LINEAR STOCHASTIC RESERVING METHODS

BY

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ABSTRACT

In this article we want to motivate and analyse a wide family of reserving models, called linear stochastic reserving methods (*LSRMs*). The main idea behind them is the assumption that the (conditionally) expected changes of claim properties during a development period are proportional to exposures which depend linearly on the past. This means the discussion about the choice of reserving methods can be based on heuristic reasons about exposures driving the claims development, which in our opinion is much better than a pure philosophic approach. Moreover, the assumptions of *LSRMs* do not include the independence of accident periods.

We will see that many common reserving methods, like the Chain-Ladder-Method, the Bornhuetter-Ferguson-Method and the Complementary-Loss-Ratio-Method, can be interpreted in this way. But using the *LSRM* framework you can do more. For instance you can couple different triangles via exposures. This leads to reserving methods which look at a whole bundle of triangles at once and use the information of all triangles in order to estimate the future development of each of them.

We will present unbiased estimators for the expected ultimate and estimators for the mean squared error of prediction, which may become an integral part of IFRS 4. Moreover, we will look at the one period solvency reserving risk, which already is an important part of Solvency II, and present a corresponding estimator.

Finally we will present two examples that illustrate some features of *LSRMs*.

KEYWORDS

Stochastic Reserving, Mean Squared Error of Prediction, Solvency Reserving Risk, Claims Development Result.

1. INTRODUCTION

A main task of actuaries is to analyse random claim properties and project their development. This often includes the combination of several sources of information, but most of the standard reserving models cannot properly combine such information. For instance, they only project payments or reported amounts separately, but cannot combine both. In recent years several authors have studied models that can be used in specific situations in order to analyse different claim properties simultaneously, see for instance Quarg-Mack [12], Halliwell [5], Dahms [3] and Wüthrich-Merz [11].

In this paper we will introduce a wide class of stochastic reserving methods that can deal with several claim properties simultaneously. The main idea behind them is the assumption that the (conditionally) expected changes of claim properties during a development period are proportional to exposures which depend linearly on the past of claim properties. Therefore, we will call such methods linear stochastic reserving methods or *LSRMs*. Another important property of *LSRMs* is that they allow for various dependencies of accident periods. Many of the classical reserving methods, like the Chain-Ladder-Method, the Complementary-Loss-Ratio-Method and the Bornhuetter-Ferguson-Method, are *LSRMs*, see Sections 2.1-2.4.

We will derive estimators for the ultimate outcome of claim properties (Section 3), analyse the overall uncertainty of these estimators (Section 4) and the one period uncertainty of the claims development result (Section 5). The analysis of the overall uncertainty may become an integral part of IFRS 4 and the analysis of the uncertainty of the claims development result already is an important part of Solvency II. Moreover, we will see that in the case of some classical reserving methods those estimators are the same as introduced before by other authors, see for instance Mack [6], Buchwalder et al. [2] and Dahms-Merz-Wüthrich [4].

In Section 6 we will present and discuss two examples of *LSRMs* based on real data. We will not discus the question which method is the best for the projection of specific data. Although this is a very important question it is too complex for this paper. Moreover, we think that for the model selection non triangle based information is of great importance, see the example of Section 6.1, and it is very difficult to include such information into an analytic triangle based rating of methods.

2. The model

Let $S^m_{i,k}$, $0 \leq m \leq M$, $0 \leq i \leq I$, $0 \leq k \leq J$, denote the incremental value of the m-th claim property of the i-th accident period during the k-th development period. We assume that $I \geq J$ and that there is no development of any claim property after development period J, which means we do not discuss any tail development. Such claim properties may be the usual candidates like payments, reported amounts and number of reported claims or even more special constructions like payments after reopening.

Our model contains three natural time lines: accident periods or rows, development periods or columns and business periods or lower-left to upper-right diagonals. We will use the indices i and h for accident periods, j and k for development periods, l and m for claim properties and n for business periods, see Figure 1.

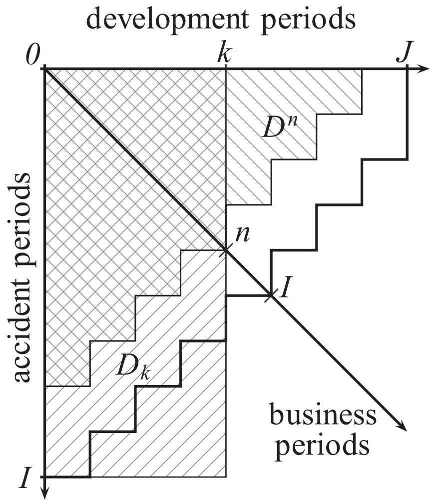

FIGURE 1: Claim property triangle.

By $\mathbb{L}^n$ and $\mathbb{L}_k$ we denote the linear spaces generated by all increments $S^m_{i,j}$ up to business period n and development period k, respectively. Moreover, by $\mathbb{L}^n_k$ we denote the linear space generated by $\mathbb{L}^n$ and $\mathbb{L}_k$, i.e.

$$\begin{aligned}
\mathbb{L}^n &:= \left\{ \sum_{m=0}^{M} \sum_{i=0}^{I} \sum_{j=0}^{(n-i)\wedge J} x^m_{i,j}\, S^m_{i,j} \,:\, x^m_{i,j} \in \mathbb{R} \right\}, \\
\mathbb{L}_k &:= \left\{ \sum_{m=0}^{M} \sum_{i=0}^{I} \sum_{j=0}^{k} x^m_{i,j}\, S^m_{i,j} \,:\, x^m_{i,j} \in \mathbb{R} \right\}, \\
\mathbb{L}^n_k &:= \left\{ \sum_{m=0}^{M} \sum_{i=0}^{I} \sum_{j=0}^{((n-i)\wedge J)\vee k} x^m_{i,j}\, S^m_{i,j} \,:\, x^m_{i,j} \in \mathbb{R} \right\},
\end{aligned} \tag{2.1}$$

where $a \wedge b$ and $a \vee b$ denote the minimum and maximum of the real numbers a and b, respectively. The σ-algebra of all information of accident period i up to development period k is denoted by $\mathcal{B}_{i,k}$. Moreover, we denote the σ-algebras generated by $\mathbb{L}^n$, $\mathbb{L}_k$ and $\mathbb{L}^n_k$ by $\mathcal{D}^n$, $\mathcal{D}_k$ and $\mathcal{D}^n_k$, respectively, i.e.

$$\mathcal{B}_{i,k} := \sigma\big(S^m_{i,j} \,:\, 0 \le m \le M,\ 0 \le j \le k\big),\ \mathcal{D}_k := \sigma(\mathbb{L}_k) = \sigma\left(\bigcup_{i=0}^{I} \mathcal{B}_{i,k}\right),$$

$$\mathcal{D}^n := \sigma(\mathbb{L}^n) = \sigma\left(\bigcup_{i=0}^{I} \mathcal{B}_{i,(n-i)\wedge J}\right), \qquad \mathcal{D}^n_k := \sigma(\mathbb{L}^n_k) = \sigma\left(\bigcup_{i=0}^{I} \mathcal{B}_{i,((n-i)\wedge J)\vee k}\right),$$

see Figure 1. We call the information $\mathcal{D}^{i+k}_k$ the past of $S^m_{i,k+1}$, $0 \le m \le M$.

Assumption 2.1. *We call the stochastic model of the increments $S_{i,k}^m$ a linear stochastic reserving method (LSRM) if there exist constants f_k^m and $\sigma_k^{m_1,m_2}$ such that*

i) for all i, m and k the expectation of the claim property $S_{i,k+1}^m$ under the condition of all information of its past $\mathcal{D}_k^{i+k}$ is proportional to an exposure $R_{i,k}^m$ contained in $\mathbb{L}^{i+k} \cap \mathbb{L}_k$, i.e.

$$E\left[S_{i,k+1}^m \middle| \mathcal{D}_k^{i+k}\right] = f_k^m R_{i,k}^m \in \mathbb{L}^{i+k} \cap \mathbb{L}_k. \tag{2.2}$$

ii) for all i, m_1, m_2 and k the covariance of the claim properties $S_{i,k+1}^{m_1}$ and $S_{i,k+1}^{m_2}$ under the condition of all information of their past $\mathcal{D}_k^{i+k}$ is proportional to an exposure $R_{i,k}^{m_1,m_2}$ contained in $\mathbb{L}^{i+k} \cap \mathbb{L}_k$, i.e.

$$\mathrm{Cov}\left[S_{i,k+1}^{m_1}, S_{i,k+1}^{m_2} \middle| \mathcal{D}_k^{i+k}\right] = \sigma_k^{m_1,m_2} R_{i,k}^{m_1,m_2} \in \mathbb{L}^{i+k} \cap \mathbb{L}_k. \tag{2.3}$$

Remark 2.2.

1. *If accident periods are independent and if all exposures $R_{i,k}^m$ and $R_{i,k}^{m_1,m_2}$ are $\mathcal{B}_{i,k}$-measurable it is enough to assume*

   i)' $\mathrm{E}\left[S_{i,k+1}^m \middle| \mathcal{B}_{i,k}\right] = f_k^m R_{i,k}^m$

   ii)' $\mathrm{Cov}\left[S_{i,k+1}^{m_1}, S_{i,k+1}^{m_2} \middle| \mathcal{B}_{i,k}\right] = \sigma_k^{m_1,m_2} R_{i,k}^{m_1,m_2}.$

2. *You can not take arbitrary values for $\sigma_k^{m_1,m_2}$ and $R_{i,k}^{m_1,m_2}$. The choice has to be consistent with the corresponding covariance properties, i.e. the matrices*

$$\left(\sigma_k^{m_1,m_2} R_{i,k}^{m_1,m_2}\right)_{0 \le m_1, m_2 \le M}$$

   have to be positive semidefinite almost surely for all i and all k.

3. *To get well defined objects we have to distinguish between the model parameters f_k^m and $\sigma_k^{m_1,m_2}$ and the method defining exposure parameters $\gamma_{i,k,h,j}^{m,l}$ and $\gamma_{i,k,h,j}^{m_1,m_2,l}$ of*

$$R_{i,k}^m =: \sum_{l=0}^{M} \sum_{h=0}^{I} \sum_{j=0}^{(i+k-h)\wedge k} \gamma_{i,k,h,j}^{m,l} S_{h,j}^l \quad \text{and} \quad R_{i,k}^{m_1,m_2} =: \sum_{l=0}^{M} \sum_{h=0}^{I} \sum_{j=0}^{(i+k-h)\wedge k} \gamma_{i,k,h,j}^{m_1,m_2,l} S_{h,j}^l, \tag{2.4}$$

   respectively.

4. *Often the choice of the exposures, i.e. of the parameters $\gamma_{i,k,h,j}^{m,l}$ and $\gamma_{i,k,h,j}^{m_1,m_2,l}$ in* (2.4)*, is of great importance. Unfortunately, we neither can provide a statistical nor a general heuristic concept for this choice. In some cases, see for instance Example 6.1, there is portfolio based information that may help with the choice of exposures. An other useful technique is backtesting that means to look for exposures for which we see now that the corresponding projections*

would have been reliable in the past. For instance, if we have been using the same LSRM for several years and always got good results, there is no reason to change the exposure.

5. *If you are only interested in estimators for the expected ultimate outcome you will not need assumption* (2.3).
6. *External given exposures may be included in a similar way as described for the Complementary-Loss-Ratio-Method, see Section 2.2.*

The following lemma contains some useful implications of Assumption 2.1.

Lemma 2.3. *Assume $S_{i,k}^m$ satisfy Assumption 2.1. Then*

a) $\mathrm{E}\left[S_{i,k+1}^m \middle| \mathcal{D}^{i+k}\right] = \mathrm{E}\left[S_{i,k+1}^m \middle| \mathcal{D}_k\right] = f_k^m R_{i,k}^m.$

b) $\mathrm{Cov}\left[S_{i,k+1}^{m_1}, S_{i,k+1}^{m_2} \middle| \mathcal{D}^{i+k}\right] = \mathrm{Cov}\left[S_{i,k+1}^{m_1}, S_{i,k+1}^{m_2} \middle| \mathcal{D}_k\right] = \sigma_k^{m_1,m_2} R_{i,k}^{m_1,m_2}.$

c) $\mathrm{Cov}\left[S_{n+1-j_1,j_1}^{m_1}, S_{n+1-j_2,j_2}^{m_2} \middle| \mathcal{D}^n\right] = 0,$ *for* $j_1 \neq j_2$.

d) provided that all exposures $R_{i,k}^m$ and $R_{i,k}^{m_1,m_2}$ are $\mathcal{B}_{i,k}$-measurable, accident periods will be uncorrelated under the knowledge of some past, i.e. for all σ-algebras $\mathcal{D}_k^n$, all $i_1 \neq i_2$ and arbitrary k_1, k_2, m_1 and m_2 we have

$$\mathrm{Cov}\left[S_{i_1,k_1}^{m_1}, S_{i_2,k_2}^{m_2} \middle| \mathcal{D}_k^n\right] = 0. \tag{2.5}$$

Proof. Since $\mathcal{D}^n$ and $\mathcal{D}_k$ are subsets of $\mathcal{D}_k^n$ and $R_{i,k}^m$ and $R_{i,k}^{m_1,m_2}$ are $\mathcal{D}^n \cap \mathcal{D}_k$-measurable parts a) and b) are direct consequences of Assumption 2.1.
For part c) assume that $j_1 > j_2$. Then $S_{n+1-j_2,j_2}^{m_1}$ is $\mathcal{D}_{j_1-1}^n$-measurable and we get

$$\mathrm{Cov}\left[S_{n+1-j_1,j_1}^{m_1}, S_{n+1-j_2,j_2}^{m_2} \middle| \mathcal{D}^n\right] = \mathrm{Cov}\left[\mathrm{E}\left[S_{n+1-j_1,j_1}^{m_1} \middle| \mathcal{D}_{j_1-1}^n\right], S_{n+1-j_2,j_2}^{m_2} \middle| \mathcal{D}^n\right] = 0,$$

where we used that $\mathrm{E}\left[S_{n+1-j_1,j_1}^{m_1} \middle| \mathcal{D}_{j_1-1}^n\right] \in \mathcal{D}^n \cap \mathcal{D}_{j_1-1} \subseteq \mathcal{D}^n \subseteq \mathcal{D}_{j_1-1}^n$.

