r(t+1), C(t+1) \right) \mid r(t) = \tilde{r}_i \right] \\ &\approx \sum_{j=1}^K g^{-1} \left(1, C(t, \tilde{r}_i), \tilde{r}_j, C(t+1, \tilde{r}_j) \right) w_{ij} p_{ij} \end{aligned} \quad (16)$$

with

$$w_{ij} = \mathbb{E} \left[\exp \left(- \int_t^{t+1} r(s) ds \right) \mid r(t) = \tilde{r}_i, r(t+1) = \tilde{r}_j, \right] \quad (17)$$

for $i, j = 1, \dots, K$, and

$$p_{ij} = \mathbb{Q}[r(t+1) \in R_j \mid r(t) = \tilde{r}_i] \quad \forall j = 1, \dots, K.$$

We now have to solve (16) for every t and every i . The derivation of an explicit formula for w_{ij} can be found in the appendix.

We now consider the function h defined in (6). As shown in the appendix, we obtain for our particular examples

$$\begin{aligned} h_3(v, r(t)) &= \mathbb{E}_t[H_3(v, \lambda(t))] = \mathbb{E}_t[\min\{v^\delta, \lambda(t)\}] \\ &= v^\delta \Phi \left(\frac{r(t) - \delta \ln v - \mu}{\sigma_\gamma} \right) \\ &\quad + \exp \left(r(t) + \frac{\sigma_\gamma^2}{2} - \mu \right) \left[1 - \Phi \left(\frac{r(t) - \delta \ln v - (\mu - \sigma_\gamma^2)}{\sigma_\gamma} \right) \right] \end{aligned} \quad (18)$$

$$\begin{aligned} h_4(v, r(t)) &= \mathbb{E}_t[H_4(v, \lambda(t))] = \mathbb{E}_t[\max\{v^\delta, \lambda(t)\}] \\ &= v^\delta \left[1 - \Phi \left(\frac{r(t) - \delta \ln v - \mu}{\sigma_\gamma} \right) \right] \\ &\quad + \exp \left(r(t) + \frac{\sigma_\gamma^2}{2} - \mu \right) \Phi \left(\frac{r(t) - \delta \ln v - (\mu - \sigma_\gamma^2)}{\sigma_\gamma} \right) \end{aligned} \quad (19)$$

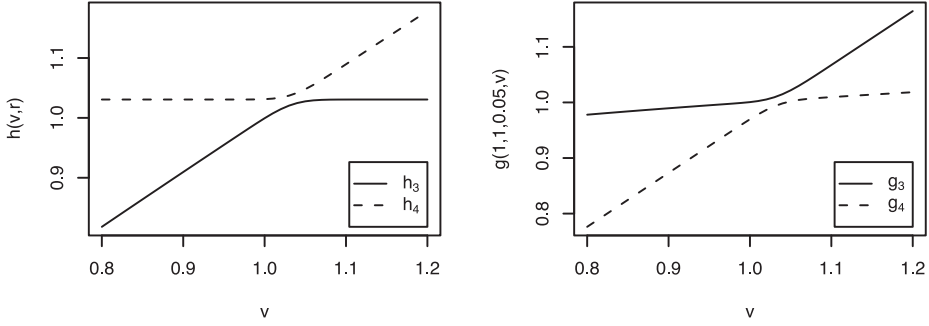


FIGURE 1: The functions h and g for the two examples H_3 and H_4 .
The value of the short rate is $r = 0.05$, and $x = v_0 = 1$.

where Φ is the distribution function of the standard normal distribution.

We now proceed by defining g_3 and g_4 as in (7),

$$g_3(x, v_0, \rho, v) = \frac{v}{xh_3(v/v_0, \rho)} \quad \text{and} \quad g_4(x, v_0, \rho, v) = \frac{v}{xh_4(v/v_0, \rho)}.$$

Although g_3 and g_4 are invertible with respect to v , we cannot find an explicit formula for the inverse functions. In our examples we will therefore invert them numerically.