In order to prove part d) take $i_1 \neq i_2$ and arbitrary k, k_1, k_2, m_1, m_2 and n. If $S_{i_1,k_1}^{m_1}$ or $S_{i_2,k_2}^{m_2}$ is measurable with respect to $\mathcal{D}_k^n$ we are done. Otherwise, $\mathcal{D}_k^n$ is a subset of $\mathcal{D}_{k_1-1}^{i+k_1-1}$ and $\mathcal{D}_{k_2-1}^{i+k_2-1}$ and $S_{i_1,k_1}^{m_1}$ is measurable with respect to the past of $S_{i_2,k_2}^{m_2}$ or vice versa. Without loss of generality assume that $S_{i_1,k_1}^{m_1}$ is $\mathcal{D}_{k_2-1}^{i+k_2-1}$-measurable. Then we get

$$\begin{aligned}\mathrm{Cov}\left[S_{i_1,k_1}^{m_1}, S_{i_2,k_2}^{m_2} \middle| \mathcal{D}_k^n\right] &= \mathrm{E}\left[\mathrm{Cov}\left[S_{i_1,k_1}^{m_1}, S_{i_2,k_2}^{m_2} \middle| \mathcal{D}_{k_2-1}^{i+k_2-1}\right] \middle| \mathcal{D}_k^n\right] \\ &\quad + \mathrm{Cov}\left[\mathrm{E}\left[S_{i_1,k_1}^{m_1} \middle| \mathcal{D}_{k_2-1}^{i+k_2-1}\right], \mathrm{E}\left[S_{i_2,k_2}^{m_2} \middle| \mathcal{D}_{k_2-1}^{i+k_2-1}\right] \middle| \mathcal{D}_k^n\right] \\ &= 0 + \mathrm{Cov}\left[S_{i_1,k_1}^{m_1}, f_{k_2-1}^{m_2} R_{i_2,k_2-1}^{m_2} \middle| \mathcal{D}_k^n\right].\end{aligned}$$

Since $R^{m_2}_{i_2,k_2-1} \in \mathcal{B}_{i_2,k_2-1}$ it is enough to show that $S^{m_1}_{i_1,k_1}$ and $S^{m_2}_{i_2,k_2-1}$ are $\mathcal{D}^n_k$-conditional uncorrelated. Iterating this procedure we will finally reach a point where $S^{m_1}_{i_1,k_1-j}$ or $S^{m_2}_{i_2,k_2-j}$ is $\mathcal{D}^n_k$-measurable, which proves (2.5). □

Remark 2.4. *Under the assumption that all exposures $R^m_{i,k}$ and $R^{m_1,m_2}_{i,k}$ are $\mathcal{B}_{i,k}$-measurable Lemma 2.3 implies that the correlation of different accident periods is determined by their first development period, i.e. there exist linear mappings* $C_{i,k} : \mathbb{R}^M \to \mathbb{R}$ such that

$$\mathrm{Cov}\Big[(S^m_{i_1,k_1})^{0\le m\le M}, (S^m_{i_2,k_2})^{0\le m\le M}\Big] = \mathrm{Cov}\Big[C_{i_1,k_1}(S^m_{i_1,0})^{0\le m\le M}, C_{i_2,k_2}(S^m_{i_2,0})^{0\le m\le M}\Big]$$

provided $i_1 \neq i_2$.

In the following sections we will discus for some well known reserving models if and how they fit into the framework of *LSRMs*.

2.1. Chain-Ladder-Method

For the Chain-Ladder-Method as analysed in Mack [6] one looks at one cumulative claim property

$$C_{i,k} := \sum_{j=0}^{k} S^0_{i,j}.$$

The assumptions for the Chain-Ladder-Method are

$i)^{CL}$ $\mathrm{E}\big[C_{i,k+1}\big|\mathcal{B}_{i,k}\big] = g_k C_{i,k}$.

$ii)^{CL}$ $\mathrm{Var}\big[C_{i,k+1}\big|\mathcal{B}_{i,k}\big] = \sigma^2_k C_{i,k}$.

$iii)^{CL}$ Accident periods are independent.

Since, $C_{i,k}$ are elements of $\mathbb{L}^{i+k}_k$ and since

$$\mathrm{E}\big[S^0_{i,k+1}\big|\mathcal{B}_{i,k}\big] = (g_k - 1)\,C_{i,k} \quad \text{and} \quad \mathrm{Var}\big[S^0_{i,k+1}\big|\mathcal{B}_{i,k}\big] = \sigma^2_k C_{i,k}$$

we see that the Chain-Ladder-Method is a *LSRM*.

2.2. Complementary-Loss-Ratio-Method

For the Complementary-Loss-Ratio-Method one looks at a claim property $S^0_{i,j}$ and an external given exposure P_i that does not develop over time. The assumptions for this method are

$i)^{LR}$ $\mathrm{E}\big[S^0_{i,k+1}\big|\mathcal{B}_{i,k}\big] = g_k P_i$.

$ii)^{LR}$ $\operatorname{Var}\left[S^0_{i,k+1}\middle|\mathcal{B}_{i,k}\right] = \sigma_k^2 P_i$.

$iii)^{LR}$ Accident periods are independent.

If we take

$$S^1_{i,k} := \begin{cases} P_i, & \text{for } k = 0, \\ 0, & \text{otherwise,} \end{cases}$$

we see that the Complementary-Loss-Ratio-Method is a *LSRM*.

Note, usually one assumes a bit less and takes unconditional expectations. The main differences between taking conditional and unconditional expectations are:

- By taking the unconditional expectation you pretend to be only interested in the overall expectation of the projected claim property, where the average is taken over all triangles, although the projected claim property may depend on the already observed triangle. In other words, the method does not use all available information and therefore may not be optimal.
- By taking conditional expectations you explicitly assume that the projected claim property does not depend on the already observed triangle.

2.3. Bornhuetter-Ferguson-Method

Here we look at one claim property $S^0_{i,k}$. Usually the Bornhuetter-Ferguson-Method is written as

$$\sum_{k=I+1-i}^{J} S^0_{i,k} = q_{I+1-i} U_i^{pri}, \tag{2.6}$$

where U_i^{pri} is a priori known estimate of the ultimate outcome, which may be motivated by pricing arguments or by external experts. Now we have to estimate the loss ratios q_k. Often the Chain-Ladder factors are used. But we can do better, see Mack [8]. We will use this idea and rewrite (2.6) as follows

$$\sum_{k=I+1-i}^{J} S^0_{i,k} = \sum_{k=I+1-i}^{J} g_{k-1} U_i^{pri}.$$

If we now look at the unknown factors g_k column by column we get

$$S^0_{i,k+1} = g_k U_i^{pri}.$$

Finally, taking conditional expectations and U_i^{pri} as external exposure we see that the Bornhuetter-Ferguson-Method can be looked at as Complementary-Loss-Ratio-Method and therefore as a *LSRM*.

2.4. Extended-Complementary-Loss-Ratio-Method

For this method we look at incremental payments $S^0_{i,k}$ and changes of the reported amounts $S^1_{i,k}$ simultaneously. The coupling exposures are the case reserves

$$R^0_{i,k} = R^1_{i,k} = R^{0,0}_{i,k} = R^{0,1}_{i,k} = R^{1,0}_{i,k} = R^{1,1}_{i,k} := \sum_{j=0}^{k} (S^1_{i,j} - S^0_{i,j}).$$

Using this we get the following *LSRM*

$i)^{ELR}$ $\mathrm{E}\left[S^m_{i,k+1}\middle|\mathcal{B}_{i,k}\right] = f^m_k \sum_{j=0}^k (S^1_{i,j} - S^0_{i,j})$ for $m \in \{0,1\}$.

$ii)^{ELR}$ $\mathrm{Cov}\left[S^{m_1}_{i,k+1}, S^{m_2}_{i,k+1}\middle|\mathcal{B}_{i,k}\right] = \sigma^{m_1,m_2}_k \sum_{j=0}^k (S^1_{i,j} - S^0_{i,j})$ for $m_1, m_2 \in \{0,1\}$.

$iii)^{ELR}$ Accident periods are independent.

Note, this method projects payments and reported amounts in a way that both projections lead to the same ultimate. For details see Dahms [3].

2.5. Munich-Chain-Ladder-Method

This method, introduced in Quarg-Mack [12], considers the Chain-Ladder-projections of cumulative payments $C_{i,k} := \sum_{j=0}^k S^0_{i,j}$ and reported amounts $I_{i,k} := \sum_{j=0}^k S^1_{i,j}$ together in order to reduce the systematic gap between the stand alone Chain-Ladder-projections, see Braun [1]. But the gap is not closed entirely.

As shown in Merz-Wüthrich [9] the Munich-Chain-Ladder-Method assumes

$i)^{MCL}$ $\mathrm{E}\left[C_{i,k+1}\middle|\mathcal{C}_k\right] = f_k C_{i,k}$ and $\mathrm{E}\left[I_{i,k+1}\middle|\mathcal{I}_k\right] = g_k I_{i,k}$,

$ii)^{MCL}$ Accident periods are independent.

Here $\mathcal{C}_k$ and $\mathcal{I}_k$ contain all information of payments and reported amounts up to development period k, respectively. Note, in i)^{MCL} you cannot extend these sigma algebras to $\mathcal{D}_k$ like we have done in Section 2.2. Moreover, instead of looking at $\mathrm{E}[C_{i,J}|\mathcal{D}_{I-i}]$ and $\mathrm{E}[I_{i,J}|\mathcal{D}_{I-i}]$, which are the orthogonal projections of $C_{i,J}$ and $I_{i,J}$, respectively, on the linear space of all $\mathcal{D}_{I-i}$-measurable, square-integrable random variables, the Munich-Chain-Ladder-Method considers the orthogonal projections on a much smaller affine subspace, for details see Merz-Wüthrich [9].

These are the main reasons why the Munich-Chain-Ladder-Method does not fit into the framework of *LSRMs*.

3. Estimators for future development

In this section we want to present estimators for the future development of claim properties, motivate them and prove some properties. In order to shorten notations we define $\frac{0}{0} := 0$.

Estimator 3.1 (of the model parameter f_k^m). *Let $S_{i,k}^m$ satisfy Assumption 2.1. Then for each set of $\mathcal{D}^n \cap \mathcal{D}_k$-measurable weights $w_{i,k}^m \geq 0$ with*

- $R_{i,k}^m = 0$ *implies* $w_{i,k}^m = 0$ *and*
- $\sum_{i=0}^{I-1-k} w_{i,k}^m = 1$ *if at least one* $R_{i,k}^m \neq 0$

we get that

$$\hat{f}_k^m := \sum_{i=0}^{I-1-k} w_{i,k}^m \frac{S_{i,k+1}^m}{R_{i,k}^m} \tag{3.1}$$

is a $\mathcal{D}_k$-conditionally unbiased estimator of the model parameter f_k^m. Moreover, for every tuple $\hat{f}_{k_1}^{m_1}, \ldots, \hat{f}_{k_r}^{m_r}$ with $k_1 < k_2 < \cdots < k_r$ we get

$$\mathrm{E}\left[\hat{f}_{k_1}^{m_1} \cdots \hat{f}_{k_r}^{m_r} \middle| \mathcal{D}_{k_1}\right] = f_{k_1}^{m_1} \cdots f_{k_r}^{m_r} = \mathrm{E}\left[\hat{f}_{k_1}^{m_1} \middle| \mathcal{D}_{k_1}\right] \cdots \mathrm{E}\left[\hat{f}_{k_r}^{m_r} \middle| \mathcal{D}_{k_1}\right], \tag{3.2}$$

which implies that the estimators are pairwise $\mathcal{D}_{k_1}$-conditionally uncorrelated.

Proof. Let us start with the derivation of (3.1):

$$\mathrm{E}\left[\hat{f}_k^m \middle| \mathcal{D}_k\right] = \sum_{i=0}^{I-1-k} w_{i,k}^m \frac{\mathrm{E}\left[\mathrm{E}\left[S_{i,k+1}^m \middle| \mathcal{D}_k^{i+k}\right] \middle| \mathcal{D}_k\right]}{R_{i,k}^m} = \sum_{i=0}^{I-1-k} w_{i,k}^m \frac{f_k^m R_{i,k}^m}{R_{i,k}^m} = f_k^m.$$

Moreover, for every tuple $\hat{f}_{k_1}^{m_1}, \ldots, \hat{f}_{k_r}^{m_r}$ *with* $k_1 < k_2 < \cdots < k_r$ we compute

$$\begin{aligned}
\mathrm{E}\left[\hat{f}_{k_1}^{m_1} \cdots \hat{f}_{k_r}^{m_r} \middle| \mathcal{D}_{k_1}\right] &= \mathrm{E}\left[\mathrm{E}\left[\hat{f}_{k_1}^{m_1} \cdots \hat{f}_{k_r}^{m_n} \middle| \mathcal{D}_{k_r}\right] \middle| \mathcal{D}_{k_1}\right] \\
&= \mathrm{E}\left[\hat{f}_{k_1}^{m_1} \cdots \hat{f}_{k_{r-1}}^{m_{r-1}} \mathrm{E}\left[\hat{f}_{k_r}^{m_r} \middle| \mathcal{D}_{k_r}\right] \middle| \mathcal{D}_{k_1}\right] \\
&= \mathrm{E}\left[\hat{f}_{k_1}^{m_1} \cdots \hat{f}_{k_{r-1}}^{m_{r-1}} \middle| \mathcal{D}_{k_1}\right] f_{k_r}^{m_r} \\
&\vdots \\
&= f_{k_1}^{m_1} \cdots f_{k_r}^{m_r},
\end{aligned}$$

which proves (3.2). □

Remark 3.2. *Assumption 2.1.ii) implies that the weights*

$$w_{i,k}^{m} := \frac{(R_{i,k}^{m})^2}{R_{i,k}^{m,m}} \left(\sum_{h=0}^{I-1-k} \frac{(R_{h,k}^{m})^2}{R_{h,k}^{m,m}} \right)^{-1}, \tag{3.3}$$

result in estimators $\hat{f}_k^m$ *with minimal variance of all estimators of the form* (3.1). *In other words the resulting estimators* $\hat{f}_k^m$ *are (homogeneous) credibility estimators. Moreover, in case of the Chain-Ladder-Method, the Complementary-Loss-Ratio-Method and the Extended-Complementary-Loss-Ratio-Method those variance minimal estimators are the well known standard estimators, see for example Mack [6] and [7] and Dahms [3].*

In order to shorten notations for further calculations we will use the linear mappings

$$F_{i,k}^{m} : \mathbb{L}^{i+k} \longrightarrow \mathbb{R} \quad \text{and} \quad \mathbf{F}^{n} : \mathbb{L}^{n} \longrightarrow \mathbb{L}^{n+1}$$

defined by the exposure parameter $\gamma_{i,k,h,j}^{m,l}$, see (2.4),

$$\left(\mathbf{F}^{n} x\right)_{i,k}^{m} := \begin{cases} x_{i,k}^{m}, & \text{for } i+k \le n, \\ f_k^m \sum_{l=0}^{M} \sum_{h=0}^{I} \sum_{j=0}^{(n-h)\wedge k} \gamma_{i,k,h,j}^{m,l} x_{h,j}^{l}, & \text{for } i+k = n+1, \end{cases} \tag{3.4}$$

$$F_{i,k}^{m} x := \left(\mathbf{F}^{i+k} x\right)_{i,k+1}^{m}.$$

Remark 3.3.

- *The mapping* $\mathbf{F}^n$ *fills the* $n+1$*-th diagonal of all claim property triangles based on all diagonals up to the n-th business period.*
- *The functional* $F_{i,k}^{m}$ *does depend on coordinates within* $\mathbb{L}^{i+k} \cap \mathbb{L}_k$*, only.*

The concatenation of linear mappings $\mathbf{F}^n$ is denoted by

$$\mathbf{F}^{n_2 \leftarrow n_1} := \begin{cases} \Pi_{\mathbb{L}^{n_2+1}}, & \text{for } n_2 < n_1, \\ \mathbf{F}^{n_2}\mathbf{F}^{n_2-1} \cdots \mathbf{F}^{n_1}, & \text{for } n_2 \ge n_1, \end{cases} \tag{3.5}$$

$$\mathbf{F}_{i,k}^{m,n} x := \left(\mathbf{F}^{i+k \leftarrow n} x\right)_{i,k+1}^{m},$$

where $\Pi_{\mathbb{L}^n}$ denotes the projection on the first n diagonals. Moreover, we will use the symbol $\mathbf{S}^n$ for the vector

$$\mathbf{S}^{n} := (S_{i,k}^{m})_{i+k \le n}^{0 \le m \le M}.$$

As a consequence we get

$$\mathrm{E}\Big[S^m_{i,k+n+1}\Big|\mathcal{D}^{i+k}_k\Big] = \mathbf{F}^{m,i+k}_{i,k+n}\,\mathbf{S}^{i+k},$$

$$\mathrm{E}\Big[\mathbf{S}^{n_1+n_2+1}\Big|\mathcal{D}^{n_1}\Big] = \mathbf{F}^{n_1+n_2\leftarrow n_1}\,\mathbf{S}^{n_1}.$$

This together with Estimator 3.1 lead to estimators for the future development of all claim properties.

Estimator 3.4 (of the future development). *Let $S^m_{i,k}$ satisfy Assumption 2.1. Then*

$$\widehat{S}^m_{i,k+1} := \widehat{\mathbf{F}}^{m,I}_{i,k}\,\mathbf{S}^I, \qquad I-i \le k < J, \tag{3.6}$$

are both $\mathcal{D}_{I-i}$ and $\mathcal{D}^I \cap \mathcal{D}_{I-i}$-conditionally unbiased estimators for $\mathrm{E}[S^m_{i,k}\,|\,\mathcal{D}_{I-i}]$, *where $\widehat{\mathbf{F}}^{m,n}_{i,k}$ is defined in the same way as $\mathbf{F}^{m,n}_{i,k}$, see* (3.4) *and* (3.5)*, but with $\widehat{f}^m_k$ instead of f^m_k.*

Proof. Since $\mathcal{D}^I \cap \mathcal{D}_{I-i}$ is a subset of $\mathcal{D}_{I-i}$ and since $\widehat{\mathbf{F}}^{m,I}_{i,k}\,\mathbf{S}^I$ is measurable with respect to $\mathcal{D}^I \cap \mathcal{D}_{I-i}$ it is enough to prove the stated $\mathcal{D}_{I-i}$-conditional unbiasedness of the estimators $\widehat{S}^m_{i,k}$.

Because each mapping $\widehat{F}^m_{i,k}$ depends linearly on $\widehat{f}^m_k$, we can rewrite the estimators as follows

$$\widehat{\mathbf{F}}^{m,I}_{i,k}\,\mathbf{S}^I = \sum_{I-i\le k_1<\cdots<k_r\le k} \widehat{f}^{m_1}_{k_1}\cdots\widehat{f}^{m_r}_{k_r}\,X^{m_1,\ldots,m_r}_{k_1,\ldots,k_r},$$

where $X^{m_1,\ldots,m_r}_{k_1,\ldots,k_r}$ are elements of $\mathcal{D}^I \cap \mathcal{D}_{k_1}$. Now the stated unbiasedness follows from the properties of $\widehat{f}^m_k$, stated in Estimator 3.1. □

In the same way we get $\mathcal{D}_{I-i}$-conditionally unbiased estimators $\widehat{R}^m_{i,k}$ and $\widehat{R}^{m_1,m_2}_{i,k}$ for the exposures $R^m_{i,k}$ and $R^{m_1,m_2}_{i,k}$ by

$$\widehat{R}^m_{i,k} := \sum_{l=0}^{M}\sum_{h=0}^{I}\sum_{j=0}^{(i+k-h)\wedge k} \gamma^{m,l}_{i,k,h,j}\,\widehat{S}^l_{h,j} \quad\text{and}\quad \widehat{R}^{m_1,m_2}_{i,k} := \sum_{l=0}^{M}\sum_{h=0}^{I}\sum_{j=0}^{(i+k-h)\wedge k} \gamma^{m_1,m_2,l}_{i,k,h,j}\,\widehat{S}^l_{h,j}, \tag{3.7}$$

respectively, with exposure parameters $\gamma^{m,l}_{i,k,h,j}$ and $\gamma^{m_1,m_2,l}_{i,k,h,j}$, see (2.4). Moreover, in order to shorten notations we will use for $k \le I-i$ the definitions

$$\widehat{S}^m_{i,k} := S^m_{i,k}, \qquad \widehat{R}^m_{i,k} := R^m_{i,k} \quad\text{and}\quad \widehat{R}^{m_1,m_2}_{i,k} := R^{m_1,m_2}_{i,k}.$$

4. Mean squared error of prediction

In the previous section we presented estimators for the ultimate outcome of claim properties. Now let us look at the (conditional) mean squared error of prediction for the estimated future development. Often we are in a situation where we are not only interested in a single claim property but in a linear combination of several claim properties, see for instance the examples presented in Section 6. Therefore, take $\mathcal{D}^I$-measurable weights α_i^m, $0 \leq i \leq I$ and $m \in \mathcal{M} \subseteq \{0, \ldots, M\}$.

We will start with a fixed accident period $i > I - J$. The corresponding mean squared error of prediction is defined by

$$\operatorname{mse}\left[\sum_{m\in\mathcal{M}}\sum_{k=I-i}^{J-1}\alpha_i^m \widehat{S}_{i,k+1}^m\right] := \mathrm{E}\left[\left(\sum_{m\in\mathcal{M}}\sum_{k=I-i}^{J-1}\alpha_i^m\left(S_{i,k+1}^m - \widehat{S}_{i,k+1}^m\right)\right)\middle|\mathcal{D}^I\right]. \tag{4.1}$$

A short calculation yields

$$\begin{aligned}\operatorname{mse}&\left[\sum_{m\in\mathcal{M}}\sum_{k=I-i}^{J-1}\alpha_i^m \widehat{S}_{i,k+1}^m\right]\\ &= \operatorname{Var}\left[\sum_{m\in\mathcal{M}}\sum_{k=I-i}^{J-1}\alpha_i^m S_{i,k+1}^m\middle|\mathcal{D}^I\right] + \left(\sum_{m\in\mathcal{M}}\sum_{k=I-i}^{J-1}\alpha_i^m \mathrm{E}\left[S_{i,k+1}^m - \widehat{S}_{i,k+1}^m\middle|\mathcal{D}^I\right]\right)^2\\ &= \quad\text{process variance} \quad + \quad \text{parameter estimation error.}\end{aligned} \tag{4.2}$$

For estimators of second moments we have to estimate the model parameters $\sigma_k^{m_1,m_2}$. If $k < J \wedge (I-1)$ one can take the following unbiased estimators

$$\hat{\sigma}_k^{m_1,m_2} := \frac{1}{Z_k^{m_1,m_2}}\sum_{i=0}^{I-1-k}\frac{R_{i,k}^{m_1}R_{i,k}^{m_2}}{R_{i,k}^{m_1,m_2}}\left(\frac{S_{i,k+1}^{m_1}}{R_{i,k}^{m_1}} - \hat{f}_k^{m_1}\right)\left(\frac{S_{i,k+1}^{m_2}}{R_{i,k}^{m_2}} - \hat{f}_k^{m_2}\right) \tag{4.3}$$

with

$$Z_k^{m_1,m_2} := \sum_{i=0}^{I-1-k}\left(1 - w_{i,k}^{m_1} - w_{i,k}^{m_2} + w_{i,k}^{m_1}w_{i,k}^{m_2}\frac{R_{i,k}^{m_1,m_2}}{R_{i,k}^{m_1}R_{i,k}^{m_2}}\sum_{h=0}^{I-1-k}\frac{R_{h,k}^{m_1}R_{h,k}^{m_2}}{R_{h,k}^{m_1,m_2}}\right).$$

As we will see later we will not need estimators of $\sigma_{J-1}^{m_1,m_2}$ for $m_1 \neq m_2$. Finally, for $m_1 = m_2$ and $I = J$ one could take the extrapolation, see Mack [6],

$$\hat{\sigma}_{J-1}^{m,m} := \min\left(\frac{(\hat{\sigma}_{J-2}^{m,m})^2}{\hat{\sigma}_{J-3}^{m,m}}, \hat{\sigma}_{J-3}^{m,m}, \hat{\sigma}_{J-2}^{m,m}\right). \tag{4.4}$$

Remark 4.1. *The estimation of the model parameters $\sigma_k^{m_1,m_2}$ is a wide field and you may often find better estimators than presented here. For instance, you may* *introduce weighted estimators and use other extrapolations. But since such customising usually depends heavily one the analysed data we will not go into details here.*

4.1. Process variance for an accident period

In order to get estimators for the process variance let us start with some computations of the expectation of products of $S_{i,k}^{m}$.

Lemma 4.2. Assume $S_{i,k}^{m}$ *satisfy Assumption 2.1. Then for all* $I+1 \le n \le I+J$ *and arbitrary* $\mathcal{D}^{n-1}$*-measurable real numbers* $g_{1,h,j}^{m}$ *and* $g_{2,h,j}^{m}$ *we get*

$$\mathrm{Cov}\left[\sum_{m_1=0}^{M}\sum_{h_1=0}^{I}\sum_{j_1=0}^{(n-h_1)\wedge J} g_{1,h_1,j_1}^{m} S_{h_1,j_1}^{m_1}, \sum_{m_2=0}^{M}\sum_{h_2=0}^{I}\sum_{j_2=0}^{(n-h_2)\wedge J} g_{2,h_2,j_2}^{m} S_{h_2,j_2}^{m_2} \middle| \mathcal{D}^{n-1}\right]$$
$$= \sum_{m_1,m_2=0}^{M}\sum_{j=n-I}^{J} g_{1,n-j,j}^{m_1}\, g_{2,n-j,j}^{m_2}\, \sigma_{j-1}^{m_1,m_2}\, R_{n-j,j-1}^{m_1,m_2}. \tag{4.5}$$

Proof. Take arbitrary $\mathcal{D}^{n-1}$-measurable real numbers $g_{1,h,j}^{m}$ and $g_{2,h,j}^{m}$. Since $S_{h,j}^{m}$ is $\mathcal{D}^{n-1}$-measurable for all $h+j \le n-1$ we get

$$\mathrm{Cov}\left[\sum_{m_1=0}^{M}\sum_{h_1=0}^{I}\sum_{j_1=0}^{(n-h_1)\wedge J} g_{1,h_1,j_1}^{m} S_{h_1,j_1}^{m_1}, \sum_{m_2=0}^{M}\sum_{h_2=0}^{I}\sum_{j_2=0}^{(n-h_2)\wedge J} g_{2,h_2,j_2}^{m} S_{h_2,j_2}^{m_2} \middle| \mathcal{D}^{n-1}\right]$$
$$= \sum_{m_1,m_2=0}^{M}\sum_{j_1,j_2=n-I}^{J} g_{1,n-j_1,j_1}^{m_1}\, g_{2,n-j_2,j_2}^{m_2}\, \mathrm{Cov}\left[S_{n-j_1,j_1}^{m_1}, S_{n-j_2,j_2}^{m_2} \middle| \mathcal{D}^{n-1}\right]$$
$$= \sum_{m_1,m_2=0}^{M}\sum_{j=n-I}^{J} g_{1,n-j,j}^{m_1}\, g_{2,n-j,j}^{m_2}\, \sigma_{j-1}^{m_1,m_2}\, R_{n-j,j-1}^{m_1,m_2},$$

where we used the covariance assumption on a *LSRM* and part c) of Lemma 2.3 for the last step. □

Now fix i_1, i_2, k_1 and k_2 with $I \le i_1 + k_1 < i_2 + k_2$. Then we get

$$\begin{aligned}
\mathrm{Cov}\left[S_{i_1,k_1+1}^{m_1}, S_{i_2,k_2+1}^{m_2} \middle| \mathcal{D}^{I}\right] &= \mathrm{Cov}\left[S_{i_1,k_1+1}^{m_1}, \mathrm{E}\left[S_{i_2,k_2+1}^{m_2} \middle| \mathcal{D}^{i_2+k_2}\right] \middle| \mathcal{D}^{I}\right]\\
&= \mathrm{Cov}\left[S_{i_1,k_1+1}^{m_1}, \mathbf{F}_{i_2,k_2}^{m_2,i_2+k_2}\, \mathbf{S}^{i_2+k_2} \middle| \mathcal{D}^{I}\right]\\
&\;\;\vdots\\
&= \mathrm{Cov}\left[S_{i_1,k_1+1}^{m_1}, \mathbf{F}_{i_2,k_2}^{m_2,i_1+k_1+1}\, \mathbf{S}^{i_1+k_1+1} \middle| \mathcal{D}^{I}\right]\\
&= \mathrm{E}\left[\mathrm{Cov}\left[S_{i_1,k_1+1}^{m_1}, \mathbf{F}_{i_2,k_2}^{m_2,i_1+k_1+1}\, \mathbf{S}^{i_1+k_1+1} \middle| \mathcal{D}^{i_1+k_1}\right] \middle| \mathcal{D}^{I}\right]\\
&\quad + \mathrm{Cov}\left[\mathbf{F}_{i_1,k_1}^{m_1,i_1+k_1}\, \mathbf{S}^{i_1+k_1}, \mathbf{F}_{i_2,k_2}^{m_2,i_1+k_1}\, \mathbf{S}^{i_1+k_1} \middle| \mathcal{D}^{I}\right].
\end{aligned}$$