For our numerical illustrations we choose the following parameters for the short rate:

$$a = 0.1, \quad b = 0.05, \quad \sigma = 0.02,$$

and

$$\tilde{r}_i = \frac{1}{100} [-10 + i/200] \quad \text{for } i = 0, \dots, 80$$

which corresponds to possible value of $r(t)$ between -10% and 30% . For the index Λ and its growth rate defined in (14) we only have to choose the parameters for γ , which are

$$\mu = 0.02 \quad \text{and} \quad \sigma_\gamma = 0.01.$$

Furthermore, we consider a pension payable in $T = 40$ years, and choose $\delta = 0.9$ in (13). Figure 1 shows the functions h and g for our examples.

We now numerically solve (16) for $t = T - 1$ using $C(t + 1, \tilde{r}_j) = 1$ for all $j = 0, \dots, 80$. We then go backwards until $t = 0$. In each step we solve (16) for each value of $\tilde{r}_0, \dots, \tilde{r}_{80}$. In this way we obtain the values of $C(t, \tilde{r}_i)$ for all $t = 0, \dots, 40$ and $i = 0, \dots, 80$. In Figure 2 we have plotted $C(0, r)$ and $C(5, r)$ as a function of the short rate $r(0)$ and $r(5)$ respectively. The left plot shows the

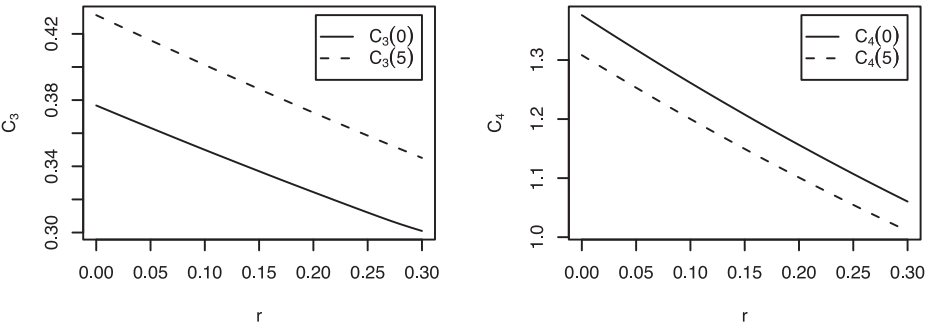


FIGURE 2: The functions $C(0, r)$ and $C(5, r)$ as a function of the short term interest rate r at time 0 and time 5, respectively. On the left-hand side C_3 is considered, corresponding to the CI function H_3 . Results for H_4 are shown on the right-hand side.

values for the CI-function H_3 . The corresponding functions for H_4 are plotted in the right plot.

It is apparent from that figure that the relative value of the pension deal, or the relative “asset-liability ratio”, $C = V/X$ is a decreasing function of the interest rate. However, the change in C due to a change in the remaining time to retirement depends on the particular CI function H . For H_3 , the relative value of the pension deal is increasing during the working life starting on a relatively low level, while the relative value of a pension deal with CI function H_4 is actually decreasing starting on a relatively high level. This reflects the guarantee given by the CI function H_4 where guaranteed benefits can not decrease in contrast to H_3 where there is no lower limit to benefits.

When the two relative value functions C , corresponding to H_3 and H_4 in figure 2 are compared, one should also look at the levels of C . It is apparent that a CI pension scheme with CI-function H_4 is about three to four times as expensive as a scheme with CI-function H_3 . The reason is, that H_4 provides a guarantee: the increase in benefits is at least equal to the increase in the index. On the other hand, the increase in the index is an upper bound for the increase in benefits in schemes with CI-function H_3 . Therefore, H_4 provides a larger payoff than H_3 for the same initial pension promise $X(0)$, and we expect the value of H_4 to be higher.

As an illustration for the proposed management strategy we first consider one particular interest rate scenario and a particular realisation of the “real interest rate” process γ before turning to a simulation study with 10,000 scenarios. For this scenario we plot the realised paths of the index Λ , the required fund value V and the corresponding declared benefits X . We set $X(0) = 1$, $\Lambda(0) = 1$ and $r(0) = 0.04$. For the CI-function H_3 we obtain $V(0) = C(0) = 0.3659$, see the left plot in figure 2, and $V(0) = C(0) = 1.3292$ for H_4 .

In the top row in figure 3 we find the trajectories of V , X and Λ for the two CI-functions. It can be seen from these plots that the benefits paid at retirement are equal to the target fund value $V(T)$ for both CI-functions. This

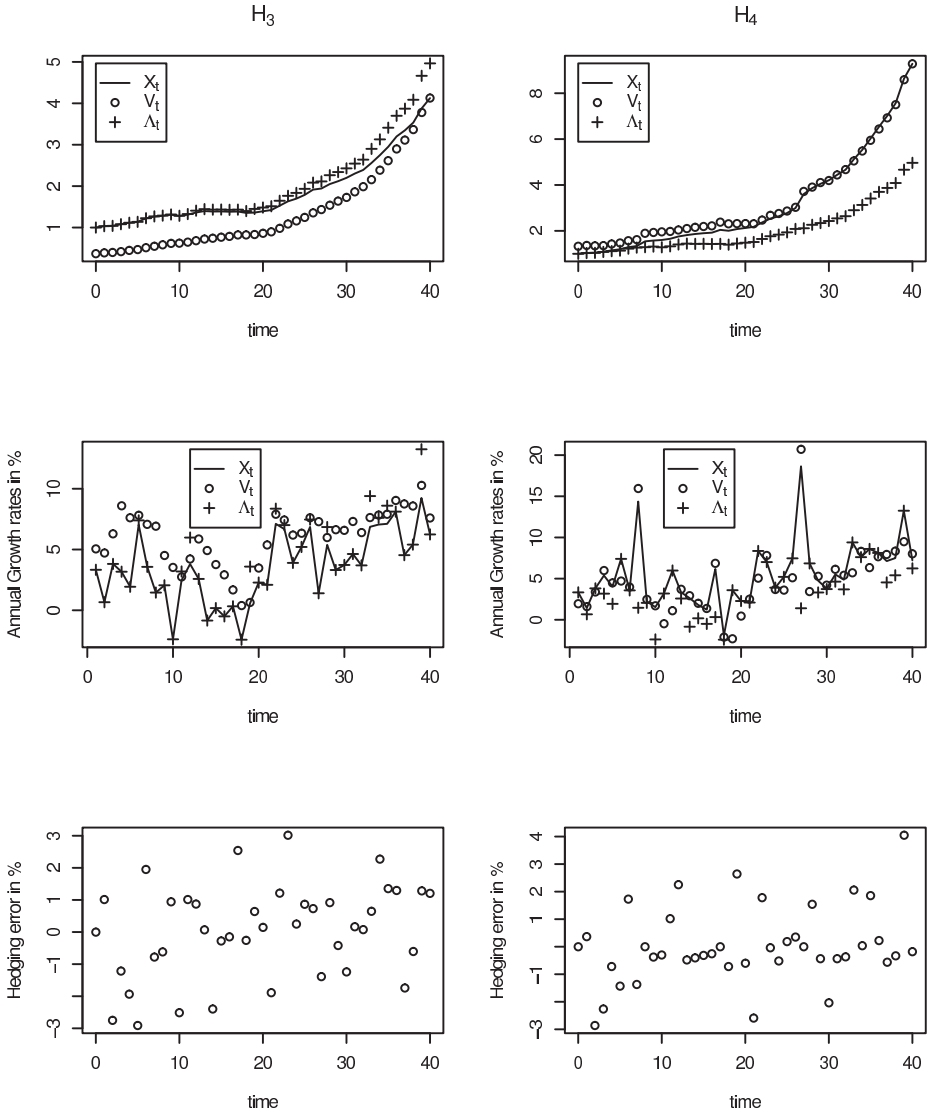


FIGURE 3: The processes X , V and Λ for a particular path of interest rates and a particular realisation of $\gamma(1), \dots, \gamma(40)$. The plots in the bottom row show the hedging error in percent.

reflects part (i) of definition 1. We also see that due to the particular CI-functions, the final benefits corresponding to H_3 (left plot) are lower than the final value of the inflation index Λ , while benefits for H_4 are higher.