An iteration of the last step leads to

$$\mathrm{Cov}\left[S^{m_1}_{i_1,k_1}, S^{m_2}_{i_2,k_2}\middle|\mathcal{D}^I\right] = \sum_{n=I+1}^{i_1+k_1+1} \mathrm{E}\left[\mathrm{Cov}\left[\mathbf{F}^{m_1,n}_{i_1,k_1}\mathbf{S}^n, \mathbf{F}^{m_2,n}_{i_2,k_2}\mathbf{S}^n\middle|\mathcal{D}^{n-1}\right]\middle|\mathcal{D}^I\right].$$

Applying the covariance formula (4.5) we can proceed with

$$\mathrm{Cov}\left[S^{m_1}_{i_1,k_1+1}, S^{m_2}_{i_2,k_2+1}\middle|\mathcal{D}^I\right]$$
$$= \sum_{n=I+1}^{i_1+k_1+1} \sum_{l_1,l_2=0}^{M} \sum_{j=n-I}^{J} \sigma^{l_1,l_2}_{j-1}\, \mathrm{E}\left[R^{l_1,l_2}_{n-j,j-1}\middle|\mathcal{D}^I\right]\left(\mathbf{F}^{m_1,n}_{i_1,k_1}\right)^{l_1}_{n-j,j}\left(\mathbf{F}^{m_2,n}_{i_2,k_2}\right)^{l_2}_{n-j,j}.$$

Using the same techniques we get similar formulas for all remaining indices i_1, i_2, k_1 and k_2 with $i_1 + k_1, i_2 + k_2 \geq I$. Finally, we replace all unknown model parameters by their estimators:

Estimator 4.3 (of the process variance of a single accident period)
Assume $S^m_{i,k}$ satisfy Assumption 2.1 and take arbitrary $\mathcal{D}^I$-measurable factors α^m_i, $m \in \mathcal{M} \subseteq \{0, \ldots, M\}$. Then the process variance of a single accident period can be estimated by

$$\widehat{\mathrm{Var}}\left[\sum_{m\in\mathcal{M}} \sum_{k=I-i}^{J-1} \alpha^m_i S^m_{i,k+1}\middle|\mathcal{D}^I\right] := \sum_{m_1,m_2\in\mathcal{M}} \alpha^{m_1}_i \alpha^{m_2}_i \sum_{k_1,k_2=I-i}^{J-1} \sum_{l_1,l_2=0}^{M} \sum_{n=I+1}^{i+(k_1\wedge k_2)+1} \sum_{j=n-1-I}^{J-1}$$
$$\hat{\sigma}^{l_1,l_2}_j \widehat{R}^{l_1,l_2}_{n-1-j,j}\left(\widehat{\mathbf{F}}^{m_1,n}_{i,k_1}\right)^{l_1}_{n-1-j,j+1}\left(\widehat{\mathbf{F}}^{m_2,n}_{i,k_2}\right)^{l_2}_{n-1-j,j+1}.$$

4.2. Parameter estimation error for an accident period

In order to get an estimator for the parameter estimation error we will apply the conditional resampling approach, see Wüthrich-Merz [10, Section 3.2.3]. Therefore, we will look at

$$\begin{aligned}\Delta^{\mathcal{M}}_i\left(\left(\hat{f}^l_k\right)^{0\leq l\leq M}_{0\leq k\leq J-1}\right) &:= \left(\sum_{m\in\mathcal{M}} \alpha^m_i \sum_{k=I-i}^{J-1} \mathrm{E}\left[\widehat{S}^m_{i,k+1} - S^m_{i,k+1}\middle|\mathcal{D}^I\right]\right)^2 \\ &= \left(\sum_{m\in\mathcal{M}} \alpha^m_i \sum_{k=I-i}^{J-1} \widehat{\mathbf{F}}^{m,I}_{i,k}\mathbf{S}^I - \sum_{m\in\mathcal{M}} \alpha^m_i \sum_{k=I-i}^{J-1} \mathbf{F}^{m,I}_{i,k}\mathbf{S}^I\right)^2\end{aligned} \tag{4.6}$$

as a function of the estimated model parameters $\hat{f}^m_k$. The conditional resampling approach means to estimate $\Delta^{\mathcal{M}}_i$ by its expected value under the resampling probability measure P^*, which is the product measure of

$$\mathrm{P}^*\left(\left(\hat{f}^m_k\right)^{1\leq m\leq M} \in A\right) := \mathrm{P}\left(\left(\hat{f}^m_k\right)^{1\leq m\leq M} \in A\middle|\mathcal{D}^I \cap \mathcal{D}_k\right).$$

We denote the expectation, variance and covariance with respect to P^* by E^*, Var^* and Cov^*, respectively.

Remark 4.4. *From the definition of the conditional resampling measure it follows that:*

1. *Under* P^* *every collection* $\{\hat{f}_{k_1}^{m_1}, \ldots, \hat{f}_{k_n}^{m_n}\}$ *with* $k_1 < \cdots < k_n$ *is a collection of independent variables.*
2. *For all* $0 \le m \le M$ *and all* $0 \le k \le J-1$ *we have* $E^*\left[\hat{f}_k^m\right] = f_k^m$.
3. *For all* $0 \le m_1, m_2 \le M$ *and all* $0 \le k \le J-1$ *we have*

$$\sigma_k^{*m_1,m_2} := Cov^*\left[\hat{f}_k^{m_1}, \hat{f}_k^{m_2}\right] = \sigma_k^{m_1,m_2} \sum_{i=0}^{I-1-k} w_{i,k}^{m_1} w_{i,k}^{m_2} \frac{R_{i,k}^{m_1,m_2}}{R_{i,k}^{m_1} R_{i,k}^{m_2}}. \tag{4.7}$$

Using Remark 4.4 we get

$$\begin{aligned} \Delta_i^{\mathcal{M}} &\approx E^*\left[\Delta_i^{\mathcal{M}}\right] = Var^*\left[\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \hat{S}_{i,k+1}^m\right] \\ &= E^*\left[\left(\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \hat{S}_{i,k+1}^m\right)^2\right] - \left(E^*\left[\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \hat{S}_{i,k+1}^m\right]\right)^2. \end{aligned} \tag{4.8}$$

In order to get an estimator for the first addend on the right hand side let us start with some computations of expectations of products of $\hat{S}_{i,k}^m$ under P^*:

$$E^*\left[\hat{S}_{i_1,k_1+1}^{m_1} \hat{S}_{i_2,k_2+1}^{m_2}\right] = E^*\left[\hat{S}_{i_1,k_1+1}^{m_1} \hat{F}_{i_2,k_2}^{m_1} \hat{\mathbf{S}}^{i_2+k_2}\right],$$

for all $i_2 + k_2 \ge I$. If $k_2 > k_1$ the variables $\hat{S}_{i_1,k_1+1}^{m_1}$ and $\hat{\mathbf{S}}^{i_2+k_2}$ do not depend on $\hat{f}_{k_2}^{m_2}$ and we can use Remark 4.4 in order to obtain

$$E^*\left[\hat{S}_{i_1,k_1+1}^{m_1} \hat{S}_{i_2,k_2+1}^{m_2}\right] = E^*\left[\hat{S}_{i_1,k_1+1}^{m_1} F_{i_2,k_2}^{m_1} \hat{\mathbf{S}}^{i_2+k_2}\right]. \tag{4.9}$$

Analogously we compute for $0 \le k \le J-1$ and $i_1, i_2 \ge I-k$

$$E^*\left[\hat{S}_{i_1,k+1}^{m_1} \hat{S}_{i_2,k+1}^{m_2}\right] = \left(1 + \rho_k^{*m_1,m_2}\right) E^*\left[F_{i_1,k}^{m_1} \hat{\mathbf{S}}^{i_1+k} F_{i_2,k}^{m_2} \hat{\mathbf{S}}^{i_2+k}\right], \tag{4.10}$$

with $\rho_k^{*m_1,m_2}$ is the covariance coefficient corresponding to $\sigma_k^{*m_1,m_2}$ defined in (4.7), i.e.

$$\rho_k^{*m_1,m_2} := \begin{cases} \dfrac{\sigma_k^{*m_1,m_2}}{f_k^{m_1} f_k^{m_2}}, & \text{for } f_k^{m_1} f_k^{m_2} \neq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4.11}$$

Now we want to take the linear operators $F_{i,k}^{m}$ out of the expectation. Therefore, we define the following linear operators:

$$\mathbf{H}_k(\tau) : \mathbb{L}_k^I \otimes \mathbb{L}_k^I \to \mathbb{L}_{k+1}^I \otimes \mathbb{L}_{k+1}^I$$

by

$$\begin{aligned}&\big(\mathbf{H}_k(\tau)\,xy\big)_{i_1,k_1,i_2,k_2}^{m_1,m_2} \\ &:= \begin{cases}\mathbf{F}_{i_1,k}^{m_1,I\vee(i_1+k_1)}x\,\mathbf{F}_{i_2,k}^{m_2,I\vee(i_2+k_2)}y, & \text{for } i_1\wedge i_2 \le I-k-1 \text{ or } k_1\wedge k_2\le k,\\ (1+\tau_{i_1,i_2,k}^{m_1,m_2})\,F_{i_1,k}^{m_1}x\,F_{i_2,k}^{m_2}y, & \text{otherwise,}\end{cases}\end{aligned} \tag{4.12}$$

where τ is a $M\times M\times I\times I\times(J-1)$ matrix of real numbers.

Note, $\mathbf{F}_{i_1,k}^{m_1,I\vee(i_1+k_1)} = F_{i_1,k}^{m_1}$, for $i_1+k_1 > I$ and $k_1 = k$, and $\mathbf{F}_{i_1,k}^{m_1,I\vee(i_1+k_1)}x = x$ in all other cases of the first line of the definition of $\mathbf{H}_k(\tau)$.

Concatenations of those operators will be denoted by

$$\begin{aligned}\mathbf{H}_{k_2\leftarrow k_1}(\tau) &:= \begin{cases}\mathbf{H}_{k_2\leftarrow k_1}(\tau)\mathbf{H}_{k_2-1}(\tau)\cdots\mathbf{H}_{k_1}(\tau), & \text{for } k_2\ge k_1,\\ \Pi_{\mathbb{L}_{k_2+1}^I\otimes\mathbb{L}_{k_2+1}^I}, & \text{otherwise,}\end{cases}\\ \mathbf{H}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\tau)x &:= \big(H_{(k_1\vee k_2)\leftarrow 0}(\tau)x\big)_{i_1,k_1+1,i_2,k_2+1}^{m_1,m_2},\end{aligned} \tag{4.13}$$

where $\Pi_{\mathbb{L}_{k_2+1}^I\otimes\mathbb{L}_{k_2+1}^I}$ denotes the projection onto $\mathbb{L}_{k_2+1}^I\otimes\mathbb{L}_{k_2+1}^I$.

Corollary 4.5. *At point $\tau = 0$ we have*

$$\mathbf{H}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(0)xy = \mathbf{F}_{i_1,k_1}^{m_1,I}x\,\mathbf{F}_{i_2,k_2}^{m_2,I}y.$$

Moreover, a linearisation of $\mathbf{H}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\tau)$ at $\tau = 0$ yields

$$\begin{aligned}&\mathbf{H}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\tau)xy - \mathbf{F}_{i_1,k_1}^{m_1,I}x\,\mathbf{F}_{i_2,k_2}^{m_2,I}y\\ &\approx \sum_{j=I-(i_1\wedge i_2)}^{k_1\wedge k_2}\sum_{l_1,l_2=0}^{M}\sum_{h_1,h_2=I-j}^{I}\Big(\mathbf{F}_{i_1,k_1}^{m_1,h_1+j+1}\Big)_{h_1,j+1}^{l_1}\Big(\mathbf{F}_{i_2,k_2}^{m_2,h_2+j+1}\Big)_{h_2,j+1}^{l_2}\\ &\qquad\cdot\tau_{h_1,h_2,j}^{l_1,l_2}\,\mathbf{F}_{h_1,j}^{l_1,I}x\,\mathbf{F}_{h_2,j}^{l_2,I}y.\end{aligned} \tag{4.14}$$

Proof. The first statement of Corollary 4.5 is a direct consequence of the definition of $\mathbf{H}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\tau)$. Moreover, $\tau_{h_1,h_2,j}^{l_1,l_2}$ is only contained within the $(l_1, l_2, h_1, j+1, h_2, j+1)$ coordinate of $\mathbf{H}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\tau)$. This proves (4.14). □

Iterating (4.9) and (4.10) we get for $I \leq i_1 + k_1, i_2 + k_2$

$$\mathrm{E}^*\left[\widehat{S}^{m_1}_{i_1,k_1+1}\,\widehat{S}^{m_2}_{i_2,k_2+1}\right] = \mathbf{H}^{m_1,m_2}_{i_1,k_1,i_2,k_2}(\boldsymbol{\rho}^*)\mathbf{S}^I\mathbf{S}^I \tag{4.15}$$

with

$$\mathbf{S}^I\mathbf{S}^I := \left(S^{m_1}_{i_1,j_1}S^{m_2}_{i_2,j_2}\right)^{0\leq m_1,m_2\leq M}_{i_1+j_1,i_2+j_2\leq I}$$

and

$$\boldsymbol{\rho}^{*\,m_1,m_2}_{i_1,i_2,k} := \rho^{*\,m_1,m_2}_k. \tag{4.16}$$

Combining (4.13) with Corollary 4.5 and replacing all unknown parameters by their estimates we get