The middle row in Figure 3 shows the growth rates of V , X and Λ . Let us first consider H_3 (left plot). It can be seen that the growth rate of declared benefits X is equal to the inflation rate in years in which the return on the fund is sufficiently high. However, there are years where the fund has performed

badly as compared to the index Λ , see for example year 12. In those years the increase in declared benefits is limited by the fund performance according to H_3 . The picture is different for H_4 (right plot). In that plot we see that the growth rates of X and Λ are equal whenever the fund underperforms relative to Λ , while X grows faster than Λ if the fund performs sufficiently well.

The bottom row in figure 3 shows the hedging error at indexation times as a percentage of the available funds $V_-(t)$ at time t , that is, the hedging error $E(t)$ as shown in that figure is defined by

$$E(t) = 100 \frac{V(t) - V_-(t)}{V_-(t)} \text{ for } t = 1, \dots, 40.$$

It is apparent from that figure that the mismatch between available funds $V_-(t)$ and required funds $V(t)$ is rather small, and that our strategy appears indeed to be self-financing on average as discussed at the end of section 4.

To shed further light on the behaviour of the proposed investment strategy we simulate 10000 scenarios for the next forty years. Each scenario consists of a trajectory for the short rate process r and the index Λ . In this simulation study we are particularly interested in the hedging errors $E(t)$ and the annual return achieved by pension fund members. We therefore calculate the following quantities for each simulated scenario:

- the average return r_V per annum over the forty years in percent,

$$r_V = 100 \frac{\log V(40) - \log V(0)}{40}.$$

Note that this is the return obtained for an initial investment of $V(0)$ ignoring surpluses or deficits at indexation times. Assuming that the pension fund member makes the initial contribution $V(0)$ and receives the pension benefit $X(T)$, the average return r_V is the annual return obtained by the member. However, r_V is not the return of a self-financing portfolio.

- the maximal hedging error $E_{\max} = \max \{E(t), t = 1, \dots, 40\}$,
- the minimal hedging error $E_{\min} = \min \{E(t), t = 1, \dots, 40\}$,
- the empirical correlation $\text{Corr}(E(t), E(t-1))$, and
- the accumulated value E_{accu} of all hedging errors as a percentage of the final fund size $V(40)$,

$$E_{\text{accu}} = \frac{100}{V(40)} \sum_{t=1}^{40} \frac{V(t) - V_-(t)}{P(t, 40)}$$

where $P(t, 40)$ is the price at time t of a zero-coupon bond with maturity $T = 40$.

These quantities are all random variables since their obtained values in a particular scenario depend on the path of r and the path of Λ observed in this

TABLE 1

RESULTS FOR THE HEDGING ERROR AND THE RETURN OBTAINED BY PENSION FUND MEMBERS
FROM 10000 SIMULATED SCENARIOS.

		r_V	E_{\max}	E_{\min}	E_{accu}	$\text{Corr}(E(t), E(t-1))$
H_3	Mean	4.71	2.94	-3.96	-1.17	-0.0206
	St. dev.	2.37	0.61	0.93	12.41	0.2197
H_4	Mean	4.60	3.66	-2.50	-0.66	-0.0247
	St. dev.	1.96	1.06	0.63	8.43	0.1707

scenario. Using the 10000 values that we obtain for each of the above quantities we calculate their averages and standard deviations over all scenarios, and report these in table 1. These numbers provide estimates for the corresponding expected values, for example $E[r_V]$, and standard deviations.

From table 1 we would estimate that the expected value $E[r_V]$ of the average annual return over the next forty years for CI function H_3 is 4.71 percent per annum, and the standard deviation of r_V is 2.37 percent. The standard error of the estimator for $E[r_V]$ is therefore approximately $2.37/\sqrt{1000} = 0.0237$.

We find that the average return obtained in a CI scheme with indexation function H_3 (4.71) is slightly higher than the one obtained from using H_4 (4.60). However, the same applies to the standard deviations of the average returns. These returns should also be compared to the yield of a zero-coupon bond. Using the Vasicek-model given in (15) with the parameters specified in (20) and $r(0) = 0.04$ we find that the annual yield of a zero-coupon bond with a maturity of 40 years is 3.49%, which is lower than the average return we obtained for the pension funds in our simulation.

From a risk management point of view, the maximal hedging error is of particular importance. For the average scenario we find that this is less than 4% of the available funds at any indexation time during the next 40 years. This seems to be a rather small value indicating that the hedging strategy works well. This conclusion is also supported by the minimum hedging error and the accumulated value E_{accu} of all hedging errors. In particular, E_{accu} is not significantly different from zero reflecting the fact that the proposed hedging strategy is self-financing on average. A further consequence of this property is that the hedging errors are virtually uncorrelated as documented in the last column of table 1.

We will now compare the two indexation schemes with respect to the return to pension fund members obtained in different interest rate scenarios. To this end we consider the average value \bar{r} of the short rate at indexation times in each scenario, that is

$$\bar{r} = \frac{1}{40} \sum_{t=1}^{40} r(t)$$

and compare it to the value of the average return r_V in each scenario. Figure 4 shows the scatter plots for the two considered CI-functions together with the

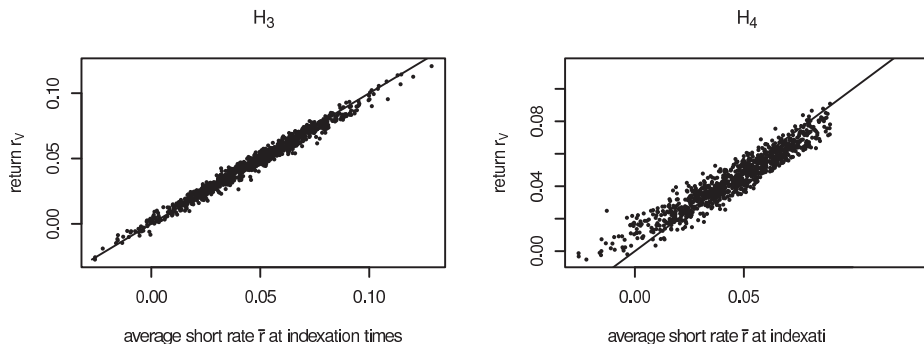


FIGURE 4: Scatter plots of the average return r_V to pension fund members and the average short rate at indexation times.

identity function (straight line). From this figure we notice that the average short rate during the next forty years varies between values just below 0 and about 12%. In particular, there are several scenarios among the 10000 simulated paths of r in which the average short rate \bar{r} is negative. This is due to the model chosen for the short rate process, and we will not exclude these scenarios from our analysis. As we can see in Figure 4, the return to members seems to be approximately the same as the average short rate. However, the variability of returns seems to be larger for CI-function H_4 than for H_3 . Although we find a larger standard deviation of returns resulting from H_3 in table 1, it seems that variation of returns for a given average interest rate \bar{r} is larger for CI-function H_4 . Another interesting feature is that for H_4 , the return r_V seems to be higher on average than \bar{r} for rather low values of \bar{r} and smaller for rather high values. This feature is less pronounced in schemes with CI-function H_3 . The study of these and other properties of particular CI-functions is beyond the scope of this paper, but some further simulation studies are needed to investigate these in more detail.

6. CONCLUSION AND FURTHER RESEARCH

The purpose of this paper was to introduce a new conditional indexation scheme for pensions and to show how the resulting risks can be managed by choosing an appropriate investment strategy for the pension fund. It appeared that no perfect match of pension fund assets and liabilities is possible with a self-financing portfolio since non-hedgeable risks have to be taken into account. However, it has been shown that a mean self-financing portfolio can be found that produces a perfect match between assets and liabilities on the cost of regular adjustments of the fund. The results show, in particular, how the feedback effect that is often found in CI schemes can be addressed, and can actually be used to reduce the risk of underfunding of pensions.

Our financial market model as well as the CI schemes considered in our simulation study are both very simple. Future research could be focused on deriving similar results for more realistic models. Since equity is an important asset class, “market rates of return should increasingly play a large role in the retirement patterns of individuals”, (MacDonald and Cairns, 2009). Therefore, it might be of particular interest to include equity into either the CI-function H or the function L that was used to model the growth rate of the index Λ .