Estimator 4.6 (of the single period parameter estimation error)

Assume $S^m_{i,k}$ satisfy Assumption 2.1 and take arbitrary $\mathcal{D}^I$-measurable factors α^m_i, $m \in \mathcal{M} \subseteq \{0, \ldots, M\}$. Then the parameter estimation error for accident period i can be estimated by

$$\hat{\Delta}^{\mathcal{M}}_i := \sum_{m_1,m_2\in\mathcal{M}} \alpha^{m_1}_i\alpha^{m_2}_i \sum_{k_1,k_2=I-i}^{J-1}\left(\widehat{\mathbf{H}}^{m_1,m_2}_{i,k_1,i,k_2}(\widehat{\boldsymbol{\rho}}^*) - \widehat{\mathbf{H}}^{m_1,m_2}_{i,k_1,i,k_2}(0)\right)\mathbf{S}^I\mathbf{S}^I.$$

where the operator $\widehat{\mathbf{H}}(\widehat{\boldsymbol{\rho}}^*)$ is defined in the same way as the operator $\mathbf{H}(\widehat{\boldsymbol{\rho}}^*)$, see (4.12) *and* (4.13)*, but with $\hat{f}^m_k$ instead of f^m_k.*
Moreover, a linear approximation for the operator $\widehat{\mathbf{H}}(\tau)$ at $\tau = 0$ leads to

$$\hat{\Delta}^{\mathcal{M}}_i \approx \sum_{m_1,m_2\in\mathcal{M}} \alpha^{m_1}_i\alpha^{m_2}_i \sum_{k_1,k_2=I-i}^{J-1}\;\sum_{j=I-i}^{k_1\wedge k_2}\;\sum_{l_1,l_2=0}^{M}\;\sum_{h_1,h_2=I-j}^{I} \hat{\rho}^{*\,l_1,l_2}_j\,\widehat{S}^{l_1}_{h_1,j+1}\widehat{S}^{l_2}_{h_2,j+1} \cdot \left(\widehat{\mathbf{F}}^{m_1,h_1+j+1}_{i,k_1}\right)^{l_1}_{h_1,j+1}\left(\widehat{\mathbf{F}}^{m_2,h_2+j+1}_{i,k_1}\right)^{l_2}_{h_2,j+1}.$$

4.3. Single period mean squared error of prediction

Combining the results of the previous two sections we obtain

Estimator 4.7 (of the mse of prediction for a single accident period)

Assume $S^m_{i,k}$ satisfy Assumption 2.1 and take arbitrary $\mathcal{D}^I$-measurable factors α^m_i, $m \in \mathcal{M} \subseteq \{0, \ldots, M\}$. Then the mean squared error of prediction for the projected claim properties of accident period i can be estimated by

$$\widehat{\mathrm{mse}}\left[\sum_{m\in\mathcal{M}}\alpha_i^m\sum_{k=I-i}^{J-1}\widehat{S}_{i,k+1}^m\right]$$

$$:=\sum_{m_1,m_2\in\mathcal{M}}\alpha_i^{m_1}\alpha_i^{m_2}\sum_{k_1,k_2=I-i}^{J-1}\Bigg[\left(\widehat{\mathbf{H}}_{i,k_1,i,k_2}^{m_1,m_2}(\widehat{\boldsymbol{\rho}}^*)-\widehat{\mathbf{H}}_{i,k_1,i,k_2}^{m_1,m_2}(0)\right)\mathbf{S}^I\mathbf{S}^I$$

$$+\sum_{l_1,l_2=0}^{M}\sum_{n=I+1}^{i+(k_1\wedge k_2)+1}\sum_{j=n-1-I}^{J-1}\widehat{\sigma}_j^{l_1,l_2}\widehat{R}_{n-1-j,j}^{l_1,l_2}\left(\widehat{\mathbf{F}}_{i,k_1}^{m_1,n}\right)_{n-1-j,j+1}^{l_1}\left(\widehat{\mathbf{F}}_{i,k_2}^{m_2,n}\right)_{n-1-j,j+1}^{l_2}\Bigg].$$

Moreover, a linear approximation for the operator $\widehat{\mathbf{H}}(\tau)$ *at* $\tau=0$ *leads to*

$$\widehat{\mathrm{mse}}\left[\sum_{m\in\mathcal{M}}\alpha_i^m\sum_{k=I-i}^{J-1}\widehat{S}_{i,k+1}^m\right]\approx\sum_{m_1,m_2\in\mathcal{M}}\alpha_i^{m_1}\alpha_i^{m_2}\sum_{k_1,k_2=I-i}^{J-1}\sum_{l_1,l_2=0}^{M}$$

$$\Bigg[\sum_{n=I+1}^{i+(k_1\wedge k_2)+1}\sum_{j=n-1-I}^{J-1}\widehat{\sigma}_j^{l_1,l_2}\widehat{R}_{n-1-j,j}^{l_1,l_2}\left(\widehat{\mathbf{F}}_{i,k_1}^{m_1,n}\right)_{n-1-j,j+1}^{l_1}\left(\widehat{\mathbf{F}}_{i,k_2}^{m_2,n}\right)_{n-1-j,j+1}^{l_2}$$

$$+\sum_{j=I-i}^{k_1\wedge k_2}\sum_{h_1,h_2=I-j}^{I}\widehat{\rho}_j^{*l_1,l_2}\widehat{S}_{h_1,j+1}^{l_1}\widehat{S}_{h_2,j+1}^{l_2}\left(\widehat{\mathbf{F}}_{i,k_1}^{m_1,h_1+j+1}\right)_{h_1,j+1}^{l_1}\left(\widehat{\mathbf{F}}_{i,k_2}^{m_2,h_2+j+1}\right)_{h_2,j+1}^{l_2}\Bigg].$$

Remark 4.8. *For the Chain-Ladder-Method the stated estimator is the same as in Buchwalder et al. [2, Approach 3] and the linear approximation is the same as in Mack [6].*
Moreover, for the Extended-Complementary-Loss-Ratio-Method the linear approximation is the same as in Dahms [3].

4.4. Overall mean squared error of prediction

Since the estimators $\widehat{S}_{i_1,k_1}^{m_1}$ and $\widehat{S}_{i_2,k_2}^{m_2}$ depend on the observed data of all accident periods they are usually not uncorrelated. Therefore, the overall mean squared error of prediction is not equal to the sum of all single period mean squared errors of prediction. As in Section 4 we can decompose the overall mean squared error of prediction as follows

$$\mathrm{mse}\left[\sum_{m\in\mathcal{M}}\sum_{i=0}^{I}\alpha_i^m\sum_{k=I-i}^{J-1}\widehat{S}_{i,k+1}^m\right]$$

$$=\mathrm{Var}\left[\sum_{m\in\mathcal{M}}\sum_{i=0}^{I}\alpha_i^m\sum_{k=I-i}^{J-1}S_{i,k+1}^m\middle|\mathcal{D}^I\right]+\left(\sum_{m\in\mathcal{M}}\sum_{i=0}^{I}\alpha_i^m\sum_{k=I-i}^{J-1}\mathrm{E}\left[S_{i,k+1}^m-\widehat{S}_{i,k+1}^m\middle|\mathcal{D}^I\right]\right)^2$$

$$=\qquad\text{process variance}\qquad+\qquad\text{parameter estimation error}$$

Using the same arguments like in Sections 4.1 and 4.2 we get

Estimator 4.9 (of the overall mean squared error of prediction)

Assume $S_{i,k}^{m}$ satisfy Assumption 2.1 and take arbitrary $\mathcal{D}^I$-measurable factors α_i^m, $m \in \mathcal{M} \subseteq \{0, \dots, M\}$. Then the overall mean squared error of prediction for the projected claim properties can be estimated by

$$\widehat{\mathrm{mse}}\left[\sum_{i=0}^{I}\sum_{m\in\mathcal{M}}\alpha_i^m\sum_{k=I-i}^{J-1}\widehat{S}_{i,k+1}^{m}\right]$$
$$:= \sum_{i_1,i_2=0}^{I}\sum_{m_1,m_2\in\mathcal{M}}\alpha_{i_1}^{m_1}\alpha_{i_2}^{m_2}\sum_{k_1=I-i_1}^{J-1}\sum_{k_2=I-i_2}^{J-1}\Bigg[\Big(\widehat{\mathbf{H}}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\widehat{\boldsymbol{\rho}}^*) - \widehat{\mathbf{H}}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(0)\Big)\mathbf{S}^I\mathbf{S}^I$$
$$+ \sum_{l_1,l_2=0}^{M}\sum_{n=I+1}^{(i_1+k_1)\wedge(i_2+k_2)+1}\sum_{j=n-1-I}^{J-1}\widehat{\sigma}_j^{l_1,l_2}\widehat{R}_{n-1-j,j}^{l_1,l_2}\left(\widehat{\mathbf{F}}_{i_1,k_1}^{m_1,n}\right)_{n-1-j,\,j+1}^{l_1}\left(\widehat{\mathbf{F}}_{i_2,k_2}^{m_2,n}\right)_{n-1-j,\,j+1}^{l_2}\Bigg].$$

Moreover, a linear approximation for the operator $\widehat{\mathbf{H}}(\tau)$ at $\tau = 0$ leads to

$$\widehat{\mathrm{mse}}\left[\sum_{i=0}^{I}\sum_{m\in\mathcal{M}}\alpha_i^m\sum_{k=I-i}^{J-1}\widehat{S}_{i,k+1}^{m}\right] \approx \sum_{i_1,i_2=0}^{I}\sum_{m_1,m_2\in\mathcal{M}}\alpha_{i_1}^{m_1}\alpha_{i_2}^{m_2}\sum_{k_1=I-i_1}^{J-1}\sum_{k_2=I-i_2}^{J-1}\sum_{l_1,l_2=0}^{M}$$
$$\Bigg[\sum_{n=I+1}^{(i_1+k_1)\wedge(i_2+k_2)+1}\sum_{j=n-1-I}^{J-1}\widehat{\sigma}_j^{l_1,l_2}\widehat{R}_{n-1-j,j}^{l_1,l_2}\left(\widehat{\mathbf{F}}_{i_1,k_1}^{m_1,n}\right)_{n-1-j,\,j+1}^{l_1}\left(\widehat{\mathbf{F}}_{i_2,k_2}^{m_2,n}\right)_{n-1-j,\,j+1}^{l_2}$$
$$+ \sum_{j=I-(i_1\wedge i_2)}^{k_1\wedge k_2}\sum_{h_1,h_2=I-j}^{I}\widehat{\rho}_j^{*l_1,l_2}\widehat{S}_{h_1,j+1}^{l_1}\widehat{S}_{h_2,j+1}^{l_2}\left(\widehat{\mathbf{F}}_{i_1,k_1}^{m_1,h_1+j+1}\right)_{h_1,j+1}^{l_1}\left(\widehat{\mathbf{F}}_{i_2,k_2}^{m_2,h_2+j+1}\right)_{h_2,j+1}^{l_2}\Bigg].$$

Remark 4.10. *For the Chain-Ladder-Method the stated estimator is the same as in Buchwalder et al. [2, Approach 3] and the linear approximation is the same as in Mack [6].*
Moreover, for the Extended-Complementary-Loss-Ratio-Method the linear approximation is the same as in Dahms [3].

5. Solvency reserving risk

In this section we want to look at what we can say at the end of business period I about the development result related to the estimates $\widehat{S}_{i,k}^{m,I+1}$ at the end of the next business period, assuming that we will take the same *LSRM*. For the projection of payments this means we want to analyse the profit or loss of the next business period related to the estimated reserves.

In order to distinguish between the objects of the previous sections, which belong to estimation period I, and the objects of the next estimation period $I + 1$, we will introduce, if necessary, an additional upper index that indicates the time which the object belongs to.

Taking the same *LSRM* means:

Assumption 5.1. *There exist $\mathcal{D}^I \cap \mathcal{D}_k$-measurable factors $0 \le w_{I-k}^{m,I+1} \le 1$ with*

- $R_{I-k,k}^m = 0$ *implies* $w_{I-k}^{m,I+1} = 0$,
- $w_{i,k}^{m,I+1} = (1 - w_{I-k}^{m,I+1})\, w_{i,k}^{m,I}$ *for* $0 \le i \le I - 1 - k$.

Remark 5.2. *The above assumption means that we do not change our (relative) believes into the old development periods and only put some credibility $w_{I-k}^{m,I+1}$ to the new encountered development.*
The variance minimizing weights, introduced in Remark 3.2, satisfy Assumption 5.1.

The estimates of the model parameters for the next period are given by

$$\hat{f}_k^{m,I+1} := \sum_{i=0}^{I-k} w_{i,k}^{m,I+1} \frac{S_{i,k+1}^m}{R_{i,k}^m}, \quad \text{for } 1 \le k \le J-1. \tag{5.1}$$

Note, the estimates $\hat{f}_k^{m,I+1}$ for the model parameters f_k^m may depend on $S_{I-k,k+1}^m$ and are therefore usually not $\mathcal{D}^I$-measurable. Their at time I expected values are

$$\bar{f}_k^m := \mathrm{E}\left[\hat{f}_k^{m,I+1} \middle| \mathcal{D}^I\right] = \left(1 - w_{I-k,k}^{m,I+1}\right) \hat{f}_k^{m,I} + w_{I-k,k}^{m,I+1} f_k^m. \tag{5.2}$$

Therefore, the estimate of the at time I expected value of the model parameter $f_k^{m,I+1}$ is

$$\hat{\bar{f}}_k^m = \hat{f}_k^{m,I}. \tag{5.3}$$

Using (5.2) we compute for the $\mathcal{D}^I$-conditional expected value of the next years projected claim properties

$$\bar{S}_{i,k+1}^m := \mathrm{E}\left[\widehat{S}_{i,k+1}^{m,I+1} \middle| \mathcal{D}^I\right] = \bar{\mathbf{F}}_{i,k}^{m,I+1} \mathbf{F}^I \mathbf{S}^I, \tag{5.4}$$

where $\bar{\mathbf{F}}_{i,k}^{m,n}$ is defined in the same way as $\mathbf{F}_{i,k}^{m,n}$, see (3.5), but with $\bar{f}_k^m$ instead of f_k^m. For the exposures we get

$$\begin{aligned} \bar{R}_{i,k}^m &:= \mathrm{E}\left[\widehat{R}_{i,k}^{m,I+1} \middle| \mathcal{D}^I\right] = \sum_{l=0}^{M} \sum_{h=0}^{I} \sum_{j=0}^{(i+k-h)\wedge k} \gamma_{i,k,h,j}^{m,l} \bar{S}_{h,j}^l \\ \bar{R}_{i,k}^{m_1,m_2} &:= \mathrm{E}\left[\widehat{R}_{i,k}^{m_1,m_2,I+1} \middle| \mathcal{D}^I\right] = \sum_{l=0}^{M} \sum_{h=0}^{I} \sum_{j=0}^{(i+k-h)\wedge k} \gamma_{i,k,h,j}^{m_1,m_2,l} \bar{S}_{h,j}^l \end{aligned} \tag{5.5}$$

with exposure parameter $\gamma_{i,k,h,j}^{m,l}$ and $\gamma_{i,k,h,j}^{m_1,m_2,l}$, see (2.4).

In order to shorten notations we define

$$\bar{\mathbf{F}}_{i,k}^{m,n} := \mathbf{F}_{i,k}^{m,n} \quad \text{and} \quad \hat{\mathbf{S}}^{n,I+1} = \bar{\mathbf{S}}^{n} := \mathbf{S}^{n}$$

for $n \leq I$, and analogously for the exposures $\widehat{R}_{i,k}^{m,I+1}$, $\bar{R}_{i,k}^{m,I+1}$, $\widehat{R}_{i,k}^{m_1,m_2,I+1}$ and $\bar{R}_{i,k}^{m_1,m_2,I+1}$.

The at time $I+1$ observed (claims) development result (CDR) of a linear combination of claim properties for a single accident period i is given by

$$\mathrm{CDR}_i^{\mathcal{M},I+1} := \sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \left(\widehat{S}_{i,k+1}^{m,I+1} - \widehat{S}_{i,k+1}^{m,I}\right), \tag{5.6}$$

where α_i^m are arbitrary $\mathcal{D}^I$-measurable real numbers. Since the estimates $\widehat{S}_{i,k}^{m,I}$ and $\widehat{S}_{i,k}^{m,I+1}$ are unbiased, the expected development result will be zero. Moreover, because of (5.3) and (5.4), the at time I estimated $\mathcal{D}^I$-conditional expected value of the CDR is zero, too.

Now, we want to look at the uncertainty of the observed development result in terms of the $\mathcal{D}^I$-conditional mean squared error of prediction.

As for the ultimate mean squared error of prediction, see Section 4, we can split the mse of the observed development result for a single accident period i into a process variance term and a parameter estimation error term:

$$\mathrm{mse}\left[\mathrm{CDR}_i^{\mathcal{M},I+1}\right] := \mathrm{E}\left[\left(\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \left(\widehat{S}_{i,k+1}^{m,I+1} - \widehat{S}_{i,k+1}^{m,I}\right) - 0\right)^2 \middle| \mathcal{D}^I\right]$$

$$= \mathrm{Var}\left[\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \widehat{S}_{i,k+1}^{m,I+1} \middle| \mathcal{D}^I\right] + \left(\mathrm{E}\left[\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \left(\widehat{S}_{i,k+1}^{m,I+1} - \widehat{S}_{i,k+1}^{m,I}\right) \middle| \mathcal{D}^I\right]\right)^2.$$

5.1. Process variance of a single period CDR

We will split the process variance term of the CDR as follows

$$\begin{aligned} &\mathrm{Var}\left[\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \widehat{S}_{i,k+1}^{m,I+1} \middle| \mathcal{D}^I\right] \\ &\quad = \mathrm{E}\left[\left(\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \widehat{S}_{i,k+1}^{m,I+1}\right)^2 \middle| \mathcal{D}^I\right] - \left(\mathrm{E}\left[\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \widehat{S}_{i,k+1}^{m,I+1} \middle| \mathcal{D}^I\right]\right)^2. \end{aligned} \tag{5.7}$$

In order to get estimators for the first addend on the right hand side let us start with some computations of $\mathcal{D}^I$-conditional expectations of products of $\widehat{S}_{i,k}^{m,I+1}$. Therefore, take $k_1 < k_2$ and $k_2 + i_2 \geq I$. Then we get

$$\mathrm{E}\Big[\widehat{S}^{m_1,I+1}_{i_1,k_1+1}\,\widehat{S}^{m_2,I+1}_{i_2,k_2+1}\Big|\mathcal{D}^I\Big] = \mathrm{E}\Big[\mathrm{E}\Big[\widehat{S}^{m_1,I+1}_{i_1,k_1+1}\,\widehat{S}^{m_2,I+1}_{i_2,k_2+1}\Big|\mathcal{D}^I_{k_2}\Big]\Big|\mathcal{D}^I\Big]$$
$$= \mathrm{E}\Big[\widehat{S}^{m_1,I+1}_{i_1,k_1+1}\,\bar{F}^{m_2}_{i_2,k_2}\,\widehat{\mathbf{S}}^{i_2+k_2,I+1}\Big|\mathcal{D}^I\Big].$$

In case of $k_1 = k_2 =: k$ we compute

$$\mathrm{E}\Big[\widehat{S}^{m_1,I+1}_{i_1,k+1}\,\widehat{S}^{m_2,I+1}_{i_2,k+1}\Big|\mathcal{D}^I\Big] = \mathrm{E}\Big[\Big(1+\bar{\rho}^{m_1,m_2}_{i_1,i_2,k}\Big)\,\bar{F}^{m_1}_{i_1,k}\,\widehat{\mathbf{S}}^{i_1+k,I+1}\bar{F}^{m_2}_{i_2,k}\,\widehat{\mathbf{S}}^{i_2+k,I+1}\Big|\mathcal{D}^I\Big]$$

with

$$\bar{\rho}^{m_1,m_2}_{i_1,i_2,k} := \begin{cases} \mathrm{Cov}\left[\dfrac{\widehat{f}^{m_1,I+1}_k\widehat{f}^{m_2,I+1}_k}{\widehat{f}^{m_1}_k\widehat{f}^{m_2}_k}\middle|\mathcal{D}^I_k\right], & \text{for } i_1,i_2 > I+1-k,\\ \mathrm{Cov}\left[\dfrac{S^{m_1}_{i_1,k+1}\widehat{f}^{m_2,I+1}_k}{f^{m_1}_k R^{m_1}_{i_1,k}\widehat{f}^{m_2}_k}\middle|\mathcal{D}^I_k\right], & \text{for } i_2 > i_1 = I+1-k,\\ \mathrm{Cov}\left[\dfrac{\widehat{f}^{m_1,I+1}_k S^{m_2}_{i_2,k+1}}{\bar{f}^{m_1}_k f^{m_2}_k R^{m_2}_{i_2,k}}\middle|\mathcal{D}^I_k\right], & \text{for } i_1 > i_2 = I+1-k,\\ \mathrm{Cov}\left[\dfrac{S^{m_1}_{i_1,k+1}S^{m_2}_{i_2,k+1}}{f^{m_1}_k R^{m_1}_{i_1,k}f^{m_2}_k R^{m_2}_{i_2,k}}\middle|\mathcal{D}^I_k\right], & \text{for } i_1 = i_2 = I+1-k,\\ 0, & \text{otherwise or denominator equals zero.} \end{cases}$$

A short calculation yields

$$\bar{\rho}^{m_1,m_2}_{i_1,i_2,k} = \begin{cases} w^{m_1,I+1}_{I-k,k}\,w^{m_2,I+1}_{I-k,k}\,\dfrac{\sigma^{m_1,m_2}_k}{\bar{f}^{m_1}_k\bar{f}^{m_2}_k}\,\dfrac{R^{m_1,m_2}_{I-k,k}}{R^{m_1}_{I-k,k}R^{m_2}_{I-k,k}}, & \text{for } i_1,i_2 > I-k,\\ w^{m_2,I+1}_{I-k,k}\,\dfrac{\sigma^{m_1,m_2}_k}{f^{m_1}_k\bar{f}^{m_2}_k}\,\dfrac{R^{m_1,m_2}_{I-k,k}}{R^{m_1}_{I-k,k}R^{m_2}_{I-k,k}}, & \text{for } i_2 > i_1 = I-k,\\ w^{m_1,I+1}_{I-k,k}\,\dfrac{\sigma^{m_1,m_2}_k}{\bar{f}^{m_1}_k f^{m_2}_k}\,\dfrac{R^{m_1,m_2}_{I-k,k}}{R^{m_1}_{I-k,k}R^{m_2}_{I-k,k}}, & \text{for } i_1 > i_2 = I-k,\\ \dfrac{\sigma^{m_1,m_2}_k}{f^{m_1}_k f^{m_2}_k}\,\dfrac{R^{m_1,m_2}_{I-k,k}}{R^{m_1}_{I-k,k}R^{m_2}_{I-k,k}}, & \text{for } i_1 = i_2 = I-k,\\ 0, & \text{otherwise or denominator equals zero.} \end{cases}$$

Finally we use the same arguments like in Section 4.2 and replace all unknown parameters by their estimators at time I. This leads to:

Estimator 5.3 (of the process variance of $\mathrm{CDR}_i^{\mathcal{M},I+1}$)

Assume $S_{i,k}^m$ satisfy Assumptions 2.1 and 5.1 and take arbitrary $\mathcal{D}^I$-measurable factors α_i^m, $m \in \mathcal{M} \subseteq \{0, \ldots, M\}$. Then the process variance of the claim development result of a single accident period can be estimated by

$$\widehat{\mathrm{Var}}\left[\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \widehat{S}_{i,k+1}^{m,I+1} \middle| \mathcal{D}^I\right]$$
$$:= \sum_{m_1,m_2\in\mathcal{M}} \alpha_i^{m_1}\alpha_i^{m_2} \sum_{k_1,k_2=I-i}^{J-1} \left(\widehat{\mathbf{H}}_{i,k_1,i,k_2}^{m_1,m_2}\left(\tfrac{1}{\boldsymbol{\rho}^*}\right) - \widehat{\mathbf{H}}_{i,k_1,i,k_2}^{m_1,m_2}(0)\right)\mathbf{S}^I\mathbf{S}^I.$$

Moreover, a linear approximation for the operator $\widehat{\mathbf{H}}(\tau)$ at $\tau = 0$ leads to

$$\widehat{\mathrm{Var}}\left[\sum_{m\in\mathcal{M}} \alpha_i^m \sum_{k=I-i}^{J-1} \widehat{S}_{i,k+1}^{m,I+1} \middle| \mathcal{D}^I\right]$$
$$\approx \sum_{m_1,m_2\in\mathcal{M}} \alpha_i^{m_1}\alpha_i^{m_2} \sum_{k_1,k_2=I-i}^{J-1} \sum_{l_1,l_2=0}^{M} \sum_{j=I-i}^{k_1\wedge k_2} \sum_{h_1,h_2=I-j}^{I} \widehat{\rho}_{h_1,h_2,j}^{l_1,l_2} \widehat{S}_{h_1,j+1}^{l_1} \widehat{S}_{h_2,j+1}^{l_2}$$
$$\cdot \left(\widehat{\mathbf{F}}_{i,k_1}^{m_1,h_1+j+1}\right)_{h_1,j+1}^{l_1} \left(\widehat{\mathbf{F}}_{i,k_2}^{m_2,h_2+j+1}\right)_{h_2,j+1}^{l_2}$$

5.2. Parameter estimation error of a single period CDR

As for the ultimate parameter estimation error in Section 4.2 we use the resampling method and estimate

$$\bar{\Delta}_i^{\mathcal{M}} := \sum_{m_1,m_2\in\mathcal{M}} \alpha_i^{m_1}\alpha_i^{m_2} \sum_{k_1,k_2=I-i}^{J-1} \left(\bar{S}_{i,k_1+1}^{m_1} - \widehat{S}_{i,k_1+1}^{m_2}\right)\left(\bar{S}_{i,k_2+1}^{m_1} - \widehat{S}_{i,k_2+1}^{m_2}\right)$$

by its expectation under the resampling measure P^*. Hence, we have to analyse terms of the form

$$\mathrm{E}^*\left[\bar{S}_{i_1,k_1+1}^{m_1}\bar{S}_{i_2,k_2+1}^{m_2}\right] - \mathrm{E}^*\left[\bar{S}_{i_1,k_1+1}^{m_1}\widehat{S}_{i_2,k_2+1}^{m_2}\right] - \mathrm{E}^*\left[\widehat{S}_{i_1,k_1+1}^{m_1}\bar{S}_{i_2,k_2+1}^{m_2}\right] + \mathrm{E}^*\left[\widehat{S}_{i_1,k_1+1}^{m_1}\widehat{S}_{i_2,k_2+1}^{m_2}\right]. \tag{5.8}$$

We already know the last addend from Section 4.2:

$$\mathrm{E}^*\left[\widehat{S}_{i_1,k_1+1}^{m_1}\widehat{S}_{i_2,k_2+1}^{m_2}\right] = \mathbf{H}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\boldsymbol{\rho}^*)\mathbf{S}^I\mathbf{S}^I.$$