As already discussed at the end of section 4 there is also the issue of surpluses and deficits at indexation times. Although we have shown that the proposed strategy minimizes the quadratic hedging error, the incompleteness of the market prevents us from finding a perfect hedge. This means that someone has to pay for deficits or receives surpluses associated with the proposed investment strategy, and that the existence of this hedging error should have an impact on contributions made by pension fund members. This question is left for future research.

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APPENDIX

Proof of the first part of Theorem 1

Note that this proof is similar to the proof of lemma 1 in Kleinow (2009), and only minor modifications have been made.

We consider any $t \in \{1, 2, \dots, T\}$ and define $Q_t[A] = E_{t-1}[1_A]$ for all $A \in \mathcal{F}_t$ where 1 denotes the indicator function. Given \mathcal{F}_{t-1} , Q_t is a probability measure on (Ω, \mathcal{F}_t) . To prove the first result in the theorem it is sufficient to show that there exists a $c_0 \in \mathbb{R}_+$ such that

$$c_0 = E_{Q_t}[Dg^{-1}(1, c_0, r(t), C)] \quad (21)$$

where the discount factor $D = B(t-1)/B(t)$, and the short rate $r(t)$ and C are \mathcal{F}_t measurable random variables. Equation (21) holds if and only if

$$1 = E_{Q_t}\left[D \frac{1}{c_0} g^{-1}\left(1, c_0, r(t), C\right)\right] = E_{Q_t}\left[Dg^{-1}\left(\frac{1}{c_0}, 1, r(t), C\right)\right] \quad (22)$$

where the second equality follows from (9). Using the notation

$$G(\rho, v) = \frac{v}{h(v, \rho)} \quad \rho \in \mathbb{R}$$

we get

$$g\left(\frac{1}{c_0}, 1, \rho, v\right) = c_0 G(\rho, v) \quad \text{and} \quad g^{-1}\left(\frac{1}{c_0}, 1, \rho, C\right) = G^{-1}(\rho, C/c_0)$$

where $G^{-1}(\rho, \cdot)$ is the inverse function of $G(\rho, \cdot)$ and g^{-1} was defined in (8). It follows from Assumption (A2) that $G^{-1}(\rho, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is strictly increasing for every ρ with

$$\lim_{c \rightarrow \infty} G^{-1}(\rho, c) = \infty \quad \text{and} \quad \lim_{c \rightarrow 0} G^{-1}(\rho, c) = 0$$

for all ρ . For a strictly increasing sequence $\{c_n\}_{n=1,2,\dots}$ of positive real numbers with $c_n \rightarrow \infty$ we conclude with the monotone convergence theorem that

$$\begin{aligned} \lim_{c_n \rightarrow \infty} E_{Q_t}\left[Dg^{-1}\left(\frac{1}{c_n}, 1, r(t), C\right)\right] &= \lim_{c_n \rightarrow \infty} E_{Q_t}\left[DG^{-1}(r(t), C/c_n)\right] \\ &= E_{Q_t}\left[D \lim_{c_n \rightarrow \infty} G^{-1}(r(t), C/c_n)\right] = 0 \end{aligned}$$

Similarly, for a strictly decreasing sequence $\{c_n\}_{n=1,2,\dots}$ of positive real numbers with $c_n \rightarrow 0$ we find that

$$\begin{aligned} \lim_{c_n \rightarrow 0} E_{Q_t} \left[Dg^{-1} \left(\frac{1}{c_n}, 1, r(t), C \right) \right] &= \lim_{c_n \rightarrow 0} E_{Q_t} \left[DG^{-1}(r(t), C/c_n) \right] \\ &= E_{Q_t} \left[D \lim_{c_n \rightarrow 0} G^{-1}(r(t), C/c_n) \right] = \infty \end{aligned}$$

Since $G^{-1}(\rho, \cdot)$ is continuous for every ρ , $E_{Q_t} \left[Dg^{-1} \left(\frac{1}{c_0}, 1, r(t), C \right) \right]$ is continuous in c_0 and an application of the intermediate value theorem completes the proof.

Proof of the second part of Theorem 1

To prove the second part of the theorem we will show that the process V has all the criteria of an optimal strategy as defined in definition 1. Parts (i) and (ii) in that definition are fulfilled because of (CI) and (12).