The other three addends of the right hand side of (5.8) will by analyse in the same way. Using the properties of the resampling measure P^* stated in Remark 4.4 we obtain

$$\mathrm{E}^*\Big[\bar{S}^{m_1}_{i_1,k_1+1}\,\bar{S}^{m_2}_{i_2,k_2+1}\Big] = \mathrm{E}^*\Big[F^{m_1}_{i_1,k_1}\,\bar{\mathbf{S}}^{i_1+k}\,\bar{S}^{m_2}_{i_2,k_2}\Big]$$

$$\mathrm{E}^*\Big[\widehat{S}^{m_1}_{i_1,k_1+1}\,\bar{S}^{m_2}_{i_2,k_2+1}\Big] = \mathrm{E}^*\Big[F^{m_1}_{i_1,k_1}\,\widehat{\mathbf{S}}^{i_1+k}\,\bar{S}^{m_2}_{i_2,k_2}\Big]$$

$$\mathrm{E}^*\Big[\bar{S}^{m_1}_{i_1,k_1+1}\,\widehat{S}^{m_2}_{i_2,k_2+1}\Big] = \mathrm{E}^*\Big[F^{m_1}_{i_1,k_1}\,\bar{\mathbf{S}}^{i_1+k}\,\widehat{S}^{m_2}_{i_2,k_2}\Big]$$

for $k_1 > k_2$ and $i_1 + k_1 \geq I$. Moreover, in case of $k_1 = k_2 =: k$ the last three identities still hold if at least one claim property lies on or above the diagonal $I + 1$. Otherwise, we get

$$\mathrm{E}^*\Big[\bar{S}^{m_1}_{i_1,k+1}\,\bar{S}^{m_2}_{i_2,k+1}\Big] = \mathrm{E}^*\Big[\Big(1+\rho^{*12\,m_1,m_2}_{i_1,i_2,k}\Big)F^{m_1}_{i_1,k}\,\bar{\mathbf{S}}^{i_1+k}F^{m_2}_{i_2,k}\,\bar{\mathbf{S}}^{i_2+k}\Big]$$

$$\mathrm{E}^*\Big[\widehat{S}^{m_1}_{i_1,k+1}\,\bar{S}^{m_2}_{i_2,k+1}\Big] = \mathrm{E}^*\Big[\Big(1+\rho^{*2\,m_1,m_2}_{i_1,i_2,k}\Big)F^{m_1}_{i_1,k}\,\widehat{\mathbf{S}}^{i_1+k}F^{m_2}_{i_2,k}\,\bar{\mathbf{S}}^{i_2+k}\Big]$$

$$\mathrm{E}^*\Big[\bar{S}^{m_1}_{i_1,k+1}\,\widehat{S}^{m_2}_{i_2,k+1}\Big] = \mathrm{E}^*\Big[\Big(1+\rho^{*1\,m_1,m_2}_{i_1,i_2,k}\Big)F^{m_1}_{i_1,k}\,\bar{\mathbf{S}}^{i_1+k}F^{m_2}_{i_2,k}\,\widehat{\mathbf{S}}^{i_2+k}\Big]$$

with

$$\rho^{*1\,m_1,m_2}_{i_1,i_2,k} := \begin{cases}\Big(1-w^{m_1,I+1}_{I-k,k}\Big)\rho^{*\,m_1,m_2}_{i_1,i_2,k}, & \text{for } i_1+k > I+1 \text{ and } i_2+k \geq I+1,\\ 0, & \text{otherwise,}\end{cases}$$

$$\rho^{*2\,m_1,m_2}_{i_1,i_2,k} := \begin{cases}\Big(1-w^{m_2,I+1}_{I-k,k}\Big)\rho^{*\,m_1,m_2}_{i_1,i_2,k}, & \text{for } i_1+k \geq I+1 \text{ and } i_2+k > I+1,\\ 0, & \text{otherwise,}\end{cases}$$

$$\rho^{*12\,m_1,m_2}_{i_1,i_2,k} := \begin{cases}\Big(1-w^{m_1,I+1}_{I-k,k}\Big)\Big(1-w^{m_2,I+1}_{I-k,k}\Big)\rho^{*\,m_1,m_2}_{i_1,i_2,k}, & \text{for } i_1+k,\ i_2+k > I+1,\\ 0, & \text{otherwise.}\end{cases}$$

Summarizing all parts and replacing all unknown parameters by their estimators yields

Estimator 5.4 (of the parameter estimation error of $\mathrm{CDR}^{\mathcal{M},I+1}_i$)

Assume $S^m_{i,k}$ satisfy Assumptions 2.1 and 5.1 and take arbitrary $\mathcal{D}^I$-measurable factors α^m_i, $m \in \mathcal{M} \subseteq \{0, \ldots, M\}$. Then the parameter estimation error of the claim development result of a single accident period can be estimated by

$$\widehat{\Delta}^{\mathcal{M}}_i := \sum_{m_1,m_2\in\mathcal{M}} \alpha^{m_1}_i\alpha^{m_2}_i \sum_{k_1,k_2=I-i}^{J-1}\Big(\widehat{\mathbf{H}}^{m_1,m_2}_{i,k_1,i,k_2}(\widehat{\boldsymbol{\rho}}^*) - \widehat{\mathbf{H}}^{m_1,m_2}_{i,k_1,i,k_2}(\widehat{\boldsymbol{\rho}}^{*1}) - \widehat{\mathbf{H}}^{m_1,m_2}_{i,k_1,i,k_2}(\widehat{\boldsymbol{\rho}}^{*2}) + \widehat{\mathbf{H}}^{m_1,m_2}_{i,k_1,i,k_2}(\widehat{\boldsymbol{\rho}}^{*12})\Big)\widehat{\mathbf{S}}^I\,\widehat{\mathbf{S}}^I.$$

Moreover, a linear approximation for the operator $\widehat{\mathbf{H}}(\tau)$ at $\tau = 0$ leads to

$$\widehat{\Delta}_i^m \approx \sum_{m_1, m_2 \in \mathcal{M}} \alpha_i^{m_1} \alpha_i^{m_2} \sum_{k_1, k_2 = I-i}^{J-1} \sum_{l_1, l_2 = 0}^{M} \sum_{j = I-i}^{k_1 \wedge k_2} \sum_{h_1, h_2 = I-j}^{I} \left(\widehat{\rho}_{h_1, h_2, j}^{*l_1, l_2} - \widehat{\rho}_{h_1, h_2, j}^{*1\, l_1, l_2} - \widehat{\rho}_{h_1, h_2, j}^{*2\, l_1, l_2} + \widehat{\rho}_{h_1, h_2, j}^{*12\, l_1, l_2}\right)$$
$$\cdot \left(\widehat{\mathbf{F}}_{i, k_1}^{m_1, h_1 + j + 1}\right)_{h_1, j+1}^{l_1} \left(\widehat{\mathbf{F}}_{i, k_2}^{m_2, h_2 + j + 1}\right)_{h_2, j+1}^{l_2} \widehat{S}_{h_1, j+1}^{l_1} \widehat{S}_{h_2, j+1}^{l_2}.$$

5.3. Mean squared error of a single period CDR

Combining the results of the previous two sections we obtain:

Estimator 5.5 (of the mean squared error of $\mathrm{CDR}_i^{\mathcal{M}, I+1}$)

Assume $S_{i,k}^m$ satisfy Assumptions 2.1 and 5.1 and take arbitrary $\mathcal{D}^I$-measurable factors α_i^m, $m \in \mathcal{M} \subseteq \{0, \ldots, M\}$. Then the mean squared error of the claim development result of a single accident period can be estimated

$$\widehat{\mathrm{mse}}\left[\mathrm{CDR}_i^{\mathcal{M}, I+1}\right] := \sum_{m_1, m_2 \in \mathcal{M}} \alpha_i^{m_1} \alpha_i^{m_2} \sum_{k_1, k_2 = I-i}^{J-1} \Big(\widehat{\mathbf{H}}_{i, k_1, i, k_2}^{m_1, m_2}(\widehat{\boldsymbol{\rho}}) - \widehat{\mathbf{H}}_{i, k_1, i, k_2}^{m_1, m_2}(0)$$
$$+ \widehat{\mathbf{H}}_{i, k_1, i, k_2}^{m_1, m_2}(\widehat{\boldsymbol{\rho}}^{*}) - \widehat{\mathbf{H}}_{i, k_1, i, k_2}^{m_1, m_2}(\widehat{\boldsymbol{\rho}}^{*1})$$
$$- \widehat{\mathbf{H}}_{i, k_1, i, k_2}^{m_1, m_2}(\widehat{\boldsymbol{\rho}}^{*2}) + \widehat{\mathbf{H}}_{i, k_1, i, k_2}^{m_1, m_2}(\widehat{\boldsymbol{\rho}}^{*12})\Big) \mathbf{S}^I \mathbf{S}^I.$$

Moreover, a linear approximation for the operator $\widehat{\mathbf{H}}(\tau)$ at $\tau = 0$ leads to

$$\widehat{\mathrm{mse}}\left[\mathrm{CDR}_i^{\mathcal{M}, I+1}\right] \approx \sum_{m_1, m_2 \in \mathcal{M}} \alpha_i^{m_1} \alpha_i^{m_2} \sum_{k_1, k_2 = I-i}^{J-1} \sum_{l_1, l_2 = 0}^{M} \sum_{j = I-i}^{k_1 \wedge k_2} \sum_{h_1, h_2 = I-j}^{I}$$
$$\left(\widehat{\rho}_{h_1, h_2, j}^{l_1, l_2} + \widehat{\rho}_{h_1, h_2, j}^{*l_1, l_2} - \widehat{\rho}_{h_1, h_2, j}^{*1\, l_1, l_2} - \widehat{\rho}_{h_1, h_2, j}^{*2\, l_1, l_2} + \widehat{\rho}_{h_1, h_2, j}^{*12\, l_1, l_2}\right)$$
$$\cdot \left(\widehat{\mathbf{F}}_{i, k_1}^{m_1, h_1 + j + 1}\right)_{h_1, j+1}^{l_1} \left(\widehat{\mathbf{F}}_{i, k_2}^{m_2, h_2 + j + 1}\right)_{h_2, j+1}^{l_2} \widehat{S}_{h_1, j+1}^{l_1} \widehat{S}_{h_2, j+1}^{l_2}.$$

5.4. Mean squared error of the overall CDR

As for the single period CDR we split the mean squared error of the overall CDR into a process variance and parameter estimation error term:

$$\mathrm{mse}\left[\sum_{i=0}^{I} \mathrm{CDR}_i^{\mathcal{M}, I+1}\right] := \mathrm{E}\left[\left(\sum_{i=0}^{I} \sum_{m \in \mathcal{M}} \alpha_i^m \sum_{k = I-i}^{J-1} \left(\widehat{S}_{i, k+1}^{m, I+1} - \widehat{S}_{i, k+1}^{m, I}\right) - 0\right)^2 \middle| \mathcal{D}^I\right]$$
$$= \mathrm{Var}\left[\sum_{i=0}^{I} \sum_{m \in \mathcal{M}} \alpha_i^m \sum_{k = I-i}^{J-1} \widehat{S}_{i, k+1}^{m, I+1} \middle| \mathcal{D}^I\right] + \left(\mathrm{E}\left[\sum_{i=0}^{I} \sum_{m \in \mathcal{M}} \alpha_i^m \sum_{k = I-i}^{J-1} \left(\widehat{S}_{i, k+1}^{m, I+1} - \widehat{S}_{i, k+1}^{m, I}\right) \middle| \mathcal{D}^I\right]\right)^2.$$

Since $\widehat{S}_{i_1,k_1}^{m_1,I+1}$ and $\widehat{S}_{i_2,k_2}^{m_2,I+1}$ depend on the new observed diagonal they are usually not $\mathcal{D}^I$-conditionally uncorrelated. Therefore, the overall process error as well as the parameter estimation error are not equal to the sum of all single period process and parameter errors, respectively.

Analogue to Sections 5.1 and 5.2 we can calculate the additional terms and get

Estimator 5.6 (of the mean squared error of the overall CDR)

Assume $S_{i,k}^m$ satisfy Assumptions 2.1 and 5.1 and take arbitrary $\mathcal{D}^I$-measurable factors α_i^m, $m \in \mathcal{M} \subseteq \{0, \ldots, M\}$. Then the mean squared error of the overall claim development result can be estimated by

$$\widehat{\mathrm{mse}}\left[\sum_{i=0}^{I} \mathrm{CDR}_i^{\mathcal{M},I+1}\right] := \sum_{i_1,i_2=0}^{I} \sum_{m_1,m_2 \in \mathcal{M}} \alpha_{i_1}^{m_1}\alpha_{i_2}^{m_2} \sum_{k_1=I-i_1}^{J-1} \sum_{k_2=I-i_2}^{J-1}$$
$$\Big(\widehat{\mathbf{H}}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\widehat{\boldsymbol{\rho}}) - \widehat{\mathbf{H}}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(0) + \widehat{\mathbf{H}}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\widehat{\boldsymbol{\rho}}^{*}) - \widehat{\mathbf{H}}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\widehat{\boldsymbol{\rho}}^{*1}) \tag{5.9}$$
$$- \widehat{\mathbf{H}}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\widehat{\boldsymbol{\rho}}^{*2}) + \widehat{\mathbf{H}}_{i_1,k_1,i_2,k_2}^{m_1,m_2}(\widehat{\boldsymbol{\rho}}^{*12})\Big)\mathbf{S}^I\mathbf{S}^I.$$

Moreover, a linear approximation for the operator $\widehat{\mathbf{H}}(\tau)$ at $\tau = 0$ leads to

$$\widehat{\mathrm{mse}}\left[\sum_{i=0}^{I} \mathrm{CDR}_i^{\mathcal{M},I+1}\right]$$
$$\approx \sum_{i_1,i_2=0}^{I} \sum_{m_1,m_2 \in \mathcal{M}} \alpha_{i_1}^{m_1}\alpha_{i_2}^{m_2} \sum_{k_1=I-i_1}^{J-1} \sum_{k_2=I-i_2}^{J-1} \sum_{l_1,l_2=0}^{M} \sum_{j=I-(i_1\wedge i_2)}^{k_1\wedge k_2} \sum_{h_1,h_2=I-j}^{I}$$
$$\left(\widehat{\rho}_{h_1,h_2,j}^{\,l_1,l_2} + \widehat{\rho}_{h_1,h_2,j}^{*l_1,l_2} - \widehat{\rho}_{h_1,h_2,j}^{*1\,l_1,l_2} - \widehat{\rho}_{h_1,h_2,j}^{*2\,l_1,l_2} + \widehat{\rho}_{h_1,h_2,j}^{*12\,l_1,l_2}\right)$$
$$\cdot \left(\widehat{\mathbf{F}}_{i_1,k_1}^{m_1,h_1+j+1}\right)_{h_1,j+1}^{l_1} \left(\widehat{\mathbf{F}}_{i_2,k_2}^{m_2,h_2+j+1}\right)_{h_2,j+1}^{l_2} \widehat{S}_{h_1,j+1}^{l_1} \widehat{S}_{h_2,j+1}^{l_2}.$$

Remark 5.7. *For the Chain-Ladder- and the Extended-Complementary-Loss-Ratio-Method the linear approximation is the same as in Buchwalder et al. [2, Approach 3] and Dahms-Merz-Wüthrich [4], respectively.*

If at time I we do not believe in the development of the next period, that means if we take all $w_{I-k,k}^{m,I+1} = 0$, the last four terms of (5.9) *and its linearisation will vanish. This means the mean squared error of the overall* CDR *is the sum of the process variance terms*

$$\widehat{\sigma}_j^{m_1,m_2} R_{h,j}^{m_1,m_2}$$

transferred to the ultimate by

$$\widehat{\mathbf{F}}_{i_1,k_1}^{m_1,h+j+1}\widehat{\mathbf{F}}_{i_2,k_2}^{m_2,h+j+1}.$$

Moreover, increasing the credibility $w_{I-k,k}^{m,I+1}$ *we give to the development of the next period will increase the part of the ultimate uncertainty that belongs to the development of the next period.*
The technically Assumption 5.1 could be weakened to arbitrary $\mathcal{D}^I \cap \mathcal{D}_k$*-measurable weights* $w_{I,k}^{m,I+1}$ *which satisfy the normalizing assumption. But in general this will lead to*

$$\widehat{\mathrm{E}}\left[\hat{f}_k^{m,I+1}\middle|\mathcal{D}^I\right] \neq \hat{f}_k^{m,I+1},$$

which means that at time I the estimated expected CDR *would not be zero. This does not get along with most of the reserving standards. Moreover, we would have to be a bit more careful with the resampling.*

6. Two examples

In the following we will present two examples of *LSRMs*. The first one illustrates the power of *LSRMs* if we want to analyse different kinds of reserves (or claims) by using different methods for the estimation without losing the ability to estimate the mean squared error of prediction of the overall ultimate outcome and of the overall claims development result. The second example shows how different methods may be mixed in order to estimate the reserves and the corresponding mean squared errors of prediction of the ultimate outcome and of the claims development result.

6.1. Example 1

The first example is an accident portfolio where we have three types of liabilities:

- **Medical expenses (ME)** will be estimated using the Chain-Ladder-Method. The motivation for the choice of this method (exposure) is mainly that it worked fine in the past. Data are provided in Table 3.
- **Payments for incapacitation for work (IW)** are by law proportional to the insured salary, which is limited to a maximum amount. Moreover, during accident period 7 the maximum insured salary has been increased by about 20%, valid for all claims happening afterwards. Therefore, we think the Complementary-Loss-Ratio-Method with the insured salary as external exposure is a good method to estimate the corresponding reserves. Data are provided in Tables 4 and 5.
- **Subrogation (Sub)** possibilities are huge. The reason is that many claims are caused by car accidents and that by law the accident insurer of the insured persons has to pay first and may take subrogation against the motor liability

insurer afterwards. Therefore, we assume that the amount of possible subrogation is proportional to the total amount that already had been paid, i.e. to ME+IW+Sub. Data are provided in Table 6.

For the coupling of those three types of payments we choose the cumulative total payments, i.e. $R_{i,k}^{m_1,m_2}$ is the sum of all payments (including subrogation) for all claims of accident period i up to development period k.

For the estimation we used the standards weights of (3.3) and the corresponding unbiased estimators for the model parameters (3.1), (4.3) and (4.4). Note, a few of the estimated correlation matrices for development period 6 and 7 are slightly non-positive defined. We believe that this is more an estimation problem than a model problem and we could change the estimated $\hat{\sigma}_k^{m_1,m_2}$ slightly in order to get always non-negative defined correlation matrices and only change the resulting MSEP and CDR by less than 0.5%. Therefore, we did not do that.

Table 1 shows the resulting estimates for the reserves, the MSEP and the CDR. In the last column we added the corresponding results of an overall Chain-Ladder-Method. Note, the difference between the shown figures and their linear approximations are less than 0.03. We see that the total reserves of the *LSRM* are much higher (11%) than the Chain-Ladder-Reserves. The main reason is that the Complementary-Loss-Ratio-Method fits the special development of the payments for IW better than the Chain-Ladder-Method. Moreover, the subrogation potential has been increased by the higher expected total payments.

Taking the Complementary-Loss-Ratio- instead of the Chain-Ladder-Method for the projection of IW is only important for the second development period. This can be verified by backtesting, but we do not have a good explanation for this behaviour. Since other parameters which have an impact on IW, like a change in the general economic situation, are not reflected within the insured salary it may be a further improvement to the model to choose the insured salary as exposure for the second development period and switch to the Chain-Ladder-Method for all other development periods.

The differences of the MSEP and the CDR between the *LSRM* and the Chain-Ladder-Method are not so significant, which confirms that neither the MSEP nor the CDR should be used to decide which method is the best. We strongly recommend to look for good exposure measures $R_{i,k}^{m}$ that can be motivated by other facts than triangle based statistics.

TABLE 1

RESULTS OF EXAMPLE 1

	ME	IW	Sub	Total	Total CL
Reserves	81'954	125'809	–46'443	161'319	144'788
MSEP	3'777	5'991	4'975	8'504	8'633
CDR	2'795	4'723	3'208	6'088	6'484

6.2. Example 2

In this example we want to show how *LSRMs* may be used in order to combine method based results with actuarial judgement. For instance, assume we have projected payments and reported amounts (or incurred) separately with some *LSRM* (the method based results). Now we look at those projections and decide about a final ultimate, which is a linear combination of the two projections (actuarial judgement). If we introduce in addition a coupling exposure $R^{0,1}_{i,k}$ we automatically get a corresponding estimate of the overall uncertainty and the uncertainty of the claims development result.

As example we take the data of Dahms [3, Example 1]. The triangles are shown in Tables 7 and 8. We will apply the following two *LSRMs*:

- **ECLRM:** The Extended-Complementary-Loss-Ratio-Method, see Section 2.4. We take the same parameter as in Dahms [3]. Note, the parameters $\hat{\sigma}^{m_1,m_2}_{i,k}$ of Dahms [3] are not the variance minimising estimators for $\sigma^{m_1,m_2}_{i,k}$ as presented in (4.3), but the effect on the estimators for the uncertainty is less than 0.5%.
- **CL:** We project payments $S^0_{i,k}$ and reported amounts $S^1_{i,k}$ separately by the Chain-Ladder-Method and couple the projections by the exposure

$$R^{0,1}_{i,k} := \sum_{j=0}^{k} S^0_{i,j} + S^1_{i,j}.$$

For the coupling of the projected estimates we take a credibility approach that is a generalisation of the credibility interpretation of the Bornhuetter-Ferguson-Method, which is the credibility mixture of a projected ultimate $C_{i,J}$ and an external given ultimate U_i. The credibility weight given to U_i is proportional to the distance of the projected ultimate and the last known value $C_{i,I-i}$. This means we look at the credibility mixture

$$\frac{C_{i,I-i}}{C_{i,J}} C_{i,J} + \frac{C_{i,J} - C_{i,I-i}}{C_{i,J}} U_i = \frac{C_{i,I-i}}{C_{i,J}} C_{i,J} + \left(1 - \frac{C_{i,I-i}}{C_{i,J}}\right) U_i.$$

This works fine as long as $C_{i,I-i} \leq C_{i,J}$. If this is not the case we could take $\frac{C_{i,J}}{C_{i,I-i}}$ instead of $\frac{C_{i,I-i}}{C_{i,J}}$. Finally, generalising the above formula to M projected ultimates we get the following credibility mixture

$$\frac{\sum_{m=0}^{M} \alpha^m_i C^m_{i,J}}{\sum_{m=0}^{M} \alpha^m_i},$$

with

$$\alpha^m_i := \left(\frac{C^m_{i,I-i}}{C^m_{i,J}} \wedge \frac{C^m_{i,J}}{C^m_{i,I-i}}\right).$$

TABLE 2

Estimates of Example 2.

	Reserves	MSEP	MSEP proxy	CDR	CDR proxy
CL Paid	10˙165˙612	1˙517˙861	1˙517˙480	1˙004˙481	1˙004˙164
CL Incurred	10˙665˙287	455˙802	455˙794	347˙709	347˙698
Mixed CL	10˙539˙276	676˙047	675˙927	478˙785	478˙688
ECLRM Paid	10˙728˙771	467˙964	467˙814	346˙640	346˙576
ECLRM Incurred	10˙728˙771	472˙131	471˙873	350˙692	350˙534
Mixed ECLRM	10˙728˙771	469˙518	469˙324	348˙110	348˙009

Table 2 shows the resulting estimates for the reserves, for the mean squared error of prediction (MSEP) and for the uncertainty of the claims development result (CDR). Moreover, the table contains the linear approximations of the presented estimators. Note, they differ from their original values by less than 0.1%.

For the estimation of the reserves within the ECLRM the credibility mixture has no effect, because this method already combines both triangles in such a way that the projection of payments lead to the same estimated reserves like the projection of reported amounts. But in order to get estimates for the MSEP and the uncertainty of the CDR such a credibility mixture may be useful, although in this example the corresponding values differ only slightly. The linear approximations are the same as presented in Dahms [3] and Dahms-Merz-Wüthrich [4].

The credibility weighted estimates for the Chain-Ladder-Methods tent more in the direction of the projection of the reported amounts. But this does not have to be the case. Although the weighted estimates for each single accident period always lie between the corresponding two estimates of the separate projections the overall estimates (for all accident periods) do not have to be between the corresponding two estimates of the separate projections.

7. Conclusion

Up to now in most cases discussions about the choice of reserving methods were more philosophic than scientific. By introducing *LSRMs* we want to encourage actuaries to spend more time on the investigation of drivers (exposures) behind the development of portfolios, claims and claim properties. If such a driver is, at least heuristically, identified and if the dependence structure is linear we have a very good reason to look at the corresponding *LSRM* for reserving purposes. This means the discussion about the choice may now be based on heuristic reasons about exposures driving the claims development, which in our opinion is much better than a pure philosophic approach.

TABLE 3

CUMULATIVE MEDICAL EXPENSES (ME) OF EXAMPLE 1.

AP \ DP	0	1	2	3	4	5	6	7	8
0	32'049'768	63'513'505	67'369'991	69'284'366	70'320'614	70'976'783	71'332'078	71'482'164	71'548'144
1	30'773'240	61'573'313	66'438'071	68'621'389	70'325'480	71'423'351	72'082'591	72'306'733	
2	29'604'850	59'596'781	64'276'364	67'367'651	69'234'941	70'321'894	71'140'995		
3	30'807'773	60'402'823	66'025'881	70'363'461	72'279'063	73'265'992			
4	30'657'470	63'369'991	68'525'705	70'632'024	72'410'130				
5	35'211'976	69'320'943	74'343'378	76'684'606					
6	37'919'094	74'154'428	80'107'831						
7	38'966'667	78'809'524							
8	39'855'014								

TABLE 4

CUMULATIVE PAYMENTS FOR INCAPACITATION FOR WORK (IW) OF EXAMPLE 1.

AP \ DP	0	1	2	3	4	5	6	7	8
0	48'628'810	100'963'185	110'554'502	114'119'367	116'645'683	117'496'665	117'604'048	117'643'102	117'662'650
1	45'155'170	95'474'850	106'938'693	112'709'009	116'331'874	117'556'952	117'963'655	118'115'590	
2	44'096'634	92'816'782	103'800'762	108'188'259	110'238'974	110'983'483	111'171'294		
3	42'129'452	89'627'198	101'297'355	105'185'576	107'232'475	108'125'979			
4	42'141'971	90'152'967	100'367'106	103'543'007	104'531'566				
5	51'009'151	103'592'399	114'433'185	117'696'951					
6	46'896'999	98'408'508	110'961'202						
7	47'179'472	102'408'320							
8	49'927'514								

TABLE 5

INSURED SALARY OF EXAMPLE 1.

AP	Salary
0	98'982
1	96'267
2	99'243
3	100'374
4	107'739
5	110'798
6	107'097
7	118'494
8	135'846

TABLE 6

CUMULATIVE SUBROGATION (SUB) OF EXAMPLE 1.

AP \ DP	0	1	2	3	4	5	6	7	8
0	–1'186'623	–8'288'444	–11'355'093	–13'701'656	–15'738'756	–19'923'012	–24'128'664	–25'753'654	–26'551'442
1	–976'191	–6'477'313	–10'229'110	–12'560'543	–15'352'201	–18'888'775	–21'542'902	–22'160'350	
2	–1'012'196	–5'763'248	–9'598'395	–13'224'706	–17'163'998	–20'189'269	–21'690'726		
3	–1'133'569	–7'513'501	–12'821'732	–17'966'920	–22'020'892	–24'322'587			
4	–1'500'880	–7'487'797	–12'636'852	–14'821'871	–16'297'604				
5	–1'396'143	–10'756'906	–17'244'816	–19'006'441					
6	–2'863'771	–11'652'478	–16'185'295						
7	–2'740'522	–9'530'119							
8	–2'035'799								

TABLE 7

CUMULATIVE PAYMENTS OF EXAMPLE 2.

AP \ DP	0	1	2	3	4	5	6	7	8	9
0	1'216'632	1'347'072	1'786'877	2'281'606	2'656'224	2'909'307	3'283'388	3'587'549	3'754'403	3'921'258
1	798'924	1'051'912	1'215'785	1'349'939	1'655'312	1'926'210	2'132'833	2'287'311	2'567'056	
2	1'115'636	1'387'387	1'930'867	2'177'002	2'513'171	2'931'930	3'047'368	3'182'511		
3	1'052'161	1'321'206	1'700'132	1'971'303	2'298'349	2'645'113	3'003'425			
4	808'864	1'029'523	1'229'626	1'590'338	1'842'662	2'150'351				
5	1'016'862	1'251'420	1'698'052	2'105'143	2'385'339					
6	948'312	1'108'791	1'315'524	1'487'577						
7	917'530	1'082'426	1'484'405							
8	1'001'238	1'376'124								
9	841'930									

TABLE 8

CUMULATIVE REPORTED AMOUNTS OF EXAMPLE 2.

AP \ DP	0	1	2	3	4	5	6	7	8	9
0	3'362'115	5'217'243	4'754'900	4'381'677	4'136'883	4'094'140	4'018'736	3'971'591	3'941'391	3'921'258
1	2'640'443	4'643'860	3'869'954	3'248'558	3'102'002	3'019'980	2'976'064	2'946'941	2'919'955	
2	2'879'697	4'785'531	4'045'448	3'467'822	3'377'540	3'341'934	3'283'928	3'257'827		
3	2'933'345	5'299'146	4'451'963	3'700'809	3'553'391	3'469'505	3'413'921			
4	2'768'181	4'658'933	3'936'455	3'512'735	3'385'129	3'298'998				
5	3'228'439	5'271'304	4'484'946	3'798'384	3'702'427					
6	2'927'033	5'067'768	4'066'526	3'704'113						
7	3'083'429	4'790'944	4'408'097							
8	2'761'163	4'132'757								
9	3'045'376									

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