For part (iii) note that it follows from (10) that

$$\begin{aligned} C(t) &= g[X(t-1), V(t-1), r(t), V_-(t)] \\ &= \frac{V_-(t)}{X(t-1) h(V_-(t)/V(t-1), r(t))} \end{aligned}$$

and, therefore,

$$\begin{aligned} V_-(t) &= X(t-1) h(V_-(t)/V(t-1), r(t)) C(t) \\ &= X(t-1) E_t [H(V_-(t)/V(t-1), \lambda(t)) | \mathcal{L}_{t-1}] C(t) \\ &= E_t [X(t-1) H(V_-(t)/V(t-1), \lambda(t)) C(t) | \mathcal{L}_{t-1}] \\ &= E_t [X(t) C(t) | \mathcal{L}_{t-1}] \\ &= E_t [V(t) | \mathcal{L}_{t-1}] \end{aligned}$$

where the third equality holds since $C(t)$ is \mathcal{F}_t -measurable. We therefore obtain (5) which is equivalent to part (iii) of definition 1.

To show part (iv) we note that

$$\begin{aligned} V(t) &= X(t) C(t) \\ &= X(t) E_t \left[\frac{B(t)}{B(t+1)} g^{-1}(1, C(t), r(t+1), C(t+1)) \right] \\ &= E_t \left[\frac{B(t)}{B(t+1)} g^{-1}(X(t), V(t), r(t+1), C(t+1)) | \mathcal{L}_t \right] \\ &= E_t \left[\frac{B(t)}{B(t+1)} V_-(t+1) | \mathcal{L}_t \right]. \end{aligned} \tag{23}$$

This and part 2. c) of the theorem complete the proof.

Derivation of the weights w_{ij} in (17)

It is well-known for the Vasicek-model, see for example Cairns (2004) p. 249, that, conditionally on $r(t) = \tilde{r}_i$, the random vector $(r(t+1), \int_t^{t+1} r(s)ds)$ is normally distributed with mean

$$\mu(i) = \begin{pmatrix} \mu_1(i) \\ \mu_2(i) \end{pmatrix} = \begin{pmatrix} b + (\tilde{r}_i - b)e^{-a} \\ b + (\tilde{r}_i - b)(1 - e^{-a})/a \end{pmatrix}$$

and covariance matrix

$$V = \begin{pmatrix} V_1 & V_{12} \\ V_{21} & V_2 \end{pmatrix} = \begin{pmatrix} \frac{\sigma^2(1 - e^{-2a})}{2a} & \frac{\sigma^2(1 - e^{-a})^2}{2a^2} \\ \frac{\sigma^2(1 - e^{-a})^2}{2a^2} & \frac{\sigma^2(4e^{-a} - e^{-2a} + 2a - 3)}{2a^3} \end{pmatrix}$$

It follows that the conditional distribution of $\int_t^{t+1} r(s)ds$ given $r(t) = \tilde{r}_i$ and $r(t+1) = \tilde{r}_j$ is a normal distribution with mean

$$\mu_3(i, j) = \mu_2(i) + \frac{V_{12}}{V_1} [\tilde{r}_j - \mu_1(i)]$$

and variance

$$V_3 = V_2 - \frac{V_{12}^2}{V_1}$$

Using the moment-generating function we obtain

$$w_{ij} = \exp\left(-\mu_3(i, j) + \frac{1}{2} V_3\right)$$

Explicit Form of h_3 and h_4 in (18) and (19)

Note that we will only consider h_4 here. The explicit form of h_3 in (18) follows with similar arguments. To obtain (19) note that

$$\begin{aligned} h_4(v, r(t)) &= E_t[\max\{v^\delta, \lambda(t)\}] \\ &= E_t[v^\delta 1_{\{\lambda(t) \leq v^\delta\}}] + E_t[\lambda(t) 1_{\{\lambda(t) > v^\delta\}}] \end{aligned} \tag{24}$$

and, since $\lambda(t) = \exp(r(t) - \gamma(t))$, we obtain

$$\lambda(t) \leq v^\delta \iff \gamma(t) \geq r(t) - \delta \ln v.$$

For the first term in (24) we obtain:

$$\begin{aligned}
 E_t \left[v^\delta 1_{\{\lambda(t) \leq v^\delta\}} \right] &= v^\delta Q \left[\gamma(t) \geq r(t) - \delta \ln v \mid r(t) \right] \\
 &= v^\delta Q \left[\frac{\gamma(t) - \mu}{\sigma_\gamma} \geq \frac{r(t) - \delta \ln v - \mu}{\sigma_\gamma} \mid r(t) \right] \\
 &= v^\delta \left[1 - \Phi \left(\frac{r(t) - \delta \ln v - \mu}{\sigma_\gamma} \right) \right]
 \end{aligned}$$

since $\gamma(t) \sim N(\mu, \sigma_\gamma^2)$ and r is a Markov process.

For the second term in (24) we obtain

$$\begin{aligned}
 E_t \left[\lambda(t) 1_{\{\lambda(t) > v^\delta\}} \right] &= E_t \left[\exp(r(t) - \gamma(t)) 1_{\{\gamma(t) < r(t) - \delta \ln v\}} \right] \\
 &= \int_{-\infty}^{r(t) - \delta \ln v} \exp(r(t) - z) f(z) dz \\
 &= \exp(r(t)) \int_{-\infty}^{r(t) - \delta \ln v} \exp(-z) f(z) dz.
 \end{aligned}$$

With some standard arguments used for calculating the moment-generating-function of a normally distributed random variable we obtain that

$$\begin{aligned}
 \int_{-\infty}^a \exp(-z) f(z) dz &= \frac{1}{\sqrt{2\pi\sigma_\gamma^2}} \int_{-\infty}^a \exp \left(\frac{-(z^2 - 2\mu z + \mu^2 + 2\sigma_\gamma^2 z)}{2\sigma_\gamma^2} \right) dz \\
 &= \frac{1}{\sqrt{2\pi\sigma_\gamma^2}} \int_{-\infty}^a \exp \left(\frac{-\left[z - (\mu - \sigma_\gamma^2) \right]^2 - \sigma_\gamma^4 + 2\mu\sigma_\gamma^2}{2\sigma_\gamma^2} \right) dz \\
 &= \exp \left(\frac{\sigma_\gamma^4 - 2\mu\sigma_\gamma^2}{2\sigma_\gamma^2} \right) \frac{1}{\sqrt{2\pi\sigma_\gamma^2}} \int_{-\infty}^a \exp \left(\frac{-\left[z - (\mu - \sigma_\gamma^2) \right]^2}{2\sigma_\gamma^2} \right) dz \\
 &= \exp \left(\frac{\sigma_\gamma^2}{2} - \mu \right) Q[\tilde{Z} \leq a]
 \end{aligned}$$

where $\tilde{Z} \sim N(\mu - \sigma_\gamma^2, \sigma_\gamma^2)$. For $a = r(t) - \delta \ln v$ we obtain for the second term in (24)

$$\begin{aligned}
 E_t \left[\lambda(t) 1_{\{\lambda(t) > v^\delta\}} \right] &= \exp(r(t)) \int_{-\infty}^{r(t) - \delta \ln v} \exp(-z) f(z) dz \\
 &= \exp \left(r(t) + \frac{\sigma_\gamma^2}{2} - \mu \right) Q[\tilde{Z} \leq r(t) - \delta \ln v] \\
 &= \exp \left(r(t) + \frac{\sigma_\gamma^2}{2} - \mu \right) \Phi \left(\frac{r(t) - \delta \ln v - (\mu - \sigma_\gamma^2)}{\sigma_\gamma} \right)
 \end{aligned}$$

and finally

$$h_4(v, r(t)) = v^\delta \left[1 - \Phi \left(\frac{r(t) - \delta \ln v - \mu}{\sigma_\gamma} \right) \right] \\ + \exp \left(r(t) + \frac{\sigma_\gamma^2}{2} - \mu \right) \Phi \left(\frac{r(t) - \delta \ln v - (\mu - \sigma_\gamma^2)}{\sigma_\gamma} \right)$$

as in (19).

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